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From higher Bruhat orders to Steenrod cup-i coproducts

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Abstract. We show that the higher Bruhat orders of Manin and Schechtman provide a useful conceptual framework for understanding Steenrod's cup-i coproducts, which are used to define the cohomology operations known as Steenrod squares. Indeed, we show that the elements of the $(i+1)$-dimensional higher Bruhat order are in bijection with all possible cup-i coproducts on the chain complex of the simplex which give a homotopy between cup-$(i-1)$ and its opposite. The Steenrod cup-i coproduct and its opposite are then given by the maximal and minimal elements of the higher Bruhat order. This correspondence uses the geometric realisation of the higher Bruhat orders in terms of tilings of cyclic zonotopes, and enables us to give conceptual proofs of the fundamental properties of the cup-i coproducts.

Keywords: Higher Bruhat orders, zonotopal tilings, cubillages, cup-i coproducts, Steenrod squares

1 Introduction

In a classical article from 1947, N. E. Steenrod introduced the cup-i products on the cochains of a simplicial complex [12]. These can allow one to distinguish between non-homotopy-equivalent spaces with isomorphic cohomology rings, such as the suspensions of $\mathbb{C}P^2$ and $S^2 \vee S^4$. More recently, they were shown to be part of an E_∞-algebra structure on the cochains of a space X [11, 1], which faithfully encodes its homotopy type when X is of finite type and nilpotent [8, 9].

The cup-0 product is the usual cup product. This is commutative at the level of cohomology, but not at the level of cochains. The cup-1 product provides an explicit homotopy between cup-0 and its opposite which witnesses this fact. However, the cup-1

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product is itself not commutative, which gives rise to the cup-2 product, and so on. In this paper, we consider cup-i coproducts, which give rise to cup-i products via linear duality.

Over 40 years later, Yu. I. Manin and V. V. Schechtman introduced the higher Bruhat orders, with an entirely different purpose [10]. Their original motivation was to study hyperplane arrangements and higher braid groups, but the higher Bruhat orders have gone on to subsequently find connections with Soergel bimodules [4], the quantum Yang–Baxter equation [3], and many other areas of mathematics besides.

The 1-dimensional higher Bruhat order is the weak Bruhat order on the symmetric group. The elements of the 2-dimensional higher Bruhat order are then equivalence classes of maximal chains in the weak order, that is, reduced expressions for the longest element, up to swapping commuting simple reflections. The covering relations of the 2-dimensional order are then given by braid moves. This pattern repeats, with the $(i+1)$-dimensional higher Bruhat orders having as elements equivalence classes of maximal chains in the i-dimensional order.

Already, one can see a resemblance between the cup-i coproducts and the higher Bruhat orders insofar as in each case the objects in a given dimension give rise to the objects in one dimension higher. In fact, it is more than a resemblance, as we show. To state our main theorem, we need the following notation.

- $\mathcal{B}([0,n], i+1)$ is the $(i+1)$-dimensional higher Bruhat poset on the base set $[0,n] := \{0, 1, \ldots, n\}$.
- $\Delta_i\colon C_\bullet(\Delta^n) \to C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ is the Steenrod cup-i coproduct on the chain complex of the n-simplex Δ^n, where we set $\Delta_{-1} = 0$.
- $T\colon C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n) \to C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ is the isomorphism given by exchange of tensor factors, with the Koszul sign rule applied.

Theorem ([7, Theorem 3.6, Theorem 3.9])**.** *For all $i \geqslant 0$, there is a bijection between elements of $\mathcal{B}([0,n], i+1)$ and coproducts Δ_i' on $C_\bullet(\Delta^n)$ satisfying the homotopy formula*

$$\partial \circ \Delta_i' - (-1)^i \Delta_i' \circ \partial = (1 + (-1)^i T)\Delta_{i-1}, \tag{1.1}$$

and containing no redundant terms, with the additional assumption in the $i = 0$ case that $\Delta_0'(p) = p \otimes p$ for all vertices p of Δ^n.

Moreover, under this bijection, the minimal and maximal elements of the higher Bruhat orders correspond to Δ_i and $T\Delta_i$, up to sign.

The Steenrod cup-i coproduct Δ_i alternates between corresponding to the minimal element of the higher Bruhat poset and the maximum, according to the parity of i. This is due to differing conventions in the two theories.

