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DÉFORMATION
QUANTIFICATION
THÉORIE DE LIE

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CHAPTER 1

INTRODUCTION

In 1997, M. Kontsevich proved that every Poisson manifold admits a formal quantization, canonical up to equivalence. In doing so he solved a longstanding problem in mathematical physics. Through his proof and his interpretation of a later proof given by Tamarkin, he also opened up new research avenues in Lie theory, quantum group theory, deformation theory and the study of operads, . . . and uncovered fascinating links of these topics with number theory, knot theory and the theory of motives. Without doubt, his work on deformation quantization will continue to influence these fields for many years to come.

In the three parts of this volume, we will

- (1) present the main results of Kontsevich's 1997 preprint and sketch his interpretation of Tamarkin's approach,
- (2) show the relevance of Kontsevich's theorem for Lie theory and
- (3) explain the idea from string theory which inspired Kontsevich's proof.

We stress that no knowledge of physics will be assumed on the part of the reader.

1.1. Hamiltonian mechanics, symplectic manifolds and Poisson manifolds

In very vague terms, the problem solved by Kontsevich consists in passing from commutative to non commutative structures, where the commutative structures have their historical origin in Hamiltonian mechanics and the non commutative ones in quantum mechanics. In this section, we review the relevant commutative structures. They are associated with classical Hamiltonian systems.

A *symplectic manifold* is a smooth (*i.e.* C^∞) manifold M endowed with a closed non degenerate 2-form ω . A typical example is the tangent manifold $M = TL$ to a manifold L with the form given by

$$\omega = \sum_{i=1}^n dp_i \wedge dq_i ,$$

where the q_i form a system of local coordinates around a point of L and the p_i a basis dual to that formed by the partial derivations with respect to the q_i . If M is an arbitrary symplectic manifold, one associates with each smooth function f on M its *Hamiltonian vector field* X_f by requiring that

$$df = \iota(X_f)\omega.$$

The description of a physical system is then given by the commutative algebra of smooth functions on the symplectic manifold M together with a fixed smooth function H , the *Hamiltonian* of the system. The points of M represent the states of the system. The smooth functions on M are the *observables* (like position, momentum, ...). If O is an observable, its time evolution is determined by *Hamilton's equations*

$$\frac{d}{dt}O(t) = X_H(O).$$

Using the notation

$$\{f, g\} = X_f(g),$$

for smooth functions f, g , we can rewrite this as

$$\frac{d}{dt}O(t) = \{H, O\}.$$

The operation

$$(f, g) \rightarrow \{f, g\}, \quad f, g \in C^\infty(M),$$

is antisymmetric and it follows from the closedness of ω that it satisfies the Jacobi identity

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}.$$

Therefore, the bracket $\{, \}$ is a *Poisson bracket*, *i.e.* it is a Lie bracket and, for each smooth function f , the map $g \mapsto \{f, g\}$ satisfies the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}, \quad g, h \in C^\infty(M).$$

Hamilton's equations only depend on the Poisson bracket, not on the symplectic form ω . It is therefore natural to enlarge the class of manifolds under consideration from symplectic manifolds to *Poisson manifolds*, *i.e.* smooth manifolds endowed with a Poisson bracket. This is also motivated by the fact that Poisson manifolds naturally arise as quotients of symplectic manifolds by symmetry groups and as the classical limit of quantum groups. Symplectic manifolds are always even-dimensional. Poisson manifolds can have arbitrary dimension. This becomes clear from the following important class of examples: let \mathfrak{g} be a finite-dimensional real Lie algebra and M its dual space over the reals. Then the space of linear functions on M is a Lie algebra (isomorphic to \mathfrak{g}) and M becomes a Poisson manifold for the unique Poisson bracket which extends the Lie bracket on the space of linear functions (*cf.* sections 2.1.3 and

5.1). Of course, M is not symplectic in general. However, it is a union of symplectic manifolds. Namely, if G is a Lie group with Lie algebra \mathfrak{g} , then each orbit of the coadjoint action of G in M carries a symplectic structure induced from the Poisson structure on M .

1.2. From Hamiltonian mechanics to quantum mechanics

As we have seen, in Hamiltonian mechanics, physical systems are described by commutative algebras, namely algebras of smooth functions on Poisson manifolds. In contrast, in quantum mechanics, physical systems are described by non commutative algebras, namely algebras of operators (=observables) on Hilbert spaces, whose rays represent the states of the system. The quantum description always yields the more accurate prediction of the results of physical experiments. Nevertheless, for systems of a macroscopic scale, the classical description is a very good approximation to the quantum description. One obtains a quantitative measure of how good the approximation is by expressing Planck's constant in a unit of action which is characteristic of the scale of the system: the smaller Planck's constant appears, the better will the classical description approximate the quantum description. Loosely speaking one often says that classical mechanics is the limit of quantum mechanics when Planck's constant tends to zero. Thus, at least from a philosophical viewpoint, the passage from the quantum to the classical description is straightforward. The reverse operation, *i.e.* producing a quantum description from a classical one, is naturally much less obvious. There have been two main approaches, namely

- (1) geometric quantization and
- (2) deformation quantization.

Geometric quantization is concerned with the explicit construction of a Hilbert space and an algebra of operators on it. It relies very much on the symmetries of a given phase space. In fact, in its main form, it applies to phase spaces which are coadjoint orbits of a Lie group. Its purely mathematical avatar is the orbit method in the representation theory of Lie groups. In this field, geometric quantization has been highly successful. On the other hand, geometric quantization has not elucidated quantum field theory and has not been successfully applied to general relativity to yield a viable theory of quantum gravity. The problem stems from the fact that geometric quantization only allows the quantization of relatively few classical observables.

Deformation quantization takes the same basic input as geometric quantization, namely a symplectic or, more generally, a Poisson manifold. In contrast, its output is a non commutative associative algebra, whose elements are viewed as quantum observables. This algebra is not given as an algebra of operators on a Hilbert space but as a formal one-parameter deformation of the algebra of smooth functions on the original Poisson manifold. By definition, the datum of such a deformation is that of

a sequence of bilinear maps

$$(f, g) \mapsto B_n(f, g), \quad f, g \text{ smooth},$$

such that $B_0(f, g) = fg$ and that the multiplication

$$f * g = \sum_{n=0}^{\infty} B_n(f, g)h^n$$

extends to an *associative* multiplication on the space $C^\infty(M)[[h]]$ of formal power series in h . The formal deformation $*$ is a *starproduct* if moreover all the B_n are bidifferential operators (*i.e.* given locally by sums of products of smooth functions with iterated partial derivatives of the two arguments). A *formal quantization* or *deformation quantization* of the given Poisson manifold M is then a starproduct such that, when h tends to zero, the bracket of the Lie algebra underlying the deformed associative algebra tends to the Poisson bracket of functions. Concretely, the last condition means that

$$B_1(f, g) - B_1(g, f) = \{f, g\}.$$

The associativity of $*$ translates into the identities

$$\sum_{j+k=n} B_j(f, B_k(g, h)) = \sum_{j+k=n} B_j(B_k(f, g), h)$$

for all $n \geq 0$. In a starproduct, one should imagine the deformation parameter h as proportional to Planck's constant so that the behaviour of the starproduct when h tends to zero reflects the idea that the classical description is the limit, when Planck's constant tends to zero, of the quantum description.

1.3. Deformation quantization before Kontsevich

Deformation quantization was first proposed in 1978 in the pioneering work [10] of Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer. This article lays down the theoretical foundations of deformation quantization and contains the first significant applications. It raises the fundamental questions about the existence and the uniqueness of a deformation quantization for a given Poisson manifold. In attempting to solve the existence problem one is naturally lead to two subproblems:

- the local problem is to construct a starproduct for a Poisson bracket defined on a small neighbourhood of the origin in a finite-dimensional vector space,
- the globalization problem is to glue starproducts given on the open sets of a covering of the manifold.

For both problems, only the case of symplectic manifolds had been treated before Kontsevich's 1997 preprint. We briefly review the main results which had been obtained and refer the reader to [25], [101] and [36] for more detailed historical accounts.

For symplectic manifolds, the local problem has a classical solution: In fact, thanks to Darboux's theorem, each symplectic manifold is locally isomorphic to the tangent manifold $M_0 = T\mathbb{R}^n$ endowed with its canonical symplectic structure. The manifold M_0 admits the *Moyal-Weyl starproduct* (cf. sections 2.1.2 and 11.3). This product comes from the composition of operators on $C^\infty(\mathbb{R}^n)$ via Weyl's identification [102] of such operators with functions on \mathbb{R}^{2n} . It was used by Moyal [79] to study quantum statistical mechanics from the viewpoint of classical phase space. Thus, to show that starproducts exist on symplectic manifolds, it remains to show that one can glue suitably chosen local Moyal-Weyl products. That this is always possible was first proved by M. De Wilde and P. Lecomte in their 1983 paper [20]. Independently, in 1985, B. V. Fedosov gave a geometric proof of the same theorem in [31], cf. also [32], [101] and [33]. Fedosov's proof lead to many major developments, cf. [25].

As for the uniqueness problem for the starproduct on a symplectic manifold, the final solution was given by Nest and Tsygan in the article [80], which built on Fedosov's work. It follows from their result and Moser's classification of nearby symplectic structures by their cohomology classes [78] that for symplectic manifolds, there is a bijection between equivalence classes of starproducts and equivalence classes of formal deformations of the given Poisson bracket.

1.4. Deformation quantization after Kontsevich

In his 1997 preprint [60], Kontsevich

- (1) constructed an explicit starproduct for an arbitrary Poisson bracket defined on an open subset of a finite-dimensional vector space,
- (2) showed how to construct a globally defined starproduct on an arbitrary Poisson manifold,
- (3) proved that for an arbitrary smooth manifold, there is a bijection between equivalence classes of formal deformations of the zero Poisson bracket and equivalence classes of starproducts.

These theorems generalize the ones which had been obtained in the symplectic case, but Kontsevich's approach was self-contained and entirely different from those of his predecessors. Using classical ideas from deformation theory due to Gerstenhaber [39], Schlessinger-Stasheff [87], Deligne [45], . . . , he deduced his results from a more general statement: the truth of the 'formality conjecture', which he had stated in 1996 in [61]. The term 'formality' comes from rational homotopy theory where it refers to the fact that the homotopy groups of certain manifolds are 'formal consequences' of their homology groups, cf. [22]. Technically speaking one says that a \mathbb{Z} -graded Lie algebra endowed with a differential (= *dg Lie algebra*) is *formal* if it is linked to its homology, viewed as a dg Lie algebra with the zero differential, by a chain of quasi-isomorphisms, *i.e.* maps of dg Lie algebras which induce isomorphisms in homology.

With each dg Lie algebra L , there is associated a deformation problem, namely that of deforming the zero solution of the Maurer-Cartan equation

$$d(x) + \frac{1}{2}[x, x] = 0, \quad x \in L^1.$$

One says that this problem is ‘controlled’ by the dg Lie algebra L . Each quasi-isomorphism of dg Lie algebras yields an equivalence of the associated deformation problems. Therefore, if L is formal, then L and its homology $H^*(L)$ control equivalent deformation problems. The problem of deforming the commutative multiplication of a smooth manifold M into a starproduct is controlled by a dg Lie algebra which Kontsevich denotes by $D_{poly}(M)$ (where the D stands for (multi-)differential operators). It turns out that its homology, denoted by $T_{poly}(M)$, controls the deformations of the zero Poisson bracket on M . Kontsevich’s formality conjecture [61] states that $D_{poly}(M)$ is formal. This then implies the equivalence of deformation problems of point (3) above. At a very concrete level, the formality of a dg Lie algebra L is reflected by the existence of a so-called L_∞ -quasi-isomorphism between $H^*(L)$ and L . This is the datum of a sequence of multilinear maps from the tensor powers of $H^*(L)$ to L which satisfy a sequence of intricate compatibility conditions. Such an L_∞ -quasi-isomorphism yields an explicit map sending each solution of the deformation problem controlled by the homology $H^*(L)$ to a solution of the deformation problem controlled by L itself. In his preprint [60], Kontsevich constructs an explicit L_∞ -quasi-isomorphism between $T_{poly}(M)$ and $D_{poly}(M)$ in the case where M is an open subset of \mathbb{R}^n . This yields an explicit formula for a starproduct which quantizes a given Poisson bracket on $M \subset \mathbb{R}^n$. In particular, this formula applies to the case where M is the dual of a finite-dimensional Lie algebra. Kontsevich constructs his L_∞ -quasi-isomorphism using a family of universal constants, the *weights* w_Γ , parametrized by certain quivers (=oriented graphs) Γ . The weight w_Γ is defined as an integral over a space (depending on Γ) of configurations of points in the plane. The compatibility conditions on the components of the L_∞ -quasi-isomorphism translate into identities satisfied by the weights w_Γ . Kontsevich derives these from Stokes’ theorem applied to certain compactifications of the configuration spaces (analogous to those of Fulton-MacPherson [38]). The L_∞ -quasi-isomorphism thus constructed for open subsets of \mathbb{R}^n is equivariant with respect to affine transformations (*i.e.* translations and linear transformations), and it is also invariant under ‘infinitesimal formal automorphisms’ of \mathbb{R}^n . Using a method reminiscent of Fedosov’s Kontsevich derives from these properties the existence of an L_∞ -quasi-isomorphism for an arbitrary smooth manifold M .

In the first two chapters of part I, we will review Kontsevich’s construction of a starproduct on an open subset M of \mathbb{R}^n and develop the material from deformation theory necessary to state the formality theorem. In the last chapter of part I, we will sketch Kontsevich’s interpretation [62] of Tamarkin’s proof [93] [49] of the formality theorem for the case where $M = \mathbb{R}^n$. In fact, Tamarkin proves a purely algebraic

statement, valid over any field of characteristic zero. His proof establishes a close link with the theory of operads and with quantum group theory. It lead Kontsevich to conjecture [62] an action of the Grothendieck-Teichmüller group [28] on the set of formality quasi-isomorphisms.