The proof of this theorem uses the geometric realisation of the higher Bruhat orders $\mathcal{B}([0,n], i+1)$ in terms of zonotopal tilings of $Z([0,n], i+1)$, the $(i+1)$-dimensional cyclic zonotope with $n+1$ vertices. We call these tilings "cubillages" and refer to their tiles as "cubes". There is a natural bijection between cubes of cubillages and basis elements of $C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$, such that the terms of $\Delta_i'([0,n])$ in $C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ is described by a set of cubes of a cubillage of $Z([0,n], i+1)$. Simplifying slightly, this is used to derive the homotopy formula as follows. Taking the boundary ∂ corresponds to taking the boundary of the cubes of the cubillage. Facets of cubes which are shared with other cubes cancel out, so that the only remaining terms come from the boundary of the zonotope. These terms turn out to be precisely $(1+(-1)^i T)\Delta_{i-1}$. Running the argument in reverse shows that all coproducts satisfying the homotopy formula must arise from cubillages, that is, elements of $\mathcal{B}([0,n], i+1)$.

This paper is an extended abstract of [7]. In Section 2, we give background on the Steenrod cup-*i* coproducts, followed by background on the higher Bruhat orders. We then outline our main results in Section 3, referring to [7] for complete proofs.

2 Background

In this section, we recall the definitions of Steenrod operations and higher Bruhat orders, and set up notation.

2.1 Steenrod cup-i coproducts

2.1.1 Chain complexes

By a (non-negatively graded) *chain complex* C we mean an $\mathbb{N}$-graded $\mathbb{Z}$-module with linear maps

$$C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} \cdots$$

satisfying $\partial_p \circ \partial_{p+1} = 0$ for each $p \in \mathbb{N}$. As usual, we refer to ∂_p as the p-th *boundary map* and suppress the subscript when convenient. A morphism of chain complexes, referred to as a *chain map*, is a morphism of $\mathbb{N}$-graded $\mathbb{Z}$-modules $f\colon C \to C'$ satisfying $\partial'_{p+1} f_{p+1} = f_p \partial_{p+1}$ for $p \in \mathbb{N}$.

The tensor product of two chain complexes X and Y is the chain complex $X \otimes Y$ whose degree r component is $(X \otimes Y)_r := \oplus_{p+q=r} X_p \otimes Y_q$, and whose differential is defined by $\partial(x \otimes y) := \partial(x) \otimes y + (-1)^{\deg(x)} x \otimes \partial(y)$. There is an isomorphism $T\colon X \otimes Y \to Y \otimes X$ defined by $T(x \otimes y) := (-1)^{\deg(x)\deg(y)} y \otimes x$.

2.1.2 Chain complex of the simplex

We denote the standard n-simplex $\Delta^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1,\ x_i \geqslant 0\}$. We refer to faces of the n-simplex using their vertex sets, where we use the notation $[p,q] := \{p, p+1, \ldots, q\}$ and $(p,q) := [p,q] \setminus \{p,q\}$. When we give a set $\{v_0, v_1, \ldots, v_q\} \subseteq [0,n]$, we mean that the elements are ordered $v_0 < v_1 < \cdots < v_q$.

We will consider the $\mathbb{Z}$-module given by the cellular chains $\mathrm{C}_\bullet(\Delta^n)$ on the standard n-simplex. This chain complex has as basis the faces of Δ^n, whose degree is given by the dimension; for example, the face $\{v_0, \ldots, v_q\}$ has dimension q. The boundary map of this chain complex is given by

$$\partial(\{v_0, \ldots, v_q\}) := \sum_{p=0}^{q} (-1)^p \{v_0, \ldots, \hat{v}_p, \ldots, v_q\}.$$

2.1.3 The Steenrod cup-i coproducts

An *overlapping partition* of $[0,n]$ is a family $\mathcal{L} = (L_0, L_1, \ldots, L_{i+1})$ of intervals $L_p = [l_p, l_{p+1}]$ such that $l_0 = 0$, $l_{i+2} = n$, and for each $0 < p < i+1$ we have $l_p < l_{p+1}$. The *Steenrod cup-i-coproduct* is the coproduct $\Delta_i \colon \mathrm{C}_\bullet(\Delta^n) \to \mathrm{C}_\bullet(\Delta^n) \otimes \mathrm{C}_\bullet(\Delta^n)$ given by the formula

$$\Delta_i([0,n]) := \sum_{\mathcal{L}} (-1)^{\varepsilon(\mathcal{L})} (L_0 \cup L_2 \cup \cdots) \otimes (L_1 \cup L_3 \cup \cdots),$$

where the sum is taken over all overlapping partitions of $[0,n]$ into $i+2$ intervals. If $n \leqslant i-1$, there are no such overlapping partitions, and the coproduct is zero. Denoting by $w_{\mathcal{L}}$ the shuffle permutation putting $0, 1, \ldots, n$ into the order

$$[0, l_1], [l_2, l_3], \ldots, (l_1, l_2), (l_3, l_4), \ldots,$$

the sign is given by $\varepsilon(\mathcal{L}) := \mathsf{sign}(w_{\mathcal{L}}) + in$. The coproduct Δ_i is then defined similarly on lower-dimensional faces, by summing over increasing overlapping partitions of their vertex sets. Steenrod [12] then shows that

$$\partial \Delta_i - (-1)^i \Delta_i \partial = (1 + (-1)^i T) \Delta_{i-1}, \tag{2.1}$$

where we set $\Delta_{-1} = 0$. We refer to this as the *homotopy formula*, since it is equivalent to saying that Δ_i gives a chain homotopy from $T\Delta_{i-1}$ to Δ_{i-1}.

2.2 Higher Bruhat orders

There are many ways of defining the higher Bruhat orders. For the purposes of this abstract, we only consider the geometric definition of the higher Bruhat orders in terms of cubillages of cyclic zonotopes due to [5, 13]. Other definitions can be found in [7].

Definition 2.1. Consider the *Veronese curve* $\xi\colon \mathbb{R} \to \mathbb{R}^{i+1}$, given by $\xi_t = (1, t, t^2, \ldots, t^i)$. The *cyclic zonotope* $Z([0,n], i+1)$ is defined to be the Minkowski sum of the line segments

$$\overline{\mathbf{0}\xi_0} + \cdots + \overline{\mathbf{0}\xi_n},$$

where $\mathbf{0}$ is the origin and $\overline{\mathbf{0}\xi_p}$ is the line segment from $\mathbf{0}$ to ξ_p. One similarly defines $Z(S, i+1)$ for $S \subseteq [0,n]$ a subset. There is a natural projection $\pi_{i+1}\colon Z([0,n], n+1) \to Z([0,n], i+1)$ given by forgetting the last $n-i$ coordinates.

Definition 2.2. A *cubillage* U of $Z([0,n], i+1)$ is a set of $(i+1)$-dimensional faces $\{F_p\}$ of $Z([0,n], n+1)$ such that π_{i+1} is a bijection when restricted to $\bigcup_p F_p$. Such faces F_p are necessarily $(i+1)$-dimensional, and we refer to them as the *cubes* of the cubillage.

A cubillage U of $Z([0,n], i+1)$ gives a subdivision of $Z([0,n], i+1)$ consisting of the images of the cubes under π_{i+1}. In the literature, cubillages are often called *fine zonotopal tilings*.

Recall that a *facet* of a polytope is a face of codimension one. The standard basis of $\mathbb{R}^{n+1}$ induces orientations of the faces of $Z([0,n], n+1)$, in the sense that the facets of a face can be partitioned into two sets, called upper facets and lower facets.

Definition 2.3. If F is a i-dimensional face of $Z([0,n], n+1)$, with G a facet of F, then G is a *lower* (resp. *upper*) facet of F if normal vectors to G which lie inside the affine span of F and point into F have positive (resp. negative) i-th coordinates.

One may similarly talk of lower and upper facets of $Z([0,n], i+1)$.

Definition 2.4 ([13, Thm. 2.1, Prop. 2.1], [5, Thm. 4.4])**.** The elements of $\mathcal{B}([0,n], i+1)$ consist of the cubillages of $Z([0,n], i+1)$. The covering relations of $\mathcal{B}([0,n], i+1)$ are given by pairs of cubillages $U \lessdot U'$ that differ by an *increasing flip*, that is when there is a $(i+2)$-face G of $Z([0,n], n+1)$ such that $(\bigcup_{F \in U} F) \setminus G = (\bigcup_{F' \in U'} F') \setminus G$ and such that U contains the lower facets of G, whereas U' contains the upper facets of G.