1.5. The tangential homology isomorphism and Lie algebras

The dg Lie algebras $T_{poly}(M)$ and $D_{poly}(M)$ which control the deformations of the zero Poisson bracket and the commutative multiplication, respectively, carry additional structures in the form of associative ‘cup products’. If π is a formal deformation of the vanishing Poisson bracket on $C^\infty(M)$ and $*$ the corresponding starproduct on $C^\infty(M)[[t]]$, then these cup products induce the algebra structures in the Poisson cohomology

$$H_{Pois}^*(C^\infty(M)[[h]], \pi)$$

and in the Hochschild cohomology

$$H_{Hoch}^*(C^\infty(M)[[h]], *)$$

(in both cases, the cochains are supposed to be continuous in the h -adic topology). Kontsevich indicates in [60] how to prove that if M is an open subset of \mathbb{R}^n , the ‘tangent map’ to his formality L_∞ -quasi-isomorphism induces an algebra isomorphism

$$H_{Pois}^*(C^\infty(M)[[h]], \pi) \rightarrow H_{Hoch}^*(C^\infty(M)[[h]], *).$$

The important point here is that the morphism is compatible with the multiplication (its bijectivity is immediate once the definitions are made precise).

In the case where M is the dual of a finite-dimensional Lie algebra \mathfrak{g} , Kontsevich’s starproduct on the algebra of formal power series $C^\infty(M)[[h]]$ induces a multiplication on the algebra $S(\mathfrak{g})[[h]]$ of polynomials in h whose coefficients are polynomial functions on M , which we identify with elements of the symmetric algebra $S(\mathfrak{g})$ on \mathfrak{g} . Kontsevich shows that the specialization at $h = 1$ of this algebra is isomorphic to the universal enveloping algebra $U(\mathfrak{g})$. The above isomorphism of cohomology algebras then yields an algebra isomorphism

$$H^*(\mathfrak{g}, S(\mathfrak{g})) \rightarrow H^*(\mathfrak{g}, U(\mathfrak{g})).$$

In particular, in degree 0, one obtains an algebra isomorphism

$$S(\mathfrak{g})^{\mathfrak{g}} \rightarrow Z(U(\mathfrak{g}))$$

from the subalgebra of invariants under \mathfrak{g} in $S(\mathfrak{g})$ to the center of the universal enveloping algebra. Kontsevich shows that it coincides with the Duflo isomorphism [29]. This is the starting point for the material we present in part II: If one interprets $S(\mathfrak{g})$ and $U(\mathfrak{g})$ as algebras of differential operators respectively on the Lie algebra \mathfrak{g} and a Lie group G with Lie algebra \mathfrak{g} , one can deduce from the Kashiwara-Vergne

conjecture [55] that the Duflo isomorphism is induced by the exponential map. The Kashiwara-Vergne conjecture is still open. However, to deduce the assertion on the Duflo isomorphism, one does not need the full strength of the conjecture. In part II, we will see how Kontsevich's methods allow one to prove a weakened form of the Kashiwara-Vergne conjecture which is still strong enough to imply the assertion on the Duflo isomorphism as well as many other algebraic and analytic consequences.

1.6. Why does it work ?

The details of Kontsevich's and Tamarkin's proofs have been checked by many mathematicians and, from this limited point of view, one can say they have been understood. The links between Tamarkin's proof, quantum group theory and the theory of operads offer hints on where to search for a more comprehensive theory which might explain the deeper reason why Kontsevich's formulae work. Another promising route to follow is certainly the one taken by Kontsevich himself: Indeed, as he points out in [60], his solution of the quantization problem was inspired by ideas from string theory in an essential manner. This link has been made explicit by Cattaneo and Felder [16]. In part III, we present their work. In their recent announcement [15], Cattaneo and Felder have used similar ideas to extend Kontsevich's formulae to coisotropic submanifolds in Poisson manifolds, and in particular to graphs of Poisson morphisms, which represents a major advance in the subject.

We see this as another piece of evidence for the fruitfulness of the dialogue between mathematicians and physicists which Kontsevich's work has stimulated and to which we hope to contribute through this volume.

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PART I

**DEFORMATION QUANTIZATION
AFTER KONTSEVICH AND TAMARKIN
(BY B. KELLER)**

CHAPTER 2

PRESENTATION OF THE MAIN RESULTS

In this chapter, we present the main results of M. Kontsevich's 1997 preprint [60] now published in [64]. We first introduce the notions of *formal deformation* and of *equivalence* between formal deformations. We will see that each *associative* formal deformation of a *commutative and associative* algebra A naturally yields a *Poisson bracket* on A . In its simplest form, Kontsevich's theorem states that each Poisson bracket on the algebra of smooth functions on a smooth manifold lifts to a formal deformation of the algebra of smooth functions on M . Moreover, this deformation is canonical up to equivalence. A *Poisson manifold* is a smooth manifold M whose algebra of smooth functions is endowed with a Poisson bracket. We examine two examples of Poisson manifolds whose associated formal deformations have been known for a long time: the Poisson manifold $M = \mathbb{R}^{2n}$ with its canonical bracket yields the Moyal-Weyl product, and the Poisson manifold given by the dual of a finite-dimensional Lie algebra yields essentially the product of the enveloping algebra. We also review a class of examples, due to O. Mathieu, of commutative algebras endowed with Poisson brackets which do not lift to formal deformations. Then we describe the explicit formulae which Kontsevich gave for the formal deformation associated with a Poisson bracket on an open subset M of \mathbb{R}^n . His construction actually yields a *starproduct*, *i.e.* an associative formal deformation given by sequence of *bidifferential operators*. We then give the more precise form of Kontsevich's theorem, which establishes a parametrization of the equivalence classes of starproducts on a smooth manifold in terms of the equivalence classes of formal deformations of the vanishing Poisson bracket. In the case where M is an open subset of \mathbb{R}^n , we state Kontsevich's theorem that the Hochschild cohomology algebra of a starproduct is isomorphic to the Poisson cohomology of the corresponding Poisson bracket. We will see how this generalizes the Duflo isomorphism which describes the center of the enveloping algebra of a finite-dimensional Lie algebra. Finally, we list the main ingredients of Kontsevich's proof and state Tamarkin's theorem, whose proof we will review in chapter 4.

In sections 2.1.1 and 2.3.1, we will largely follow the lucid presentation of [81].

2.1. Every Poisson manifold admits a formal quantization

2.1.1. Deformations and Poisson structures. — Let k be a commutative ring and A a k -algebra (i.e. a k -module endowed with a k -bilinear map from $A \times A$ to A). Denote by $k[[t]]$ the ring of formal power series in an indeterminate t , and by $A[[t]]$ the $k[[t]]$ -module of formal power series

$$\sum_{n=0}^{\infty} a_n t^n$$

with coefficients in A . Let $*$ be a *formal deformation of the multiplication of A* , i.e. a $k[[t]]$ -bilinear map

$$A[[t]] \times A[[t]] \rightarrow A[[t]]$$

such that we have

$$u * v \equiv uv \pmod{tA[[t]]}$$

for all power series $u, v \in A[[t]]$. The product of two elements a, b of A is then of the form

$$a * b = ab + B_1(a, b)t + \cdots + B_n(a, b)t^n + \cdots$$

for a sequence of k -bilinear maps B_i , and these determine the product $*$ because it is $k[[t]]$ -bilinear. We put $B_0(a, b) = ab$ and we write

$$* = \sum_{n=0}^{\infty} B_n t^n.$$

Let J be the group of $k[[t]]$ -module automorphisms g of $A[[t]]$ such that

$$g(u) \equiv u \pmod{tA[[t]]}$$

for all $u \in A[[t]]$. We define two formal deformations $*$ and $*'$ to be *equivalent* if there is an element $g \in J$ such that

$$g(u * v) = g(u) *' g(v)$$

for all $u, v \in A[[t]]$. Note that, for $g \in J$ and $a \in A$, we have

$$g(a) = a + g_1(a)t + g_2(a)t^2 + \cdots + g_n(a)t^n + \cdots$$

for certain k -linear maps $g_i : A \rightarrow A$, and that these determine g because it is $k[[t]]$ -linear.

One can show (*cf.* Cor. 3.4.5) that if A is associative and admits a unit element 1_A , then each associative formal deformation $*$ of the multiplication of A admits a unit element 1_* . Moreover, such an associative formal deformation $*$ is always equivalent to an associative formal deformation $*'$ such that $1_{*'} = 1_A$.

Suppose that A is associative and commutative.

Lemma 2.1.1. — *Let $*$ be an associative (but not necessarily commutative) formal deformation of the multiplication of A . For $a, b \in A$, put $\{a, b\} = B_1(a, b) - B_1(b, a)$.*

- a) *The map $\{, \}$ is a Poisson bracket on A , i.e. a k -bilinear map such that*
 - *the bracket $\{, \}$ is a Lie bracket and*
 - *for all $a, b, c \in A$, we have $\{a, bc\} = \{a, b\}c + b\{a, c\}$.*
- b) *The bracket $\{, \}$ only depends on the equivalence class of $*$.*

Proof. — a) The map

$$(u, v) \mapsto \frac{1}{t} (u * v - v * u)$$

clearly defines a Lie bracket on $A[[t]]$. Let us denote it by $[,]$. The bracket $\{, \}$ equals the reduction modulo t of $[,]$. Therefore, it is still a Lie bracket. The second equality follows from

$$[u, vw] = [u, v]w + u[v, w]$$

for all $u, v, w \in A[[t]]$.

- b) If $g \in J$ yields the equivalence of $*$ with $*'$, then we have

$$B_1(a, b) + g_1(ab) = B'_1(a, b) + g_1(a)b + ag_1(b)$$

for all $a, b \in A$. Thus the difference $B_1(a, b) - B'_1(a, b)$ is symmetric in a, b and does not contribute to $\{, \}$. ✓

Theorem 2.1.2 (Kontsevich [60]). — *If A is the algebra of smooth functions on a differentiable manifold M , then each Poisson bracket on A lifts to an associative formal deformation.*

In other words, the map

$$\{\text{equivalence classes of formal deformations of } A\} \longrightarrow \{\text{Poisson brackets on } A\}$$

is surjective if A is the algebra of smooth functions on a differentiable manifold M . Moreover, Kontsevich constructs a section of this map. His construction is canonical and explicit for $M = \mathbb{R}^n$; it is canonical (up to equivalence) for general manifolds M . Below, we give two simple examples of formal deformations arising from Kontsevich's construction for $M = \mathbb{R}^n$ where the Poisson bracket is given respectively by a constant and by a linear bivector field. We also give a class of examples, due to Mathieu [74], of finite-dimensional Poisson algebras whose brackets do not lift to formal deformations.

2.1.2. Example: The Moyal-Weyl product. — Let $M = \mathbb{R}^2$. Consider the Poisson bracket given by

$$\{f, g\} = \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_1},$$

where f and g are smooth functions on M . Then, up to equivalence, Kontsevich's construction yields the associative (!) formal deformation given by

$$f * g = \sum_{n=0}^{\infty} \frac{\partial^n f}{\partial x_1^n} \frac{\partial^n g}{\partial x_2^n} \frac{t^n}{n!}.$$

More generally, for $n \geq 2$, let $M = \mathbb{R}^n$, and let (τ_{ij}) be an antisymmetric $n \times n$ -matrix with real coefficients. Consider the Poisson bracket defined by

$$\{f, g\} = \sum \tau_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Let (σ_{ij}) be an $n \times n$ -matrix with real coefficients whose antisymmetrization equals (τ_{ij}) . Then the formula

$$f * g = \sum_{n=0}^{\infty} \sum_{i_1, j_1, \dots, i_n, j_n} \sigma_{i_1 j_1} \sigma_{i_2 j_2} \dots \sigma_{i_n j_n} \frac{\partial^n f}{\partial x_{i_1} \dots \partial x_{i_n}} \frac{\partial^n g}{\partial x_{j_1} \dots \partial x_{j_n}} \frac{t^n}{n!}$$

defines an associative formal deformation. Its equivalence class is independent of the choice of (σ_{ij}) and equals that of Kontsevich's canonical deformation. If we choose (σ_{ij}) antisymmetric, we obtain what is known as the Moyal-Weyl product associated with $\{, \}$.

2.1.3. Example: The dual of a Lie algebra. — Let \mathfrak{g} be a finite-dimensional real Lie algebra. Let \mathfrak{g}^* denote the dual over \mathbb{R} of \mathfrak{g} and A the algebra of smooth functions on \mathfrak{g}^* . The space of linear functions on \mathfrak{g}^* naturally identifies with \mathfrak{g} and hence carries a natural bracket. This bracket extends to a unique Poisson bracket on the algebra of smooth functions on \mathfrak{g}^* . Explicitly, for f, g smooth and $x \in \mathfrak{g}^*$, we have

$$\{f, g\}(x) = x \left([(df)_x, (dg)_x] \right),$$

where we identify the differentials $(df)_x$ and $(dg)_x$ with elements of \mathfrak{g} . Kontsevich's construction yields a canonical associative product $*$ on $A[[t]]$. This product is closely linked to that of the enveloping algebra of \mathfrak{g} : Let $S(\mathfrak{g})$ be the symmetric algebra on \mathfrak{g} . We identify it with the algebra of polynomial functions on \mathfrak{g}^* . It is not hard to show that the subspace $B = S(\mathfrak{g})[[t]]$ of $A[[t]]$ formed by the polynomials in t whose coefficients are polynomial functions on \mathfrak{g}^* is a subalgebra for $*$. Moreover, the inclusion $\mathfrak{g} \rightarrow B$ induces an isomorphism

$$U_{hom}(\mathfrak{g}) \xrightarrow{\sim} B,$$

where $U_{hom}(\mathfrak{g})$ is the homogeneous enveloping algebra, i.e. the $\mathbb{R}[[t]]$ -algebra generated by \mathfrak{g} with relations $XY - YX - t[X, Y]$, $X, Y \in \mathfrak{g}$. Thus the quotient $B/(t-1)B$ is isomorphic to the enveloping algebra $U(\mathfrak{g})$.

2.1.4. Mathieu's examples [74]. — Let \mathfrak{g} be a finite-dimensional real Lie algebra such that $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is simple and not isomorphic to $\mathfrak{sl}_n(\mathbb{C})$ for any $n \geq 2$. The bracket of \mathfrak{g} uniquely extends to a Poisson bracket on the symmetric algebra $S(\mathfrak{g})$. The ideal I of $S(\mathfrak{g})$ generated by all monomials of degree 2 is a Poisson ideal (i.e. we have $\{f, I\} \subset I$ for all $f \in S(\mathfrak{g})$). So the quotient $A = S(\mathfrak{g})/I$ becomes a (finite-dimensional) Poisson algebra. Assume that its bracket comes from a formal deformation $*$. Let B be the associative algebra $A[[t]]$ endowed with $*$ and let B_L be the Lie algebra obtained by endowing B with the commutator with respect to its multiplication. Then B_L is a formal deformation of the Lie algebra $A_L = (A, \{, \})$. Now A_L is isomorphic to $\mathbb{R} \oplus \mathfrak{g}$. Since $\mathfrak{g} \otimes \mathbb{C}$ is simple, we have $H^2(A_L, A_L) = 0$ so that A_L is rigid and B_L is isomorphic to $A_L \otimes_{\mathbb{R}} \mathbb{R}[[t]]$ (as a Lie algebra over $\mathbb{R}[[t]]$). Let K be the algebraic closure of the fraction field of $\mathbb{R}[[t]]$. By extending the scalars to K we find that $B_L \otimes_{\mathbb{R}[[t]]} K$ is isomorphic to $A_L \otimes_{\mathbb{R}[[t]]} K = K \oplus (\mathfrak{g} \otimes_{\mathbb{R}} K)$. But $B_L \otimes_{\mathbb{R}[[t]]} K$ is the Lie algebra associated with the finite-dimensional associative K -algebra $B \otimes_{\mathbb{R}[[t]]} K$. Since K is algebraically closed, this algebra is isomorphic to $M \oplus J$, where M is a product of matrix rings over K and J is nilpotent. Therefore, the only simple quotients of its associated Lie algebra are isomorphic to $\mathfrak{sl}_n(K)$ for certain $n \geq 2$. However, by assumption, $A_L \otimes_{\mathbb{R}[[t]]} K$ admits the simple quotient $\mathfrak{g} \otimes_{\mathbb{R}} K$. This contradiction shows that the bracket of A cannot come from a formal deformation.

2.2. Kontsevich's explicit formula

Let $d \geq 1$ be an integer and M a non empty open subset of \mathbb{R}^d . Let A be the algebra of smooth functions on M and $\{, \}$ a Poisson bracket on A . In this situation, Kontsevich [60] gives an explicit formula for a canonical formal quantization $*_K$. We describe his formula in this section.

Recall that derivations of the algebra A are given by vector fields on M (*cf.* for example problème 1 in section 17.14 of volume III of [23]). From this fact, one deduces the

Lemma 2.2.1. — *There are unique smooth functions α^{ij} , $1 \leq i < j \leq d$, such that for all smooth functions f and g , we have*

$$\{f, g\} = \sum_{i < j} \alpha^{ij} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial x_j} \right).$$

As in section 2.1.1, we write the quantization to be constructed in the form

$$*_K = \sum_{n=0}^{\infty} B_n t^n.$$

For smooth functions f, g , we will express $B_n(f, g)$ as a linear combination of sums of products of partial derivatives of the α^{ij} and of f and g . To describe the terms explicitly, we need a little combinatorics: A *quiver* Γ is given by

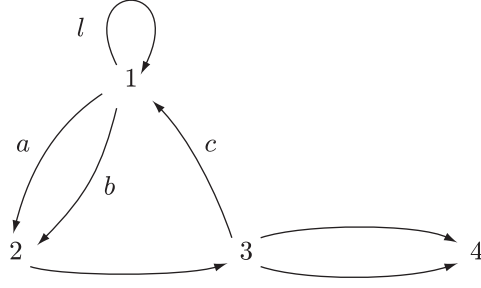


FIGURE 1. A quiver

- a set Γ_0 , whose elements are called the *vertices* of Γ ,
- a set Γ_1 , whose elements are called the *arrows* of Γ ,
- two maps $s : \Gamma_1 \rightarrow \Gamma_0$ and $t : \Gamma_1 \rightarrow \Gamma_0$, which, with an arrow, associate its *source* and its *target*.

In practice, quivers are given by drawings as in figure 1, where Γ_0 equals $\{1, 2, 3, 4\}$, Γ_1 has 7 elements and, for example, $s(c) = 3$, $t(c) = 1$. An arrow l of a quiver is a *loop* if $s(l) = t(l)$. A pair (a, b) of distinct arrows is a *double arrow* if $s(a) = s(b)$ and $t(a) = t(b)$.

Let $n \geq 0$. We define G_n to be the set of quivers Γ such that

- $\Gamma_0 = \{1, \dots, n\} \cup \{L, R\}$, where L and R are two symbols,
- $\Gamma_1 = \{a_1, b_1, \dots, a_n, b_n\}$, where the a_i and b_i are symbols,
- for each i , we have $s(a_i) = s(b_i) = i$, and
- Γ has neither loops nor double arrows.

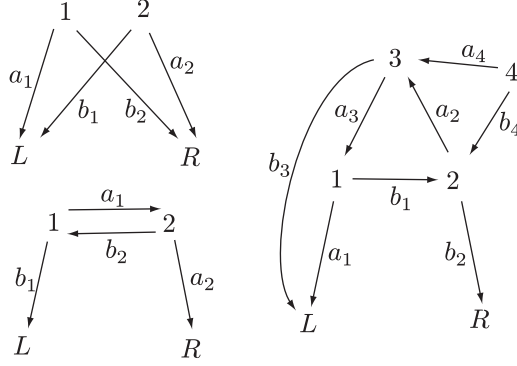
The unique quiver in G_0 has only the two vertices L and R and no arrow. The set G_1 contains exactly two quivers, namely

$$L \xleftarrow{a_1} 1 \xrightarrow{b_1} R \quad \text{and} \quad L \xleftarrow{b_1} 1 \xrightarrow{a_1} R .$$

Figure 2 shows two among the 36 quivers in G_2 and one among the 160000 quivers in G_4 . In general, G_n contains $(n(n+1))^n$ quivers. Given smooth functions f, g and a quiver $\Gamma \in G_n$, we will define a function $B_{\Gamma, \alpha}(f, g)$. For example, for the last quiver of figure 2, we have

$$B_{\Gamma, \alpha}(f, g) = \sum (\partial_{i_3} \alpha^{i_1, j_1}) (\partial_{j_1} \partial_{j_4} \alpha^{i_2, j_2}) (\partial_{i_2} \partial_{i_4} \alpha^{i_3, j_3}) (\partial_{i_1} \partial_{j_3} f) (\partial_{j_2} g) ,$$

where $i_1, j_1, \dots, i_4, j_4$ range from 1 to d (the dimension of $M \subset \mathbb{R}^d$) and ∂_k denotes the partial derivative with respect to x_k . Note that each vertex v of the quiver corresponds to a factor and that the partial derivatives with respect to x_{i_k} or x_{j_k} correspond to the edges a_k or b_k with target v . For a general quiver Γ in G_n , we

FIGURE 2. Quivers in G_n for $n = 2$ and $n = 4$

define $B_{\Gamma, \alpha}(f, g)$ to be

$$\sum \left(\prod_{i=1}^n \left(\prod_{a \in \Gamma(? , i)} \partial_{I(a)} \right) \alpha^{I(a_i), I(b_i)} \right) \left(\prod_{a \in \Gamma(? , L)} \partial_{I(a)} \right) (f) \left(\prod_{a \in \Gamma(? , R)} \partial_{I(a)} \right) (g),$$

where $\Gamma(? , v)$ denotes the set of arrows with target v and the sum ranges over all maps I from Γ_1 to $\{1, \dots, d\}$.

We will define

$$B_n = \sum_{\Gamma \in G_n} w_{\Gamma} B_{\Gamma, \alpha}$$

for certain universal constants $w_{\Gamma} \in \mathbb{R}$ called the Kontsevich *weights*, which we now construct. For this, let \mathcal{H} denote the upper half plane $\text{im } z > 0$. We endow \mathcal{H} with the hyperbolic metric. Its geodesics are the vertical half lines and the half circles whose center is on the real axis. For two distinct points p, q of \mathcal{H} , we define $l(p, q)$ to be the geodesic from p to q and we define $l(p, \infty)$ to be the vertical half line going from p to infinity. We denote by $\varphi(p, q)$ the angle from $l(p, \infty)$ to $l(p, q)$. As we see from figure 3, we have

$$\varphi(p, q) = \arg\left(\frac{q-p}{q-\bar{p}}\right) = \frac{1}{2i} \log \left(\frac{q-p}{q-\bar{p}} \cdot \frac{\bar{q}-p}{\bar{q}-\bar{p}} \right).$$

This shows that $(p, q) \mapsto \varphi(p, q)$ is analytic. It is also clear that it admits a continuous extension to the set of pairs of complex numbers (p, q) such that $\text{im } p \geq 0$, $\text{im } q \geq 0$ and $p \neq q$.

Now for $n \geq 0$, let \mathcal{H}_n be the set of n -tuples (p_1, \dots, p_n) of distinct points of \mathcal{H} . Given Γ in G_n , we interpret \mathcal{H}_n geometrically as the set of all ‘geodesic drawings’ of Γ in the closure of \mathcal{H} : the vertices $1, \dots, n$ of Γ correspond to the p_i , the vertices L and R to the points 0 and 1 of the real axis and each arrow of Γ is represented by a geodesic segment from its source point to its target point, *cf.* figure 4. With this in

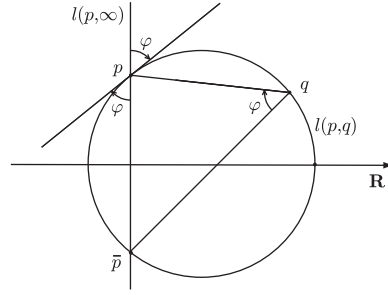


FIGURE 3. Planimetry

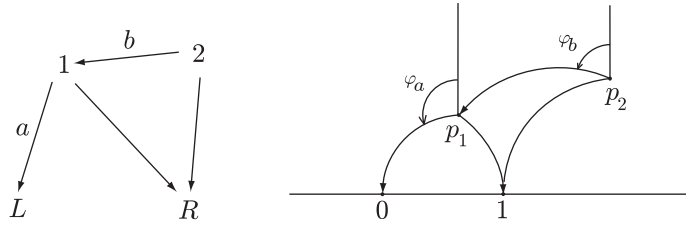


FIGURE 4. Geodesic drawing of a quiver

mind, for each arrow a of Γ , we define the function $\varphi_a : \mathcal{H}_n \rightarrow \mathbb{R}$ by

$$\varphi_a(p_1, \dots, p_n) = \varphi(p_{s(a)}, p_{t(a)}),$$

where we put $p_L = 0$ and $p_R = 1$. Finally, we define

$$w_\Gamma = \frac{1}{(2\pi)^{2n}} \int_{\mathcal{H}_n} \bigwedge_{i=1}^n (d\varphi_{a_i} \wedge d\varphi_{b_i}).$$

Note that the integrand is a $2n$ -form, and the integral is taken over a naturally oriented $2n$ -dimensional manifold.

Lemma 2.2.2. — *The integral converges absolutely.*

For this, one shows that the integrand admits a continuous extension to a compactification of \mathcal{H}_n , cf. [60].

Theorem 2.2.3 (Kontsevich [60]). — *The formula*

$$f *_K g = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\Gamma \in G_n} w_\Gamma B_{\Gamma, \alpha}(f, g)$$

defines a formal quantization of the given Poisson bracket. Its equivalence class is independent of the choice of coordinates in M .

The essential point is that $*_K$ is associative. Kontsevich deduces this from Stokes' theorem applied to compactifications of configuration spaces \mathcal{H}_n . He remarks in passing that the formula has a certain physical interpretation. This claim was made precise by Cattaneo and Felder in [16], *cf.* part III.

2.3. A more precise version of Kontsevich's theorem and its link with the Duflo isomorphism

2.3.1. Formal quantizations. — Let M be a differentiable manifold and A the algebra of smooth functions on M . Let $m \geq 1$. A *multidifferential operator* on M is a map $P : A^m \rightarrow A$ compatible with restrictions to open subsets and such that, in each system x_1, \dots, x_n of local coordinates on M , we have

$$P(f_1, \dots, f_m) = \sum a_{\nu_1, \dots, \nu_m} \left(\frac{\partial^{|\nu_1|}}{\partial x_1^{\nu_1}} f_1 \right) \cdots \left(\frac{\partial^{|\nu_m|}}{\partial x_1^{\nu_m}} f_m \right),$$

where the ν_i are multi-indices and the a_{ν_1, \dots, ν_m} are smooth functions which vanish except for a finite set of (ν_1, \dots, ν_m) .

A *star product* on M is an associative formal deformation $* = \sum B_n t^n$ such that the B_n are bidifferential operators. Note that the multiplication B_0 of A is such an operator. Let J_d denote the group of $\mathbb{R}[[t]]$ -module automorphisms $g = \sum g_n t^n$ of $A[[t]]$ such that g_0 is the identity and all g_n are differential operators. Two star products $*$ and $*'$ are *equivalent* if there is a $g \in J_d$ such that $g(u * v) = g(u) *' g(v)$ for all $u, v \in A[[t]]$.

As in lemma 2.1.1, each star product $*$ on M gives rise to a Poisson bracket $\{, \}$. We call $*$ a *formal quantization* of $\{, \}$. The Moyal-Weyl product (2.1.2) is an example.