In Figure 1 we illustrate a pair of cubillages, where the right-hand cubillage is an increasing flip of the left. Here we have illustrated the cubillages using their images under π_2, but really they are sets of faces of the four-dimensional zonotope $Z([0,3], 4)$. We have labelled the vertices $\xi_A = \sum_{a \in A} \xi_a$ of $Z([0,3], 4)$ which lie in the cubillage by dropping the 'ξ' and only retaining the subscript 'A'; we will continue to do this.

The cyclic zonotope $Z([0,n], i+1)$ possesses two canonical cubillages. One is given by the set of faces $U_{\min}$ of $Z([0,n], n+1)$ that project to the lower facets of $Z([0,n], i+2)$ under the projection π_{i+2}. This is known as the *lower cubillage*. The other is given by the set of faces $U_{\max}$ that project to the upper facets of $Z([0,n], i+2)$, which we call the *upper cubillage*. The lower cubillage $U_{\min}$ of $Z([0,n], i+1)$ gives the unique minimum of the poset $\mathcal{B}([0,n], i+1)$, and the upper cubillage $U_{\max}$ gives the unique maximum. We have the following important theorem.

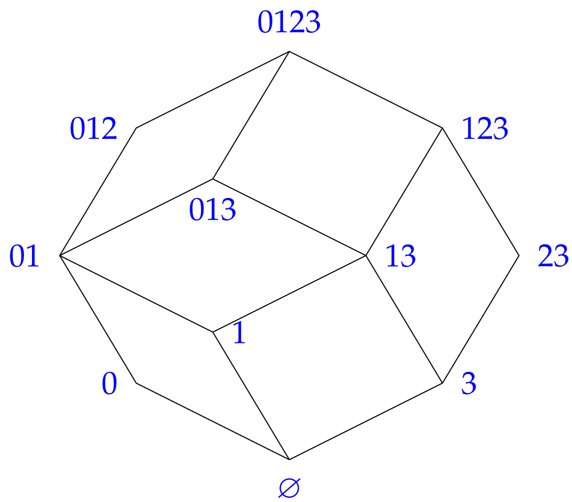

Figure 1: A pair of cubillages of $Z([0,3],2)$ such that the right is an increasing flip of the left.

Theorem 2.5 ([10, Thm. 2.3])**.** *There is a bijection between equivalence classes of maximal chains in $\mathcal{B}([0,n],i+1)$ and elements of $\mathcal{B}([0,n],i+2)$.*

The idea is that the $(i+2)$-dimensional faces which give covering relations in a maximal chain in $\mathcal{B}([0,n],i+2)$ give a cubillage of $Z([0,n],i+2)$. The equivalence relation mentioned in the theorem identifies maximal chains such that the set of these $(i+2)$-dimensional faces is the same.

Every $(i+1)$-dimensional face of $Z([0,n],n+1)$ is given by a Minkowski sum

$$\xi_A + \sum_{l \in L} \overline{\mathbf{0}\xi_l}$$

for some subset $L \in \binom{[0,n]}{i+1}$ and $A \subseteq [0,n] \setminus L$. We call L the set of *generating vectors* and A the *initial vertex*. In Figure 2, we illustrate the initial vertices and generating vectors of the cubillages from Figure 1. The initial vertices are labelled in blue, and the generating vectors are labelled in red in the centre of the cube. Given a cubillage $U \in \mathcal{B}([0,n],i+1)$, we write A_L^U for the initial vertex of the cube with generating vectors L in U. We also write $B_L^U := [0,n] \setminus (L \cup A_L^U)$ for the vectors which are neither generating vectors nor present in the initial vertex.

Given a cubillage U of $Z([0,n],i+1)$ and $k \in [0,n]$, we write U/k for the cubillage of $Z([0,n] \setminus k, i+1)$ given by the set of faces of $Z([0,n] \setminus k, n+1)$ which results from taking U and contracting all edges given by the vector ξ_k until they have length zero. For a