Theorem 2.3.1 (Kontsevich [60]). — a) *Each Poisson bracket on A admits a formal quantization, canonical up to equivalence.*

b) *There is a bijection $[\pi] \mapsto [*_\pi]$ from the set of equivalence classes of Poisson brackets*

$$\pi = 0 + \pi_1 t + \cdots + \pi_n t^n + \cdots$$

*on the commutative $\mathbb{R}[[t]]$ -algebra $A[[t]]$ to the set of equivalence classes of star products on M . Moreover, if π corresponds to $*_\pi$, then the Poisson bracket on A associated with $*_\pi$ equals the coefficient π_1 of t in π .*

In b), two Poisson brackets are *equivalent* if they are conjugate by an $\mathbb{R}[[t]]$ -algebra automorphism belonging to J_d . The canonical quantization of a given Poisson bracket $\{, \}$ in a) is obtained by applying b) to $\pi = \{, \} t$.

2.3.2. Isomorphism of cohomology algebras. — Let π be as in theorem 2.3.1 and let $n \geq 2$. By reduction, π defines a Poisson bracket on $A[[t]]/(t^n)$. In particular, this space becomes a Lie algebra over $\mathbb{R}[t]/(t^n)$. The associated Chevalley-Eilenberg complex (*cf.* section 3.5.2) with coefficients in $A[[t]]/(t^n)$ admits a subcomplex, which we denote by

$$C_{Pois}(A[[t]]/(t^n), \pi),$$

whose p th component is formed by the cochains which are derivations in each argument (for the commutative multiplication of $A[[t]]/(t^n)$). We define

$$(2.3.1) \quad C_{Pois}(A[[t]], \pi)$$

to be the inverse limit of the system of complexes $C_{Pois}(A[[t]]/(t^n), \pi)$, $n \geq 2$.

Let $*$ be a formal deformation of the multiplication of A and let $n \geq 2$. By reduction, the multiplication $*$ defines an associative $\mathbb{R}[t]/(t^n)$ -algebra structure on $A[[t]]/(t^n)$. We denote by

$$C_{star}(A[[t]]/(t^n), *)$$

the subcomplex of the associated Hochschild complex (*cf.* section 3.2) with coefficients in $A[[t]]/(t^n)$ whose p -cochains have coefficients which are p -differential operators. We define

$$(2.3.2) \quad C_{star}(A[[t]], *)$$

to be the inverse limit of the system of complexes $C_{star}(A[[t]]/(t^n), *)$, $n \geq 2$. Let μ denote the commutative multiplication of $A[[t]]$. The complexes (2.3.1) and (2.3.2) are endowed with (associative) cup products extending the multiplications μ and $*$ on their 0th components. Their homologies

$$H_{Pois}^*(A[[t]], \pi) \text{ and } H_{star}^*(A[[t]], *)$$

become graded commutative algebras when endowed with the multiplications induced by the cup products.

Theorem 2.3.2 (Kontsevich [60]). — *Suppose that M is an open subset of \mathbb{R}^n . Then for each Poisson bracket π as in the preceding theorem, there is a canonical quasi-isomorphism*

$$\Psi_\pi : C_{Pois}(A[[t]], \pi) \rightarrow C_{star}(A[[t]], *_\pi)$$

which induces an algebra isomorphism

$$H_{Pois}^*(A[[t]], \pi) \rightarrow H_{star}^*(A[[t]], *_\pi).$$

The existence of the quasi-isomorphism Ψ_π follows easily from Kontsevich's formality theorem 2.4.1 below, *cf.* section 3.6. In contrast, the fact that Ψ_π is compatible with the algebra structure in cohomology is highly non trivial. In the case where M

is the dual of a finite-dimensional Lie algebra (cf. 2.1.3) and $\pi = t\{, \}$, it is easy to see that Ψ_π induces an algebra isomorphism

$$H^*(\mathfrak{g}, S(\mathfrak{g})) \xrightarrow{\sim} HH^*(U(\mathfrak{g}), U(\mathfrak{g})),$$

where the right hand side denotes the Hochschild cohomology algebra of $U(\mathfrak{g})$. In particular, in degree 0, we obtain an algebra isomorphism

$$S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} Z(U(\mathfrak{g}))$$

from the algebra of \mathfrak{g} -invariant polynomials on \mathfrak{g}^* to the center of $U(\mathfrak{g})$. Kontsevich shows that it coincides with the Duflo isomorphism [29].

2.4. On the proofs

2.4.1. Deformation theory. — Let A be the algebra of smooth functions on a differentiable manifold M . The main theorem 2.3.1 asserts that two deformation problems are equivalent: that of deforming the zero Poisson bracket on A and that of deforming the commutative multiplication μ on A . Now ‘every’ deformation problem can be described in terms of a differential graded Lie algebra (=dg Lie algebra). In our case, we denote the corresponding dg Lie algebras by $L_{Pois}(M)$ and $L_{star}(M)$; their underlying complexes are

$$C_{Pois}(A, 0)[1] \quad \text{and} \quad C_{star}(A, \mu)[1],$$

where, for a complex K , we denote by $K[1]$ the *shifted complex*: $K[1]^p = K^{p+1}$, $d_{K[1]} = -d_K$.

In deformation theory, one shows that if $f : L \rightarrow L'$ is a quasi-isomorphism of differential graded Lie algebras (i.e. a morphism inducing isomorphisms in homology), then f induces an equivalence between the corresponding deformation problems. Therefore, to prove the main theorem 2.3.1, it is enough to show that there is a chain

$$L_{Pois}(M) \leftarrow L_1 \rightarrow \dots \leftarrow L_n \rightarrow L_{star}(M)$$

of quasi-isomorphisms. Now in general, one can show that the existence of such a chain linking two dg Lie algebras L and L' is equivalent to that of an L_∞ -*quasi-isomorphism* $\mathcal{U} : L \rightarrow L'$, i.e. a sequence of morphisms

$$\mathcal{U}_n : L^{\otimes n} \rightarrow L', \quad n \geq 1,$$

which are homogeneous of degree $1 - n$, graded antisymmetric and satisfy a sequence of compatibility conditions with the brackets and the differentials of L and L' . We will review the relevant facts from deformation theory in more detail in chapter 3.

2.4.2. Kontsevich's proof. — In the case where M is an open subset of \mathbb{R}^n , Kontsevich [60] explicitly constructed an L_∞ -quasi-isomorphism

$$(2.4.1) \quad \mathcal{U}^M : L_{Pois}(M) \rightarrow L_{star}(M).$$

using integrals on configuration spaces. His formulas allowed him to write down an explicit star product for a given Poisson bracket on $M \subset \mathbb{R}^n$ and in particular for the canonical Poisson bracket on the dual of a finite dimensional Lie algebra (*cf.* 2.1.3). The resulting formulas are explained in part II. They are given a physical interpretation (which was apparently at the origin of Kontsevich's discovery) in part III.

Kontsevich's L_∞ -morphism (2.4.1) is equivariant with respect to the group of affine transformations of \mathbb{R}^n . Using this fact and a sophisticated gluing procedure Kontsevich proved the

Theorem 2.4.1 (Formality Theorem [60]). — *For each differentiable manifold M , there is an L_∞ -quasi-isomorphism $\mathcal{U}^M : L_{Pois}(M) \rightarrow L_{star}(M)$.*

2.4.3. Formality. — A dg Lie algebra L is *formal* if it is linked to the dg Lie algebra H^*L (endowed with $d = 0$ and the bracket induced from that of L) by a chain of quasi-isomorphisms (equivalently: by an L_∞ -quasi-isomorphism). The Hochschild-Kostant-Rosenberg theorem [51] yields that, for a differentiable manifold M , the homology of $L_{star}(M)$ is isomorphic to $L_{Pois}(M)$. Therefore, Kontsevich's formality theorem 2.4.1 means that for each differentiable manifold M , the dg Lie algebra $L_{star}(M)$ is formal.

2.4.4. Tamarkin's proof. — In [93], D. Tamarkin gave a new proof of Kontsevich's formality theorem 2.4.1 for the case of $M = \mathbb{R}^n$. More precisely, he proved the following purely algebraic statement: Let k be a field of characteristic 0, let V be a finite-dimensional k -vector space and SV the symmetric algebra on V . The problem of deforming the multiplication of SV is described by the dg Lie algebra

$$L_{alg}(V^*) = C_{Hoch}(SV, \mu)[1],$$

i.e. the shifted Hochschild complex endowed with the Gerstenhaber bracket, *cf.* section 3.3.

Theorem 2.4.2 (Tamarkin [93]). — *The dg Lie algebra $L_{alg}(V^*)$ is formal.*

It is easy to see, *cf.* Lemma 4.1.2, that for $k = \mathbb{R}$, the dg Lie algebra $L_{alg}(V^*)$ is linked to $L_{star}(V^*)$ by a chain of quasi-isomorphisms. Thus, Tamarkin's theorem is equivalent to the formality theorem for $k = \mathbb{R}$ and $M = \mathbb{R}^n$. We outline Tamarkin's proof in chapter 4.

2.5. Notes

Kontsevich's theorem 2.1.2 solves a conjecture which goes back to the pioneering work [10] by Bayen-Fronsdal-Lichnerowicz-Sternheimer. An account of the history and the motivations from physics can be found in [25] and [101]. Kontsevich's proof [60] of the isomorphism of cohomology algebras of Theorem 2.3.2 was made precise by Manchon-Torossian [71]. T. Mochizuki [77] proves that this isomorphism lifts to an A_∞ -quasi-isomorphism.

Kontsevich deduces the formality theorem for arbitrary Poisson manifolds from the case of an open set in \mathbb{R}^n . The proof of this 'globalization theorem' in [60] is not very detailed. More details are given in the appendix to [62]. Alternative approaches to globalization are due to Cattaneo-Felder-Tomassini [17] and V. Dolgushev [26].

In an algebraic context, the quantization problem was studied by Kontsevich [63] and A. Yekutieli [103].

Covariant versions of the formality theorem were conjectured by B. Tsygan [97] and proved by B. Shoikhet [89].

In studying the non uniqueness of the formality morphism Kontsevich has discovered surprising links to motives and the Grothendieck-Teichmueller group, *cf.* [62] and [92].

CHAPTER 3

DEFORMATION THEORY

In this chapter, we present the deformation theoretic framework of Kontsevich's formality theorem. We show how each dg Lie algebra yields a deformation functor and present the dg Lie algebras appearing in the study of the deformations of Poisson brackets and star products on a smooth manifold. We define the notion of formality and explain its link with that of an L_∞ -morphism. Following V. Hinich we show how these notions naturally appear in the study of Quillen's equivalence between dg Lie algebras and certain cocommutative coalgebras. Finally, we give Kontsevich's geometric interpretation of these coalgebras as formal manifolds.

3.1. Notations

Let k be a commutative ring. A *graded k -module* is a sequence $K = (K^p)$, $p \in \mathbb{Z}$, of k -modules K^p . A *morphism of degree n* between graded k -modules is a sequence $f : K \rightarrow L$ of morphisms $f^p : K^p \rightarrow L^{p+n}$. Such morphisms are composed in the natural way. A *complex* K is a graded k -module endowed with a *differential*, *i.e.* an endomorphism $d : K \rightarrow K$ of degree 1 such that $d^2 = 0$. The *suspension* or *shift* of a graded k -module K is the graded k -module denoted by SK or $K[1]$ with $(K[1])^p = K^{p+1}$, $p \in \mathbb{Z}$. If K is a complex with differential d , its suspension $SK = K[1]$ is endowed with the differential $-d$.

The *tensor product* $L \otimes K$ of two graded k -modules is the \mathbb{Z} -graded k -module with components

$$(L \otimes K)^n = \bigoplus_{p+q=n} L^p \otimes_k K^q.$$

The *tensor product of two morphisms* f and g is defined by

$$(f \otimes g)(x \otimes y) = (-1)^{pq} f(x) \otimes g(y),$$

where g is of degree p and x of degree q . The flip $\tau : L \otimes K \rightarrow K \otimes L$ is defined by

$$\tau(x \otimes y) = (-1)^{pq} y \otimes x ,$$

where x is of degree p and y of degree q . Let L be a graded k -module. A multiplication map $\mu : L \otimes L \rightarrow L$ is *graded commutative* if $\mu \circ \tau = \mu$. The *tensor coalgebra* $T^c(L)$ is the direct sum of the tensor powers $L^{\otimes n}$, $n \geq 0$. Its comultiplication is defined by

$$\Delta(x_1, \dots, x_n) = \sum_{i=0}^n (x_1, \dots, x_i) \otimes (x_{i+1}, \dots, x_n)$$

and its counit $\eta : T^c(L) \rightarrow k$ is the canonical projection. The flips yield an action of the symmetric group S_n on the n th tensor power of L , for each n . The *symmetric coalgebra* $\text{Sym}^c(L)$ is the subcoalgebra of $T^c(L)$ whose underlying graded module is the sum of the fixed point modules of S_n on $L^{\otimes n}$, $n \geq 0$.

3.2. R -deformations and the Hochschild complex

Let k be a field and A an associative k -algebra with multiplication μ . Let R be a commutative local k -algebra whose maximal ideal \mathfrak{m} is finite-dimensional over k (and thus nilpotent). The truncated polynomial rings $k[t]/(t^n)$, $n \geq 1$, are good examples to keep in mind. An R -*deformation* of the multiplication of A is an associative R -bilinear multiplication $*$ on $A \otimes_k R$ which, modulo \mathfrak{m} , reduces to the multiplication μ of A , i.e. the square

$$\begin{array}{ccc} (A \otimes_k R) \otimes_R (A \otimes_k R) & \longrightarrow & A \otimes_k A \\ \downarrow * & & \downarrow \mu \\ A \otimes_k R & \longrightarrow & A \end{array}$$

commutes. An *infinitesimal deformation* is a $k[t]/(t^2)$ -deformation. Two R -deformations are *equivalent* if there is an R -module automorphism $g : A \otimes_k R \rightarrow A \otimes_k R$ which, modulo \mathfrak{m} , reduces to the identity of A , such that

$$g(u * v) = g(u) *' g(v)$$

for all u, v in $A \otimes_k R$. Note that, by R -bilinearity, an R -deformation is determined by the restriction of $*$ to $A \otimes_k A$ and, in fact, by its component

$$A \otimes_k A \rightarrow A \otimes_k \mathfrak{m}.$$

We denote by $\text{Defo}(A, R)$ the set of equivalence classes of R -deformations of A . In fact, we obtain a functor, the *deformation functor associated with the associative algebra* A

$$\mathcal{R} \rightarrow \text{Sets} , \quad R \mapsto \text{Defo}(A, R) ,$$

where \mathcal{R} denotes the *category of test algebras*, i.e. of commutative local k -algebras with finite-dimensional maximal ideal.