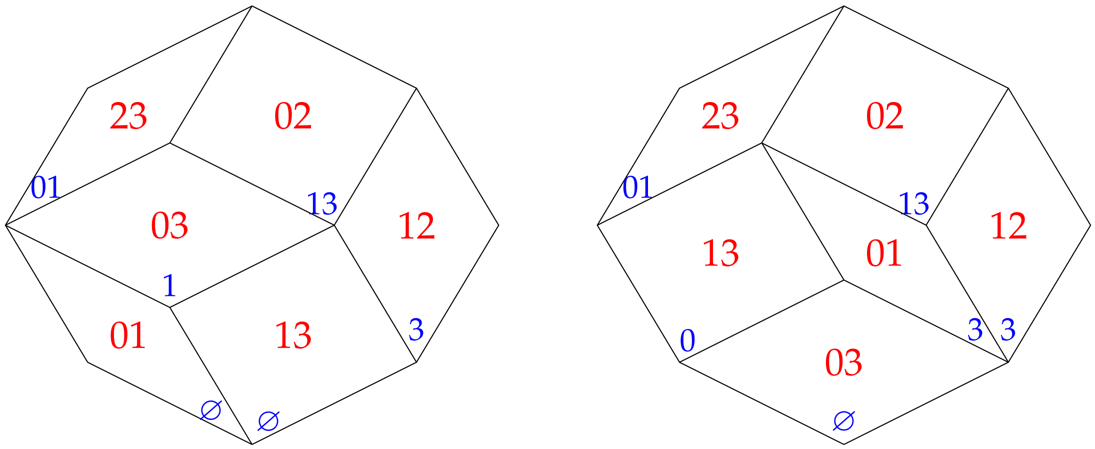

Figure 2: The pair of cubillages, with initial vertices and generating vectors indicated.

more precise construction, see [2, (4.3)], or for a construction using a different realisation of the higher Bruhat orders, see [7, Section 2.2.2].

3 Coproducts from cubillages

In this section, we show how one can construct a coproduct $\Delta_i^U\colon \mathrm{C}_\bullet(\Delta^n) \to \mathrm{C}_\bullet(\Delta^n) \otimes \mathrm{C}_\bullet(\Delta^n)$ from any cubillage $U \in \mathcal{B}([0,n], i+1)$. We show that all these coproducts give homotopies between Δ_{i-1} and $T\Delta_{i-1}$ and that all coproducts for which this is true arise from cubillages, provided they contain no redundant terms.

A central observation is as follows. Basis elements of $\mathrm{C}_\bullet(\Delta^n) \otimes \mathrm{C}_\bullet(\Delta^n)$ are of the form $X \otimes Y$ for X and Y non-empty faces of Δ^n. Given a basis element $X \otimes Y \in \mathrm{C}_\bullet(\Delta^n) \otimes \mathrm{C}_\bullet(\Delta^n)$, we can always write $X = L \cup A$ and $Y = L \cup B$, where L, A, and B are pairwise disjoint. Given a subset $S \subseteq [0,n]$, we then say that $L \cup A \otimes L \cup B$ is *supported on* S if $L \cup A \cup B = S$.

Proposition 3.1. *There is a bijection between faces of $Z(S, |S|)$ excluding $\varnothing$ and S and basis elements of $\mathrm{C}_\bullet(\Delta^n) \otimes \mathrm{C}_\bullet(\Delta^n)$ which are supported on S, given by sending a face with initial vertex A and generating vectors L to $L \cup A \otimes L \cup B$, where $B := S \setminus (L \cup A)$.*

Hence, we may identify basis elements of $\mathrm{C}_\bullet(\Delta^n) \otimes \mathrm{C}_\bullet(\Delta^n)$ with the corresponding faces of $Z(S, |S|)$, in particular in the case $S = [0,n]$.

We illustrate in Figure 3 the elements of $\mathrm{C}_\bullet(\Delta^3) \otimes \mathrm{C}_\bullet(\Delta^3)$ assigned by Proposition 3.1 to the cubes of our running pair of cubillages. Compare this to the description of the generating vectors and initial points in Figure 2. The coproduct Δ_i^U associated to a cubillage U of $Z([0,n], i+1)$ is defined by setting $\Delta_i^U([0,n])$ as the sum of elements of $\mathrm{C}_\bullet(\Delta^n) \otimes \mathrm{C}_\bullet(\Delta^n)$ assigned to the cubes of U.

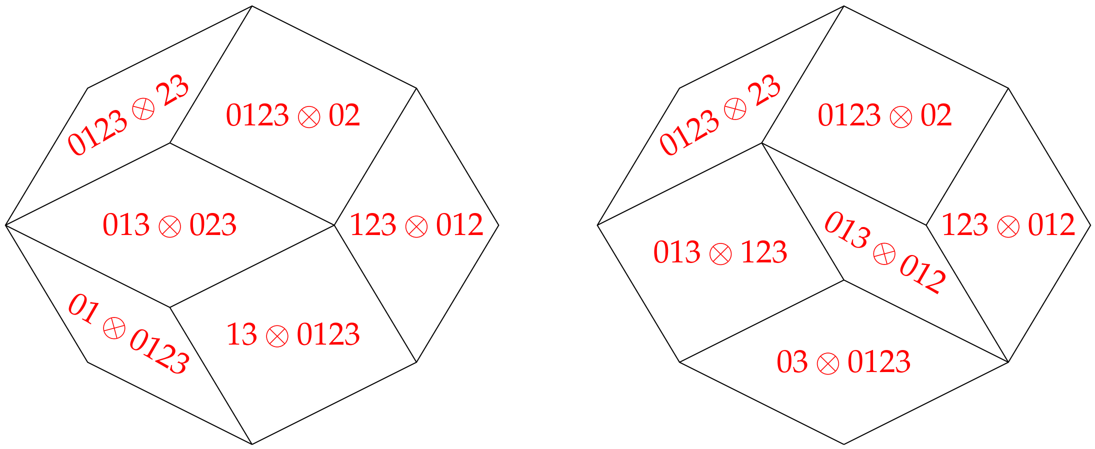

Figure 3: The terms of $C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ corresponding to the maximal cubes.

Construction 3.2. For any $U \in \mathcal{B}([0,n], i+1)$, where $n \geqslant i$, we now define the cup-i coproduct

$$\Delta_i^U \colon C_\bullet(\Delta^n) \to C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n).$$

We define Δ_i^U on the top face of Δ^n by the formula

$$\Delta_i^U([0,n]) := \sum_{L \in \binom{[0,n]}{i+1}} (-1)^{\varepsilon(L \cup A_L^U \otimes L \cup B_L^U)} L \cup A_L^U \otimes L \cup B_L^U,$$

where

$$\varepsilon(L \cup A_L^U \otimes L \cup B_L^U) := \sum_{b \in B_L^U} |A_L^U|_{<b} + \sum_{l \in L} |L|_{<l} + (n+1)|A_L^U|.$$

For codimension one faces, we define

$$\Delta_i^U([0,n] \setminus \{i\}) := \Delta_i^{U/i}([0,n] \setminus \{i\}).$$

In this way, we inductively extend the definition to lower-dimensional faces too. Once we reach a non-empty subset $S \subseteq [0,n]$ with $|S| \leqslant i$, we define $\Delta_i^U(S) := 0$, since in this case $\mathcal{B}(S, i+1)$ is empty.

The reader may wish to ignore the sign $\varepsilon(L \cup A_L^U \otimes L \cup B_L^U)$ for the purposes of this abstract. Many explicit sign calculations are carried out in [7, Appendix A].

3.1 Comparing with original Steenrod cup-i coproducts

Due to differing conventions, the Steenrod cup-i coproducts alternate between the minimal element of the higher Bruhat orders and the maximal element, according to the parity of i.

Theorem 3.3. *For i even, we have*

$$\Delta_i^{U_{\min}} = (-1)^{i/2}\Delta_i \quad \text{and} \quad \Delta_i^{U_{\max}} = (-1)^{i/2}T\Delta_i,$$

whilst for i odd we have

$$\Delta_i^{U_{\min}} = (-1)^{\lceil i/2 \rceil}T\Delta_i \quad \text{and} \quad \Delta_i^{U_{\max}} = (-1)^{\lfloor i/2 \rfloor}\Delta_i.$$

Sketch. Cubes of $U_{\min}$ or $U_{\max}$ with generating vectors L have initial points given by taking alternating parts of increasing overlapping paritions whose overlaps are given by L. This means that the terms of the coproducts coincide, up to applying T and adding a sign. □

3.2 Deriving the homotopy formula

The boundary of a term in the coproduct has a neat description in terms of the cubillage. Recalling Proposition 3.1, we may talk of upper and lower facets of a basis element F of $C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$, meaning the respective terms corresponding to the upper and lower facets of the face of $Z([0,n], n+1)$ corresponding to F. The following proposition follows from direct computation.