If $*$ is a formal deformation (in the sense of 2.1.1) then, for each $n \geq 1$, its reduction modulo (t^n) is a $k[t]/(t^n)$ -deformation. We obtain a map

$$\{\text{formal deformations}\} \longrightarrow \varprojlim \{k[t]/(t^n)\text{-deformations}\}.$$

It is not hard to see that this map is bijective and that the equivalence relations on both sides correspond to each other. Thus, the study of formal deformations reduces to that of the deformation functor.

Let us take a closer look at infinitesimal deformations: an infinitesimal deformation $*$ is determined by a k -linear map $B_1 : A \otimes_k A \rightarrow A$ such that

$$a * b = ab + B_1(a, b)t$$

for all a, b in A . The associativity of $*$ translates into

$$(3.2.1) \quad aB_1(b, c) - B_1(ab, c) + B_1(a, bc) - B_1(a, b)c = 0$$

for all a, b, c in A and two infinitesimal deformations corresponding to B_1 and B'_1 are equivalent iff there is a k -linear map $g_1 : A \rightarrow A$ such that

$$(3.2.2) \quad B'_1(a, b) - B_1(a, b) = ag_1(b) - g_1(ab) + g_1(a)b$$

for all a, b in A .

The *Hochschild complex* of A is the complex $C(A, A)$ with vanishing components in degrees $p < 0$ and whose p th component, for $p \geq 0$, is the space $\text{Hom}_k(A^{\otimes p}, A)$. By definition, the differential of a p -cochain f is the $(p+1)$ -cochain defined⁽¹⁾ by

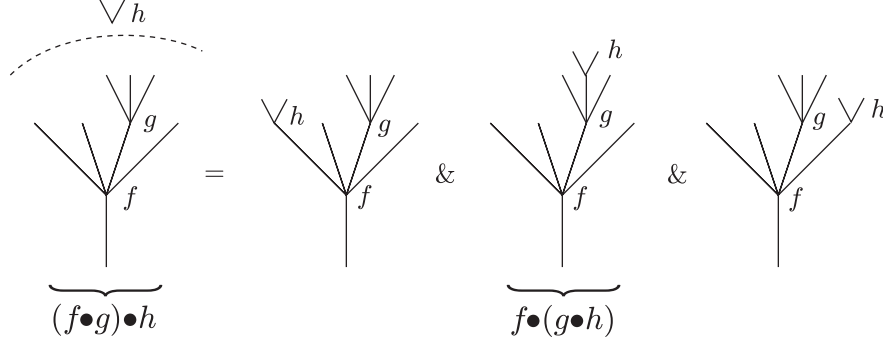
$$\begin{aligned} (-1)^p(df)(a_0, \dots, a_p) &= a_0f(a_1, \dots, a_p) - \sum_{i=0}^{p-1} (-1)^i f(a_0, \dots, a_i a_{i+1}, \dots, a_p) \\ &\quad + (-1)^{p-1} f(a_0, \dots, a_{p-1})a_p. \end{aligned}$$

The *Hochschild cohomology* $HH(A, A)$ of A (with coefficients in A) is by definition the homology of the Hochschild complex.

It follows from the formulas (3.2.1) and (3.2.2) that there is a canonical bijection between the set of equivalence classes of infinitesimal deformations and the space $HH^2(A, A)$. It is also useful to note that the 1-cocycles of $C(A, A)$ are precisely the derivations of A with the 1-coboundaries corresponding to the inner derivations. Finally, the 0-coboundaries vanish and the space of 0-cocycles equals the center of A .

In order to describe non infinitesimal deformations of A , we need a finer structure on the Hochschild complex, namely the Gerstenhaber bracket.

⁽¹⁾The sign differs by a factor $(-1)^p$ from that in [14]. This is justified by formula (3.3.1) below and the relation of the Hochschild complex with the Hochschild resolution of the bimodule A .

FIGURE 1. Terms occurring in $(f \bullet g) \bullet h$

3.3. The Gerstenhaber bracket

We keep the notations of the preceding section. Let f be a Hochschild p -cochain and g a q -cochain. The *Gerstenhaber product* of f by g is the $(p + q - 1)$ -cochain defined by

$$(f \bullet g)(a_1, \dots, a_{p+q-1}) = \sum_{i=0}^p (-1)^{i(q+1)} f(a_1, \dots, a_i, g(a_{i+1}, \dots, a_{i+q}), a_{i+q+1}, \dots, a_{p+q-1})$$

The Gerstenhaber product is not associative in general. However, its associator

$$A(f, g, h) = (f \bullet g) \bullet h - f \bullet (g \bullet h)$$

is (super) symmetric in g and h in the sense that

$$A(f, g, h) = (-1)^{(q-1)(r-1)} A(f, h, g)$$

for a q -cochain g and an r -cochain h , *cf.* figure 1. One checks that this implies that the *Gerstenhaber bracket* defined by

$$[f, g] = f \bullet g - (-1)^{(p-1)(q-1)} g \bullet f$$

satisfies the (super) Jacobi identity (3.3.2) below. Moreover, the Hochschild differential is expressed in terms of the Gerstenhaber bracket and the multiplication μ of A as

$$(3.3.1) \quad df = -[\mu, f].$$

It follows that the shifted (*cf.* section 3.1) Hochschild complex

$$L_{As}(A) = C(A, A)[1]$$

endowed with the Gerstenhaber bracket is a differential graded Lie algebra in the sense of the

Definition 3.3.1. — A \mathbb{Z} -graded Lie algebra is a \mathbb{Z} -graded vector space L endowed with a Lie bracket, i.e. a linear map

$$[\cdot, \cdot] : L \otimes L \rightarrow L$$

homogeneous of degree 0 which is

- antisymmetric, i.e. $[x, y] = (-1)^{pq}[y, x]$ for all $x \in L^p$ and all $y \in L^q$ and
- satisfies the Jacobi identity

$$(3.3.2) \quad [x, [y, z]] = [[x, y], z] + (-1)^{pq}[y, [x, z]]$$

for all $x \in L^p$, $y \in L^q$ and $z \in L$.

A differential graded (=dg) Lie algebra is a \mathbb{Z} -graded Lie algebra L endowed with a differential d which is a derivation with respect to the bracket, i.e.

$$d([x, y]) = [dx, y] + (-1)^p[x, dy]$$

for all $x \in L^p$ and $y \in L$.

Note that if L is a dg Lie algebra, L^0 is an ordinary Lie algebra. For example, if $L = L_{As}(A)$, then $L^0 = \mathfrak{gl}(A)$.

Now let $R \in \mathcal{R}$ be a test algebra (section 3.2). Let $*$ be an R -deformation and let $B : A \otimes A \rightarrow A \otimes \mathfrak{m}$ be such that

$$a * b = ab + B(a, b)$$

for all $a, b \in A$. We view B as a homogeneous element of degree 1 of the dg Lie algebra $L_{As}(A) \otimes_k \mathfrak{m}$. Then one checks that the associativity of $*$ is expressed by the Maurer-Cartan equation

$$dB + \frac{1}{2}[B, B] = 0,$$

where we suppose that the ground field k is not of characteristic 2. We see that the solutions of this equation bijectively correspond to the R -deformations of the multiplication of A . The aim of the next section is to express the equivalence of two R -deformations in terms of the dg Lie algebra $L_{As}(A) \otimes_k \mathfrak{m}$.

3.4. The Maurer-Cartan equation

>From now on, we suppose that the ground field k is of characteristic 0.

Let L be a differential graded Lie algebra (cf. section 3.3). Let $MC(L)$ denote the set of solutions $x \in L^1$ of the Maurer-Cartan equation

$$(3.4.1) \quad d(x) + \frac{1}{2}[x, x] = 0.$$

For $x \in MC(L)$, define $T_x MC(L)$ to be the space of vectors $X \in L^1$ such that

$$d(X) + [x, X] = 0.$$

Note that if L^1 is finite-dimensional, then $MC(L)$ is an intersection of quadrics and $T_x MC(L)$ is the (scheme-theoretic) tangent space at x of the algebraic variety $MC(L)$. For $x, y \in L$, put $(\text{ad}x)(y) = [x, y]$.

Lemma 3.4.1. — *Let $x \in MC(L)$.*

- a) *The map $d_x = d + \text{ad}x$ satisfies $d_x^2 = 0$.*
- b) *A vector $X \in L^1$ belongs to $T_x MC(L)$ iff $d_x X = 0$.*
- c) *For $X_0 \in L^0$, the map $y \mapsto d_y X_0$ yields a vector field on $MC(L)$, i.e. $d_y X_0 \in T_y MC(L)$ for all $y \in MC(L)$.*

Proof. — a) is an easy computation, b) is immediate from a) and c) follows from b) and a). √

Now suppose that L^0 is a nilpotent Lie algebra (*i.e.* there is an $N \gg 0$ such that each composition of at least N maps $\text{ad}(X)|_{L^0}$, $X \in L^0$, vanishes). Suppose moreover that the action of L^0 on L^1 is nilpotent (*i.e.* $\text{ad}X_0$ induces a nilpotent endomorphism of L^1 for each $X_0 \in L^0$). Denote by $\text{Aff}(L^1)$ the group of affine transformations of the vector space L^1 (the semi-direct product of the group of translations by that of linear transformations). Thanks to our nilpotency hypotheses, the Lie algebra antihomomorphism

$$L^0 \rightarrow \text{Lie}(\text{Aff}(L^1)), \quad X_0 \mapsto (x \mapsto d_x X_0 = dX_0 + [x, X_0])$$

integrates to a group antihomomorphism

$$\exp(L^0) \rightarrow \text{Aff}(L^1)$$

so that we obtain a right action of $\exp(L^0)$ on L^1 by affine automorphisms. By point c) of the lemma, this action leaves $MC(L)$ invariant, so that we have a well defined orbit set $MC(L)/\exp(L^0)$. Notice that for $x \in MC(L)$, the ‘normal space’ to the orbit $x \exp(L^0)$ at x is

$$T_x MC(L)/T_x(x \exp(L^0)) = (\ker d_x)/(\text{im } d_x) = H^1(L, d_x).$$

Now let $R \in \mathcal{R}$ be a test algebra (section 3.2) and L an arbitrary dg Lie algebra. We define

$$MC(L, R) = MC(L \otimes_k \mathfrak{m}).$$

Clearly, the dg Lie algebra $L \otimes_k \mathfrak{m}$ satisfies the nilpotency assumptions we made above. We can thus define

$$\overline{MC}(L, R) = MC(L \otimes_k \mathfrak{m})/\exp(L^0 \otimes_k \mathfrak{m}).$$

This definition is motivated by the

Lemma 3.4.2. — *Let A be an associative algebra. Then the dg Lie algebra $L_{As}(A)$ controls the deformations of the multiplication of A , i.e. there are bijections*

$$\text{Defo}(A, R) \rightarrow \overline{MC}(L_{As}(A), R)$$

functorial in $R \in \mathcal{R}$.

To prove the lemma, one checks that the bijection given at the end of section 3.3 is compatible with the equivalence relations.

A morphism of dg Lie algebras $f : L_1 \rightarrow L_2$ is a linear map homogeneous of degree 0 such that

$$f \circ d = d \circ f \text{ and } f([x, y]) = [f(x), f(y)]$$

for all $x, y \in L_1$. It is a *quasi-isomorphism of dg Lie algebras* if it induces an isomorphism in homology.

Theorem 3.4.3 (Quasi-isomorphism theorem). — *Let $f : L_1 \rightarrow L_2$ be a quasi-isomorphism of dg Lie algebras and let $R \in \mathcal{R}$. Then f induces a bijection*

$$(3.4.2) \quad \overline{MC}(L_1, R) \xrightarrow{\sim} \overline{MC}(L_2, R).$$

Note that the conclusion of the theorem concerns the solutions of systems of quadratic equations whereas the hypothesis that f is a quasi-isomorphism is *linear* in nature. The following proposition can be interpreted by saying that the map 3.4.2 induces bijections in the ‘differential graded tangent spaces’. It is proved by considering the (finite!) filtrations induced by the $\mathfrak{m}^i \subset \mathfrak{m}$.

Proposition 3.4.4. — *Let $f : L_1 \rightarrow L_2$ be a quasi-isomorphism of dg Lie algebras and let $R \in \mathcal{R}$. Then for each $x \in MC(L_1, R)$, the morphism*

$$(3.4.3) \quad (L_1 \otimes \mathfrak{m}, d + \text{ad}(x)) \longrightarrow (L_2 \otimes \mathfrak{m}, d + \text{ad}(f(x)))$$

induced by f is a quasi-isomorphism.

As a simple application of the quasi-isomorphism theorem, we prove the

Corollary 3.4.5. — *Let A be an associative algebra with unit 1. For each $R \in \mathcal{R}$, each R -deformation of the multiplication of A is equivalent to an R -deformation admitting the unit 1.*

Proof. — Let $L_{As,1}(A)$ be the subspace of $L_{As}(A)$ generated by all cochains f which vanish if one of their arguments equals 1. Then the subspace $L_{As,1}(A)$ is a dg Lie subalgebra of $L_{As}(A)$. For $R \in \mathcal{R}$, the subset $MC(L_{As,1}(A), R)$ of $MC(L_{As}(A), R)$ corresponds precisely to the R -deformations admitting the unit 1. Now by [14, Ch. IX], the inclusion of $L_{As,1}(A)$ into $L_{As}(A)$ is a quasi-isomorphism. Therefore, the claim follows from the quasi-isomorphism theorem. \checkmark

3.5. Deformations of star products, Lie brackets, Poisson brackets

As we have seen in Lemma 3.4.2, the R -deformations of an associative algebra A are controlled by the dg Lie algebra $L_{As}(A)$. Similarly, there are dg Lie algebras $L_{star}(M)$ and $L_{Pois}(M)$ which control the deformation problems appearing in Kontsevich's theorem 2.3.1, where M is a differentiable manifold. We will now describe these dg Lie algebras in more detail. Let A be the algebra of smooth functions on M .