Proposition 3.4. *Let $F = L \cup A_L^U \otimes L \cup B_L^U$ be a basis element of $C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ supported on $[0,n]$. Adopting the notation $F/k := L \cup A_L^{U/k} \otimes L \cup B_L^{U/k}$, we have that*

$$\partial((-1)^{\varepsilon(F)}F) = \sum_{\substack{G \\ \text{lower facet}}} (-1)^{\varepsilon(G)}G + \sum_{\substack{H \\ \text{upper facet}}} (-1)^{\varepsilon(H)+1}H + \sum_{k \in [0,n] \setminus L} (-1)^{\varepsilon(F/k)+k+i+2}F/k.$$

Remark 3.5. Note that terms in $\partial((-1)^{\varepsilon(F)}F)$ are of one of the forms $L \setminus \{k\} \cup A_L^U \otimes L \cup B_L^U$, $L \cup A_L^U \otimes L \setminus \{k\} \cup B_L^U$, $L \cup A_L^U \setminus \{k\} \otimes L \cup B_L^U$, or $L \cup A_L^U \otimes L \cup B_L^U \setminus \{k\}$. The first and second of these may either be lower facets or upper facets, whilst the third and fourth give terms in the last sum in Proposition 3.4.

Showing that the coproduct Δ_i^U satisfies the homotopy formula is now straightforward.

Theorem 3.6 ([7, Theorem 3.6]). *For any $U \in \mathcal{B}([0,n], i+1)$, and for any $i \geqslant 0$, we have that*

$$\partial \circ \Delta_i^U - (-1)^i \Delta_i^U \circ \partial = (1 + (-1)^i T)\Delta_{i-1}^{U_{\min}}.$$

Sketch. Using Proposition 3.4, when we expand $\partial \circ \Delta_i^U([0,n])$ we see that terms from shared facets of cubes lying inside $Z([0,n], i+1)$ cancel, and that we are left with terms of the form $(-1)^{\varepsilon(F/k)+k+i+2}F/k$ and terms corresponding to the upper and lower facets of $Z([0,n], i+1)$. The former terms cancel with those from $(-1)^i\Delta_i^U \circ \partial$, and the latter terms give the right-hand side. □

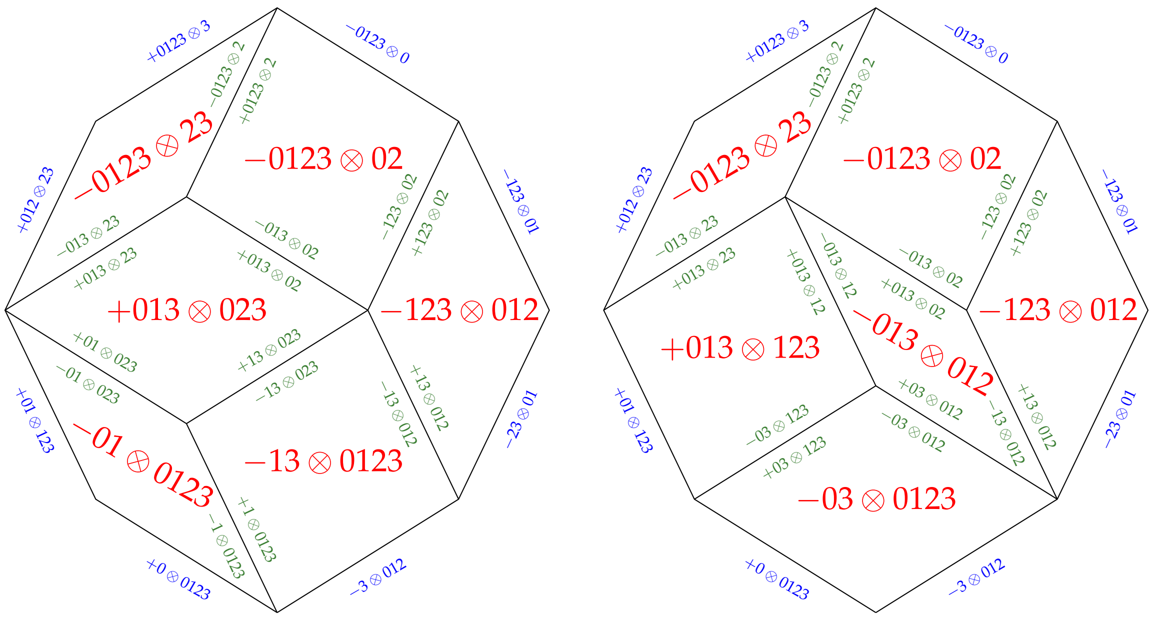

Figure 4: Illustrating why the homotopy formula holds.