3.5.1. Star products. — Let $k = \mathbb{R}$ and $R \in \mathcal{R}$. An R -star product on M is an R -deformation $*$ of the multiplication of A such that the map

$$(a, b) \mapsto (\text{id} \otimes \varphi)(a * b)$$

is a bidifferential operator for each linear form φ on \mathfrak{m} . Two R -starproducts are *equivalent* if there is an R -linear map $g : A \otimes R \rightarrow A \otimes R$ as in section 3.2 such that the map

$$a \mapsto (\text{id} \otimes \varphi)(g(a))$$

is a differential operator for each linear form φ on \mathfrak{m} . Let $L_{star}(M)$ be the subspace of $L_{As}(A)$ whose p -cochains are p -differential operators. Then $L_{star}(M)$ is a dg Lie subalgebra of $L_{As}(A)$ and, for each $R \in \mathcal{R}$, there is a canonical bijection between

$$\overline{MC}(L_{star}(M), R)$$

and the set of equivalence classes of R -star products on M . In other words, $L_{star}(M)$ controls the problem of deforming the commutative multiplication of M into a star product. Following [60] we use the notation

$$D_{poly}(M) = L_{star}(M).$$

3.5.2. Lie brackets and the Chevalley-Eilenberg complex. — Suppose that k is a field of characteristic 0. Let \mathfrak{g} be a Lie algebra over k . For $R \in \mathcal{R}$, the set of equivalence classes of R -deformations of the Lie bracket of \mathfrak{g} is defined in analogy with the case of an associative multiplication (*cf.* section 3.2). The *Chevalley-Eilenberg complex* $C_{CE}(\mathfrak{g}, \mathfrak{g})$ of \mathfrak{g} has the components $\text{Hom}_k(\Lambda^p \mathfrak{g}, \mathfrak{g})$ in degrees $p \geq 0$ and vanishing components in negative degrees. Its differential is defined⁽²⁾ by

$$\begin{aligned} (-1)^p(df)(X_0, \dots, X_p) &= \sum_{i < j} (-1)^{i+j+1} f([X_i, X_j], \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_p) \\ &\quad - \sum_i (-1)^i [X_i, f(X_0, \dots, \widehat{X}_i, \dots, X_p)]. \end{aligned}$$

As usual, the symbol \widehat{X} indicates that X is to be omitted. The *Richardson-Nijenhuis product* of a p -cochain f by a q -cochain g is the $(p+q-1)$ -cochain defined by

$$(f \bullet g) = \sum \text{sign}(\sigma) f(g(X_{\sigma(1)}, \dots, X_{\sigma(q)}), X_{\sigma(q+1)}, \dots, X_{\sigma(p+q-1)}),$$

⁽²⁾with the same modification of the sign as for the Hochschild differential in section 3.2

where σ runs through the permutations which are increasing on $\{1, \dots, q\}$ and $\{q + 1, \dots, p + q - 1\}$. The *Richardson-Nijenhuis bracket* of a p -cochain f by a q -cochain g is the commutator

$$[f, g] = f \bullet g - (-1)^{(p-1)(q-1)} g \bullet f.$$

If we let β denote the Lie bracket of \mathfrak{g} , we have

$$df = -[\beta, f].$$

As in section 3.3, one checks that the shifted complex

$$L_{Lie}(\mathfrak{g}) = C_{CE}(\mathfrak{g}, \mathfrak{g})[1]$$

endowed with the Richardson-Nijenhuis bracket is a dg Lie algebra. It controls the R -deformations of the Lie bracket of \mathfrak{g} .

3.5.3. Poisson brackets. — Suppose that k is a field of characteristic 0 and A a Poisson algebra over k . For $R \in \mathcal{R}$, a *Poisson R -deformation of the bracket of A* is an R -linear Poisson bracket π on the commutative algebra $A \otimes_k R$ which, modulo \mathfrak{m} , reduces to the bracket of A . Note that we deform only the bracket, the commutative multiplication of A remains unchanged. Two Poisson R -deformations are *equivalent* if they are conjugate by an automorphism of the commutative algebra $A \otimes_k R$ which, modulo \mathfrak{m} , induces the identity on A .

Let $L_{Pois}(A)$ be the subspace of $L_{Lie}(A)$ formed by the cochains which are derivations in each argument. Then $L_{Pois}(A)$ is a differential graded Lie subalgebra of $L_{Lie}(A)$. The dg Lie algebra $L_{Pois}(A)$ controls the Poisson deformations of the bracket of A .

Now suppose that $k = \mathbb{R}$ and that A is the algebra of smooth functions on a Poisson manifold M . Let $T_{poly}(M)$ be the graded space with vanishing components in degrees < -1 and whose p th component is the space of $(p + 1)$ -polyvector fields on M , i.e. the space $\Gamma(M, \Lambda^{(p+1)}TM)$ of global sections of the $(p + 1)$ th exterior power of the tangent bundle TM of M . For vector fields ξ_i and functions f_j , we define

$$(\xi_1 \wedge \dots \wedge \xi_p)(f_1 \wedge \dots \wedge f_p) = \frac{1}{p!} \det(\xi_i(f_j)).$$

This yields a canonical isomorphism of $L_{Pois}^p(A)$ with $T_{poly}(M)^p$. Thus we obtain a dg Lie algebra structure on the graded space $T_{poly}(M)$. Its bracket is uniquely determined by the following conditions:

$$(3.5.1) \quad T_{poly}^0(M) \text{ is the Lie algebra of vector fields on } M,$$

$$(3.5.2) \quad [\xi, f] = \xi(f),$$

$$(3.5.3) \quad [\alpha, \beta \wedge \gamma] = [\alpha, \beta] \wedge \gamma + (-1)^{(p-1)q} \beta \wedge [\alpha, \gamma],$$

where ξ is a vector field, f a smooth function, and α, β, γ are polyvector fields of degree p, q and r , respectively.

3.6. Quasi-isomorphisms and L_∞ -morphisms of dg Lie algebras

We will sketch a conceptual approach to the notions and results of this section in section 3.7.

Let L_1 and L_2 be two dg Lie algebras. By definition, an L_∞ -morphism $f : L_1 \rightarrow L_2$ is given by a sequence of maps

$$f_n : L_1^{\otimes n} \rightarrow L_2, \quad n \geq 1,$$

homogeneous of degree $1 - n$ and such that the following conditions are satisfied:

- The morphism f_n is graded antisymmetric, i.e. we have

$$f_n(x_1, \dots, x_i, x_{i+1}, \dots, x_n) = -(-1)^{|x_i||x_{i+1}|} f_n(x_1, \dots, x_{i+1}, x_i, \dots, x_n)$$

for all homogeneous x_1, \dots, x_n of L_1 .

- We have $f_1 \circ d = d \circ f_1$, i.e. the map f_1 is a morphism of complexes.
- We have

$$f_1([x_1, x_2]) = [f_1(x_1), f_1(x_2)] + d(f_2(x_1, x_2)) + f_2(d(x_1), x_2) + (-1)^{|x_1|} f_2(x_1, d(x_2))$$

for all homogeneous x_1, x_2 in L . This means that f_1 is compatible with the brackets up to a homotopy given by f_2 . In particular, f_1 induces a morphism of graded Lie algebras from H^*L_1 to H^*L_2 .

- More generally, for each $n \geq 1$ and all homogeneous elements x_1, \dots, x_n of L_1 , we have

$$(3.6.1) \quad (-1)^n \sum_{i < j} (-1)^s f_{n-1}([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_n) \\ = \frac{1}{2} \sum_{p+q=n} \sum_{\sigma} (-1)^{pn+t} [f_p(x_{\sigma(1)}, \dots, x_{\sigma(p)}), f_q(x_{\sigma(p+1)}, \dots, x_{\sigma(n)})] + \\ d(f_n(x_1, \dots, x_n)) - (-1)^{n-1} \sum_{i=1}^n (-1)^u f_n(x_1, \dots, d(x_i), \dots, x_n).$$

Here, σ runs through all (p, q) -shuffles, i.e. all permutations of $\{1, \dots, n\}$ which are increasing on $\{1, \dots, p\}$ and on $\{p+1, \dots, p+q\}$; the exponents s, t and u are respectively the numbers of transpositions of odd elements in passing from (x_1, \dots, x_n) to $(x_i, x_j, x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_n)$, from $(f_p, f_q, x_1, \dots, x_n)$ to

$$(f_p, x_{\sigma(1)}, \dots, x_{\sigma(p)}, f_q, x_{\sigma(p+1)}, \dots, x_{\sigma(n)})$$

and from (d, x_1, \dots, x_n) to $(x_1, \dots, d, x_i, \dots, x_n)$.

Roughly speaking, an L_∞ -morphism is a map between dg Lie algebras which is compatible with the brackets up to a given coherent system of higher homotopies. An L_∞ -quasi-isomorphism is an L_∞ -morphism whose first component is a quasi-isomorphism. The importance of this notion is apparent from the

Theorem 3.6.1. — *The following are equivalent*

- (i) *There is an L_∞ -quasi-isomorphism $L_1 \rightarrow L_2$.*
- (ii) *There is a diagram of two quasi-isomorphisms of dg Lie algebras*

$$L_1 \leftarrow L_3 \rightarrow L_2.$$

- (iii) *There is a chain of quasi-isomorphisms of dg Lie algebras*

$$L_1 \leftarrow L_3 \rightarrow L_4 \leftarrow \cdots \rightarrow L_n \leftarrow L_2$$

The dg Lie algebras L_1 and L_2 are *homotopy equivalent* if they satisfy the conditions of the theorem. A dg Lie algebra L is *formal* if it is homotopy equivalent to its homology H^*L (viewed as a dg Lie algebra with vanishing differential). The quasi-isomorphism theorem 3.4.3 implies that homotopy equivalent dg Lie algebras yield equivalent deformation problems. More precisely, we have the

Theorem 3.6.2 (L_∞ -quasi-isomorphism theorem). — *Let $f : L_1 \rightarrow L_2$ be an L_∞ -quasi-isomorphism of dg Lie algebras. Then, for each test algebra $R \in \mathcal{R}$ (section 3.2), the map*

$$x \mapsto \sum_{n \geq 1} \frac{1}{n!} f_n(x, \dots, x)$$

induces a bijection

$$\overline{MC}(L_1, R) \xrightarrow{\sim} \overline{MC}(L_2, R).$$

Here and below, the R -multilinear extension of $f_n : L_1^{\otimes n} \rightarrow L_2$ to the n th tensor power over R of $L_1 \otimes_k \mathfrak{m}$ is still denoted by f_n . In analogy with proposition 3.4.4, we also have quasi-isomorphisms in the ‘differential graded tangent spaces’:

Proposition 3.6.3. — *Let $f : L_1 \rightarrow L_2$ be an L_∞ -quasi-isomorphism of dg Lie algebras and let $R \in \mathcal{R}$. Then, for each $x \in MC(L_1, R)$, the map*

$$y \mapsto \sum_{n \geq 1} \frac{1}{(n-1)!} f_n(x, \dots, x, y)$$

is a quasi-isomorphism

$$(L_1 \otimes \mathfrak{m}, d + \text{ad}(x)) \longrightarrow (L_2 \otimes \mathfrak{m}, d + \text{ad}(f(x))).$$

3.7. Formal deformation theory via Quillen’s equivalence

In this section, we describe the framework for formal deformation theory provided by Quillen’s equivalence between the homotopy category of dg Lie algebras and that of (certain) dg cocommutative coalgebras. In this way, we will obtain a better understanding of L_∞ -morphisms and L_∞ -algebras, *i.e.* a formal manifolds [60]. We follow V. Hinich’s article [48].

3.7.1. From test algebras to test coalgebras. — Let k be a field of characteristic 0 and A an associative k -algebra. Let $R = k \oplus \mathfrak{m}$ be a test algebra (cf. section 3.2). Since R is finite-dimensional, its dual space $C = \text{Hom}_k(R, k)$ is naturally a coalgebra. We have a natural isomorphism of vector spaces

$$A \otimes_k R \xrightarrow{\sim} \text{Hom}_k(C, A)$$

and the canonical multiplication on the left corresponds to the *convolution product* defined by

$$f \cdot g = \mu \circ (f \otimes g) \circ \Delta$$

for all f, g linear maps $C \rightarrow A$, where μ is the multiplication of A and Δ the comultiplication on C . The R -deformations of A then correspond to the associative $\text{Hom}_k(C, k)$ -bilinear multiplications on $\text{Hom}_k(C, A)$ which induce the multiplication of A after passage to the quotient

$$\text{Hom}_k(C, A) \rightarrow \text{Hom}_k(k, A) = A.$$

This description has the advantage that it naturally generalizes to certain infinite dimensional coalgebras. For example, if C is the coalgebra $k[T]$ with

$$\Delta(f(T)) = f(T \otimes 1 + 1 \otimes T),$$

then $\text{Hom}_k(C, A)$ identifies with the power series algebra $A[[t]]$ and our description yields precisely the associative formal deformations of the multiplication of A (in the sense of 2.1.1). The appropriate class of *test coalgebras* to consider is that of *cocommutative cocomplete augmented coalgebras*, i.e. cocommutative coassociative coalgebras C endowed with a counit $\eta : C \rightarrow k$ and an augmentation $\varepsilon : k \rightarrow C$ such that, if $\overline{C} = C/\varepsilon(C)$ is the *reduction* of C , then each element of \overline{C} is annihilated by a sufficiently high iterate $\overline{C} \rightarrow \overline{C}^{\otimes i}$ of the comultiplication induced by Δ . Note that the dual of a test coalgebra C is a complete local ring with maximal ideal $\text{Hom}_k(\overline{C}, k)$.

3.7.2. The Maurer-Cartan equation. — Let L be a dg Lie algebra. For a test algebra $R = k \oplus \mathfrak{m}$, the set

$$\overline{MC}(R, L) = \overline{MC}(L \otimes_k \mathfrak{m})$$

of equivalence classes of solutions of the Maurer-Cartan equation (3.4.1) only depends on the ‘piece’

$$L^0 \rightarrow L^1 \rightarrow L^2$$

of the dg Lie algebra L . In order to capture the whole information given by L , we have to allow R to have components in several degrees and to have a non zero differential. If we combine this observation with the remarks of the preceding paragraph, we arrive at the notion of a *dg test coalgebra*, i.e. a test coalgebra endowed with a \mathbb{Z} -grading

and a coalgebra differential $d : C \rightarrow C$ of degree 1. This means that $d^2 = 0$ and that d is a *coderivation*, *i.e.*

$$\Delta \circ d = (\text{id} \otimes d + d \otimes \text{id}) \circ \Delta.$$

For a dg test coalgebra C , the graded space $\text{Hom}_k(C, L)$, whose n -th component consists of the homogeneous k -linear maps of degree n , becomes a dg Lie algebra for the differential

$$d(f) = d \circ f - (-1)^{|f|} f \circ d$$

and the convolution bracket

$$[f, g] = [,]_L \circ (f \otimes g) \circ \Delta.$$

We define the set of *twisting cochains* $C \rightarrow L$ to be the set

$$\text{Tw}(C, L) = MC(\text{Hom}_k(C, L))$$

of solutions of the Maurer-Cartan equation (3.4.1). For example, if we take $C = k[T]$ (as in 3.7.1) concentrated in degree 0 and with $d = 0$ and if $L = L_{As}(A)$ for an associative algebra A (as in section 3.3), then $\text{Tw}(C, L)$ naturally identifies with the associative formal deformations of the multiplication of A (as in paragraph 2.1.1).

3.7.3. The bar and the cobar constructions. — Let Lie be the category of dg Lie algebras and Cog that of dg test coalgebras.

Lemma and Definition 3.7.1. — a) For $L \in \text{Lie}$, the functor

$$\text{Tw}(?, L) : \text{Cog}^{op} \rightarrow \text{Sets}$$

is representable. We denote a representative by BL .

b) For $C \in \text{Com}$, the functor

$$\text{Tw}(C, ?) : \text{Lie} \rightarrow \text{Sets}$$

is representable. We denote a representative by ΩC .

c) We have canonical bijections

$$\text{Hom}_{\text{Lie}}(\Omega C, L) = \text{Tw}(C, L) = \text{Hom}_{\text{Cog}}(C, \Omega L).$$

In particular, B and Ω are a pair of adjoint functors between Lie and Cog .

Note that c) is a reformulation of a) and b). Explicitly, part a) of the Lemma claims that there is a dg test coalgebra BL and a twisting cochain $\tau : BL \rightarrow L$ which is *universal*, *i.e.* for each twisting cochain $\tau' : C \rightarrow L$ there is a unique morphism of dg test coalgebras $f : BL \rightarrow C$ such that $\tau' = \tau \circ f$. Concretely, BL is given by the *bar construction on L* , *i.e.* the graded symmetric (section 3.1) coalgebra $\text{Sym}^c(L[1])$ on the suspension $L[1]$ of L endowed with the unique coderivation d such that the evident morphism $\tau : \text{Sym}^c(L[1]) \rightarrow L$ of degree 1 becomes a twisting cochain:

$$d_L \circ \tau + \tau \circ d - \frac{1}{2}[\tau, \tau] = 0.$$

Note that if L is concentrated in degree 0, the underlying complex of BL is the homological Chevalley-Eilenberg complex which computes $H^*(L, k)$.

Dually, part b) claims that there is a dg Lie algebra ΩC and a twisting cochain $\tau : C \rightarrow \Omega C$ which is co-universal. Explicitly, ΩC is given by the free graded (section 3.1) Lie algebra on $C[-1]$ endowed with the unique derivation d such that the evident morphism $\tau : C \rightarrow \Omega C$ of degree 1 becomes a twisting cochain.

3.7.4. Quillen's equivalence. — A morphism $f : C \rightarrow C'$ of Cog is a *weak equivalence* if Ωf is a quasi-isomorphism. The following lemma is not hard to show:

Lemma 3.7.2. — a) *The functor $B : \text{Lie} \rightarrow \text{Cog}$ takes quasi-isomorphisms to weak equivalences.*

b) *For each $L \in \text{Lie}$, the adjunction morphism $\Omega BL \rightarrow L$ is a quasi-isomorphism, and for each $C \in \text{Cog}$, the adjunction morphism $C \rightarrow B\Omega C$ is a weak equivalence.*

Let $\text{Ho}(\text{Lie})$ be the *localization* of the category Lie at the class of quasi-isomorphisms, *i.e.* the category whose objects are the same as those of Lie and whose morphisms are obtained from those of Lie by formally inverting all quasi-isomorphisms. Analogously, let $\text{Ho}(\text{Cog})$ be the localization of Cog at the class of weak equivalences. We refer to these localizations as *homotopy categories*. From the lemma, we immediately obtain the

Theorem 3.7.3 ([85],[48]). — *The functors B and Ω induce quasi-inverse equivalences between homotopy categories $\text{Ho}(\text{Lie})$ and $\text{Ho}(\text{Cog})$.*

Quillen's equivalence is the equivalence $B : \text{Ho}(\text{Lie}) \rightarrow \text{Ho}(\text{Cog})$.

3.7.5. Morphisms in the homotopy categories. — Let L', L be dg Lie algebras. In general, the map

$$\text{Hom}_{\text{Lie}}(L', L) \rightarrow \text{Hom}_{\text{Ho}(\text{Lie})}(L', L)$$

will not be surjective. However, if $L' = \Omega C$ for some $C \in \text{Cog}$, it is surjective and we can describe the image in terms of equivalence classes of solutions of the Maurer-Cartan equation: Put

$$\overline{\text{Tw}}(C, L) = \overline{MC}(\text{Hom}_k(C, L)).$$

Theorem 3.7.4 ([48]). — *The maps*

$$\text{Hom}_{\text{Lie}}(\Omega C, L) \rightarrow \text{Hom}_{\text{Ho}(\text{Lie})}(\Omega C, L) \text{ and } \text{Hom}_{\text{Cog}}(C, BL) \rightarrow \text{Hom}_{\text{Ho}(\text{Cog})}(C, BL)$$

are surjective and we have bijections

$$\text{Hom}_{\text{Ho}(\text{Lie})}(\Omega C, L) = \overline{\text{Tw}}(C, L) = \text{Hom}_{\text{Ho}(\text{Cog})}(C, BL).$$

3.7.6. Formal deformation problems. — A *formal deformation problem* is a representable functor

$$F = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Cog})}(\?, C) : \mathrm{Ho}(\mathrm{Cog})^{op} \rightarrow \mathrm{Sets}.$$

If L is a dg Lie algebra, the problem of deforming the zero solution of the Maurer-Cartan equation in L is the functor

$$\overline{\mathrm{Tw}}(\?, L) : \mathrm{Ho}(\mathrm{Cog})^{op} \rightarrow \mathrm{Sets}.$$

It is a formal deformation problem since it is represented by ΩL , by theorem 3.7.4. Conversely, this theorem yields that for each formal deformation problem F there is a dg Lie algebra L , unique up to isomorphism in $\mathrm{Ho}(\mathrm{Lie})$, such that

$$F \xrightarrow{\sim} \overline{\mathrm{Tw}}(\?, L).$$

In summary, we have bijections between isomorphism classes of formal deformation problems, homotopy types of dg Lie algebras and homotopy types of dg cocomplete cocommutative coalgebras.

3.7.7. Link with L_∞ -morphisms. — Let L and L' be dg Lie algebras. In section 3.6, we have defined the notion of L_∞ -morphism from L to L' .

Lemma 3.7.5. — *There is a canonical bijection between the set of L_∞ -morphisms $L \rightarrow L'$ and the set of morphisms of dg coalgebras $BL \rightarrow BL'$. Under this bijection, the L_∞ -quasi-isomorphisms correspond to the weak equivalences. The dg Lie algebras L and L' are homotopy equivalent iff they are isomorphic in the homotopy category $\mathrm{Ho}(\mathrm{Lie})$.*

Let us deduce the L_∞ -quasi-isomorphism theorem 3.6.2: If R is a test algebra, then we have a bijection (cf. paragraph 3.7.1)

$$\overline{MC}(L, R) = \overline{\mathrm{Tw}}(DR, L) = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Cog})}(DR, BL),$$

where $DR = \mathrm{Hom}_k(R, k)$ is the coalgebra dual to the (finite-dimensional) algebra R . The lemma shows that the right hand side is preserved under L_∞ -quasi-isomorphisms. It is easy to check the explicit formula in theorem 3.6.2.

3.7.8. L_∞ -algebras and fibrant coalgebras. — Let $C \in \mathrm{Cog}$.

Proposition 3.7.6. — *The following are equivalent*

- (i) *There is a graded vector space L such that the underlying graded augmented coalgebra of C is isomorphic to $\mathrm{Sym}^c(L[1])$.*

- (ii) For each morphism $i : D \rightarrow E$ of Cog such that i is injective (on the underlying vector spaces) and $\Omega(i)$ is a quasi-isomorphism, and for each morphism $f : D \rightarrow C$ of Cog, there is a morphism $h : E \rightarrow C$ such that $h \circ i = f$.

$$\begin{array}{ccc} D & \xrightarrow{f} & C \\ i \downarrow & \nearrow h & \\ E & & \end{array}$$

Suppose that C is *fibrant*, i.e. it satisfies the properties of the proposition. Then the graded space $L[1]$ of (i) is isomorphic to the space $\text{Prim}(C)$ of primitive elements of C (i.e. the kernel of the map $\overline{C} \rightarrow \overline{C} \times \overline{C}$ induced by the comultiplication). Note however that there is no canonical isomorphism between C and $\text{Sym}^c(\text{Prim}(C))$. The differential d of C yields a sequence of graded maps

$$Q_n : L^{\otimes n} \rightarrow L, \quad n \geq 1,$$

which are homogeneous of degree $2 - n$, graded antisymmetric and satisfy quadratic equations which express the fact that $d^2 = 0$. The first two of these equation imply that $Q_1^2 = 0$ (so (L, Q_1) is a complex) and that $Q_2 : L \otimes L \rightarrow L$ is a map of complexes which induces a graded Lie bracket in homology. By definition, the space L endowed with the Q_n becomes an L_∞ -algebra. Each dg Lie algebra is naturally an L_∞ -algebra but there are many other examples. By definition, *morphisms between L_∞ -algebras* correspond bijectively to morphisms between the corresponding objects of Cog. Thus we have a fully faithful functor

$$B_\infty : \{L_\infty\text{-algebras}\} \rightarrow \text{Cog},$$

which extends the bar construction $B : \text{Lie} \rightarrow \text{Cog}$. It is easy to see that the category of L_∞ -algebras admits products and that the product of L_1 with L_2 is $L_1 \oplus L_2$ with the natural maps Q_n . An L_∞ -algebra L is *linear contractible* if $Q_n = 0$ for $n \geq 2$ and the complex (L, Q_1) is contractible. It is *minimal* if Q_1 vanishes.

Proposition 3.7.7. — *Each L_∞ -algebra L is isomorphic to the product $M \oplus C$ of a minimal L_∞ -algebra M and a linear contractible L_∞ -algebra C .*

It follows from the proposition that an L_∞ -algebra C is contractible iff $B_\infty C$ is isomorphic to zero in $\text{Ho}(\text{Cog})$. Moreover, an L_∞ -algebra M is minimal iff we have

$$f : M \rightarrow M \text{ is invertible} \Leftrightarrow B_\infty(f) \text{ becomes invertible in } \text{Ho}(\text{Cog}).$$

3.7.9. Formal manifolds. — A formal (graded) manifold is a graded cocomplete coalgebra C which is isomorphic to the symmetric coalgebra $\text{Sym}^c(V)$ of some graded vector space V . However, the isomorphism $C \xrightarrow{\sim} \text{Sym}^c(V)$ is not part of the structure. For a formal manifold P , a P -point of C is a morphism of coaugmented coalgebras $P \rightarrow C$. The formal manifold C comes with a distinguished point $k \rightarrow C$ given

by the coaugmentation. Its *tangent space at the distinguished point* is

$$T_0C = \text{Prim}(C) = \ker(\overline{C} \rightarrow \overline{C} \otimes \overline{C}).$$

Let $d : C \rightarrow C$ be a coalgebra differential, homogeneous of degree 1. Geometrically, we view d as a vector field of degree 1 on the formal manifold C satisfying $[d, d] = 0$. Let us call the datum of C with d a *Q-manifold*. If L is a graded vector space such that $C \cong \text{Sym}^c(L[1])$, then d corresponds to a structure of L_∞ -algebra on L . By lemma 3.7.5, morphisms of Q -manifolds are in bijection with L_∞ -morphisms. One defines the notion of homotopy between morphisms of Q -manifolds using polynomial families of morphisms. Then one can show [48] that the homotopy category of Q -manifolds is equivalent to the homotopy categories $\text{Ho}(\text{Cog})$ and $\text{Ho}(\text{Lie})$.

3.8. Notes

According to Goldman and Millson [45], the philosophy of controlling deformation problems by differential graded Lie algebras is due to Schlessinger-Stasheff [87] and P. Deligne. The quasi-isomorphism theorem 3.4.3 is stated and proved in this generality in [60]. The material presented in section 3.7 is due to Quillen [85], Hinich-Schechtman [50], Kontsevich [60], Hinich [48], ...

CHAPTER 4

ON TAMARKIN'S APPROACH

In this chapter, we sketch Tamarkin's approach to a purely algebraic version of Kontsevich's theorem for \mathbb{R}^n . We follow Kontsevich's interpretation [62]. We recall the necessary material from operad theory and from the theory of iterated loop spaces.