Example 3.7. In Figure 4, we illustrate the proof of Theorem 3.6 for our two coproducts from Figure 3. The boundary of each red term from the cubes of the cubillage consists of terms coming from the facets of the cube F, along with terms of the form $(-1)^{\varepsilon(F/k)+k+i+2}F/k$. While we do not illustrate these latter terms, it can be seen that when we have two cubes sharing a facet inside the zonotope, the terms on the facet given by each cube have opposite sign, and so cancel. We are left with the terms on the boundary of the zonotope. The left-hand facets are the lower facets and correspond to terms of the usual Δ_0 cup-coproduct, whereas the terms on the right-hand facets are the terms of $-T\Delta_0$. Note that the terms on the boundary remain the same despite the different cubillages.

In fact, Construction 3.2 comprises *all* coproducts that satisfy the homotopy formula, up to redundancies. The idea is to run the proof of Theorem 3.6 in reverse, so that if a coproduct satisfies the homotopy formula, then the cubes corresponding to its terms must have come from a cubillage.

Theorem 3.8 ([7, Theorem 3.9]). *Suppose that we have a degree-i coproduct $\Delta'_i\colon \mathrm{C}_\bullet(\Delta^n) \to \mathrm{C}_\bullet(\Delta^n) \otimes \mathrm{C}_\bullet(\Delta^n)$ with $i \geqslant 0$, such that*

$$\partial \circ \Delta'_i - (-1)^i \Delta'_i \circ \partial = (1 + (-1)^i T)\Delta^{U_{\min}}_{i-1}. \tag{3.1}$$

1. If, for $i > 0$, we have that for all non-empty $S \subseteq [0, n]$, $\Delta'_i(S)$ has a minimal number of *terms amongst coproducts which satsify this formula, then we have that $\Delta'_i = \Delta^U_i$ for some $U \in \mathcal{B}([0,n], i+1)$.*

2. *For $i = 0$, if we have that $\Delta'_i(p) = p \otimes p$ for all $p \in [0,n]$ and that $\Delta'_i(S)$ otherwise has a minimal number of terms for non-empty $S \subseteq [0,n]$, then we have $\Delta'_i = \Delta^U_i$ for some $U \in \mathcal{B}([0,n], 1)$.*

3.3 Extensions of the construction

There are several natural ways our construction can be extended.

1. One can consider cup-i coproducts on a whole simplicial complex, rather than a single simplex. In this case, for each maximal simplex in the simplicial complex, one can choose an element of the higher Bruhat orders. If these chosen elements agree on the intersections of the maximal simplices in an appropriate way, then they define a cup-i coproduct on the chain complex of the whole simplicial complex. The homotopy formula can then be verified simplex-by-simplex, as in Theorem 3.6.

   One interesting question is whether particular families of simplicial complexes give rise to other familiar objects on the combinatorial side.

2. It is natural to also consider cup-i coproducts on singular homology. Here, there are infinitely many singular simplices; the only feasible option is to assign the same element of the higher Bruhat orders to each of them. One can show that the only consistent way to do this is by assigning all of them the minimal elements, or all of them the maximal elements. Thus, the only cup-i coproducts that exist in the singular case are the Steenrod ones.

3. Instead of defining a homotopy from $T\Delta_{i-1}$ to Δ_{i-1}, one can instead consider homotopies from $T\Delta^U_{i-1}$ to Δ^U_{i-1}. Here the relevant posets are the "re-oriented higher Bruhat orders".

4. The cup-i coproducts give rise to the cohomology operations known as *Steenrod squares*. These operations are defined in mod 2 cohomology by $\mathrm{Sq}^i([\alpha]) = [\alpha \smile_i \alpha]$, where $\smile_i$ is the product which is the linear dual of the coproduct Δ_i. We show that different choices of elements of the higher Bruhat orders always produce the same Steenrod squares. There also exist mod p cohomology operations, which were described in terms of multi-arity versions of the cup-i products in [6].

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