4.1. Tamarkin's theorem

Let k be a field of characteristic 0, let V be a finite-dimensional k -vector space and SV the symmetric algebra on V . The problem of deforming the multiplication of SV is described by the dg Lie algebra

$$L_{As}(SV) = C(SV, SV)[1],$$

i.e. the shifted Hochschild complex endowed with the Gerstenhaber bracket (*cf.* section 3.3).

Theorem 4.1.1 (Tamarkin [93]). — *The dg Lie algebra $L_{As}(SV)$ is formal.*

4.1.1. Kontsevich's formality theorem follows for $M = \mathbb{R}^n$. — We keep the above notations and let $k = \mathbb{R}$. We consider the dual $M = V^*$ as a Poisson manifold with vanishing bracket. Let $L_{As,md}(SV)$ be the subcomplex of $L_{As}(SV)$ whose components are formed by the cochains which are multidifferential operators with polynomial coefficients. The following lemma results from suitable variants of the Hochschild-Kostant-Rosenberg theorem [51].

Lemma 4.1.2. — *$L_{As,md}(SV)$ is a dg Lie subalgebra of $L_{As}(SV)$ and of $L_{star}(V^*)$. Moreover, both inclusions are quasi-isomorphisms.*

It now follows from Tamarkin's theorem that $L_{star}(V^*)$ is linked to its homology by a chain of quasi-isomorphisms of dg Lie algebras, so that we obtain Kontsevich's formality theorem 2.4.1 using the quasi-isomorphism theorem 3.4.3.

4.1.2. Outline of Tamarkin's approach. — We essentially follow Kontsevich's presentation [62]. The basic idea (which, according to [93], goes back to B. Tsygan) consists in using the additional structure present on the Hochschild complex in the form of the cup product. We use the notations introduced in section 3.1.

A *Gerstenhaber algebra* is given by a \mathbb{Z} -graded vector space G , a graded commutative associative multiplication on G and a Lie bracket on $G[1]$ such that for each $x \in G^p$, the bracket $[x, ?]$ is a derivation of degree $p + 1$ of the associative algebra G . A *dg Gerstenhaber algebra* is a Gerstenhaber algebra endowed with a differential which is a derivation for both operations, the multiplication and the bracket.

If A is an associative algebra, the *cup product* of a Hochschild p -cochain f by a q -cochain g is the $(p + q)$ -cochain $f \cup g$ defined by

$$(f \cup g)(a_1, \dots, a_{p+q}) = (-1)^{pq} f(a_1, \dots, a_p) g(a_{p+1}, \dots, a_{p+q}).$$

Lemma 4.1.3 (Gerstenhaber [39]). — a) *Endowed with the cup product and the Hochschild differential the Hochschild complex becomes a dg associative algebra.*

b) *Hochschild cohomology endowed with the cup product and the Gerstenhaber bracket is a Gerstenhaber algebra.*

Remark 4.1.4. — It is important to note that the Hochschild complex itself is *not*, in general, a Gerstenhaber algebra for the cup product and the Gerstenhaber bracket. For example, the cup product of cochains is not commutative in general.

Let us examine the Gerstenhaber algebra structure on the Hochschild cohomology of a *commutative* k -algebra A . It is easy to see that we have an isomorphism of Lie algebras

$$\mathrm{Der}_k(A, A) \xrightarrow{\sim} HH^1(A, A),$$

where $\mathrm{Der}_k(A, A)$ denotes the space of k -linear derivations from A to itself. The bracket on $\mathrm{Der}_k(A, A)$ admits a unique extension which makes the exterior algebra $\Lambda_A \mathrm{Der}_k(A, A)$ into a Gerstenhaber algebra (where the elements of $\mathrm{Der}_k(A, A)$ are in degree 1). The above isomorphism uniquely extends to a morphism of Gerstenhaber algebras

$$\Lambda_A \mathrm{Der}_k(A, A) \rightarrow HH^*(A, A).$$

By the Hochschild-Kostant-Rosenberg theorem [51], this is an isomorphism if k is perfect and A is the algebra of polynomial functions on a smooth affine variety over k . It is also invertible if k is an arbitrary field and $A = SV$ the symmetric algebra on a finite-dimensional vector space V (use the Koszul resolution as in [67, 3.3.3]). Thus we have an isomorphism of Gerstenhaber algebras

$$\Lambda_{SV} \mathrm{Der}_k(SV, SV) \xrightarrow{\sim} HH^*(SV, SV).$$

A *quasi-isomorphism* of dg Gerstenhaber algebras is a morphism of dg Gerstenhaber algebras which induces isomorphisms in homology. A dg Gerstenhaber algebra G is *formal* if it is linked to its homology H^*G (considered as a dg Gerstenhaber algebra with vanishing differential) by a sequence of quasi-isomorphisms of dg Gerstenhaber algebras.

Proposition 4.1.5. — *Let V be a finite-dimensional vector space and G a dg Gerstenhaber algebra such that H^*G is isomorphic to $HH^*(SV, SV)$. Then G is formal.*

Though non-trivial, the proposition is not deep. Proofs can be found in section 3 of [93], in [49], or in [43]. The deep part of Tamarkin's contribution is contained in the following theorem.

Theorem 4.1.6. — *For each associative (not necessarily commutative) k -algebra A , there is a dg Gerstenhaber algebra \tilde{G} such that*

- a) $H^*\tilde{G}$ is isomorphic to $HH^*(A, A)$ as a Gerstenhaber algebra and
- b) \tilde{G} is linked to $L_{As}(A)$ by a sequence of quasi-isomorphisms of dg Lie algebras.

Together, the proposition and the theorem imply Tamarkin's formality theorem 2.4.2: Indeed, it follows from a) and the proposition that for $A = SV$, the dg Gerstenhaber algebra \tilde{G} is formal. In particular, it is formal as dg Lie algebra. So we obtain sequences of quasi-isomorphisms of dg Lie algebras

$$H^*(\tilde{G})[1] \rightsquigarrow \tilde{G}[1] \rightsquigarrow L_{As}(SV).$$

Since we have isomorphisms of dg Lie algebras

$$H^*L_{As}(SV) = HH^*(SV, SV)[1] = H^*(\tilde{G})[1],$$

it follows that $L_{As}(SV)$ is formal as a dg Lie algebra, as claimed by theorem 2.4.2.

Tamarkin's proof of theorem 4.1.6 uses the language of operads [75]. Its two main ingredients are the following theorems

- 1) Deligne's question [21] has a positive answer: There is a homotopy action of the (normalized singular chain operad of) the little squares operad on the Hochschild cochain complex of any associative algebra.
- 2) The little squares operad is formal.

In the sequel, we will succinctly introduce the language of operads, present these two theorems and show how they imply theorem 4.1.6.

4.2. Operads

4.2.1. A first example: The associative operad. — Let k be a field, $\text{Vec } k$ the category of k -vector spaces and \mathcal{A} the category of associative (non unital) k -algebras. For $n \geq 1$, we consider the functor

$$T_n : \mathcal{A} \rightarrow \text{Vec } k, \quad A \mapsto A^{\otimes n}.$$

A *natural n-ary operation* is a morphism of functors $T_n \rightarrow T_1$, i.e. a morphism of vector spaces

$$A^{\otimes n} \rightarrow A$$

which is functorial in the algebra A . For example, the multiplication of A yields a natural binary operation and the identical map a natural unary operation. Denote by $\text{As}(n)$ the space of natural n -ary operations. It is a right Σ_n -module: If $\lambda \in \text{As}(n)$ and σ is a permutation of $\{1, \dots, n\}$, then the operation $\lambda\sigma$ is defined by

$$a_1 \otimes \dots \otimes a_n \mapsto \lambda(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}).$$

If we have natural operations $\lambda \in \text{As}(n)$ and $\mu_i \in \text{As}(k_i)$, $1 \leq i \leq n$, then the composition $\lambda(\mu_1, \dots, \mu_n)$ defined by

$$a_1 \otimes \dots \otimes a_N \mapsto \lambda(\mu_1(a_1, \dots, a_{k_1}), \dots, \mu_n(a_{N-k_n+1}, \dots, a_N)), \quad N = \sum k_i,$$

belongs to $\text{As}(k_1 + \dots + k_n)$. Thus we obtain a composition map

$$\gamma : \text{As}(n) \otimes \text{As}(k_1) \otimes \dots \otimes \text{As}(k_n) \rightarrow \text{As}(k_1 + \dots + k_n).$$

It is clear that composition is compatible in a suitable way with the actions of the symmetric groups and that the identity of A yields a 'neutral element' for the composition. The spaces $\text{As}(n)$, $n \geq 1$, together with the actions of the symmetric groups, the composition and the identity morphism form the *associative operad*. It is not hard to show that $\text{As}(n)$ is in fact a free $k[\Sigma_n]$ -module and to describe the compositions explicitly, cf. for example [13, 0.10].

4.2.2. Operads and their algebras. — More generally, an *operad of vector spaces* is given by a sequence $O(n)$, $n \geq 1$, of vector spaces, a right action of Σ_n on $O(n)$ for each $n \geq 1$, composition maps

$$\gamma : O(n) \otimes O(k_1) \otimes \dots \otimes O(k_n) \rightarrow O(k_1 + \dots + k_n)$$

for all integers $n, k_1, \dots, k_n \geq 1$, and a distinguished element $1 \in O(1)$. One imposes natural conditions to the effect that

- composition is compatible with the actions of the symmetric groups,
- composition is associative,
- composition admits 1 as a neutral element.

(The reader can find the complete definition in [44], for example). *Morphisms of operads* are defined in the natural way.

If V is a vector space, the *endomorphism operad* $\text{Endop}(V)$ has the components $\text{Hom}(V^{\otimes n}, V)$ with the natural action of Σ_n , $n \geq 1$, and the natural composition. If O is an operad, an *algebra over O* (=O-algebra) is a vector space A together with a morphism of operads

$$\rho : O \rightarrow \text{Endop}(A).$$

For example, one can check that the algebras over the associative operad are precisely the associative algebras. Similarly, there is the *commutative operad* Com with $\text{Com}(n) = k$ (the trivial module) for all $n \geq 1$ whose algebras are precisely the commutative k -algebras. Another example is the *Lie operad* Lie whose algebras are precisely the Lie algebras over k .

If O is an operad and $\text{Alg}(O)$ the category of algebras over O , the forgetful functor

$$\text{Alg}(O) \rightarrow \text{Vec } k$$

admits a left adjoint: the free algebra functor, which takes a vector space V to

$$F(O, V) = \sum_{n \geq 1} O(n) \otimes_{\Sigma_n} V^{\otimes n}.$$

This shows that the Σ_n -module $O(n)$ can be recovered from the free algebra $F(O, k^n)$ as the $(1, \dots, 1)$ -component of the natural \mathbb{N}^n -grading. For example, we thus obtain a description of $\text{Lie}(n)$, $n \geq 1$, as the $(1, \dots, 1)$ -component of the free Lie algebra on n generators.

The definition of an operad still makes sense if we replace the category of vector spaces by that of topological spaces and the tensor product by the cartesian product. This yields the notion of a *topological operad* and of an algebra over such an operad. More generally, we may replace the category of vector spaces by any symmetric monoidal category (cf. [69]). We thus obtain the notion of *graded operad* and *differential graded (=dg) operad*. The *Gerstenhaber operad* Gerst is the graded operad whose algebras are the Gerstenhaber algebras (4.1.2). We have a natural morphism of graded operads $\text{Com} \rightarrow \text{Gerst}$. The restriction of a Gerstenhaber algebra along this morphism is its underlying commutative algebra. Similarly, we have a canonical morphism $\Sigma \text{Lie} \rightarrow \text{Gerst}$, where ΣLie denotes the graded operad whose algebras are the suspensions $L[1]$, where L is a (graded) Lie algebra. More generally, for any graded operad O , the *suspended operad* ΣO whose algebras are the suspensions $A[1]$ of O -algebras A , is given by

$$(\Sigma O)(n) \xrightarrow{\sim} O(n)[n-1] \otimes \text{sgn}_n,$$

where sgn_n is the sign representation of the symmetric group Σ_n . We have

$$\text{Gerst}(2) = \text{Lie}(2)[1] \otimes \text{sgn}_2 \oplus \text{Com}(2).$$

If O is a topological operad, then, thanks to the Künneth theorem, the (singular) homologies $H_*(O(n), k)$ naturally form a graded operad. More subtly, the *normalized singular chain complexes* $N_*(O(n), k)$ (the quotients of the complexes of singular chains by all degenerate chains) form a dg operad, thanks to the Eilenberg-Zilber theorem (cf. [68, VIII.8]).

4.2.3. Little squares. — The *little squares operad* \mathcal{C}_2 is an example of a topological operad. It is defined as follows: Let J be the open unit interval $]0, 1[$. A *little square*

is an affine embedding with parallel oriented axes of J^2 into itself. In other words, it is a map

$$c : J^2 \rightarrow J^2, (t_1, t_2) \rightarrow ((1 - t_1)x_1 + t_1y_1, (1 - t_2)x_2 + t_2y_2),$$

where $0 \leq x_i < y_i \leq 1$. The n th component of the *little squares operad* \mathcal{C}_2 is the set of all n -tuples (c_1, \dots, c_n) of little squares with disjoint images. We identify this set with a subspace of the space of maps (with the compact open topology) from the disjoint union of n copies of J^2 to J^2 . The group Σ_n acts on $\mathcal{C}_2(n)$ by permuting the squares:

$$(c_1, \dots, c_n) \sigma = (c_{\sigma(1)}, c_{\sigma(2)}, \dots, c_{\sigma(n)}).$$

For $c \in \mathcal{C}_2(n)$ and $d_i \in \mathcal{C}_2(k_i)$, $1 \leq i \leq n$, the composition $\gamma(c, d_1, \dots, d_n)$ is defined via the composition of maps

$$\coprod_{i=1}^n \coprod_{k_i} J^2 \xrightarrow{(d_1, \dots, d_n)} \coprod_n J^2 \xrightarrow{c} J^2$$

Finally, the unit element is the identity map $\text{id} \in \mathcal{C}_2(1)$. By sending each square to its center we obtain a homotopy equivalence between $\mathcal{C}_2(n)$ and the space of n -tuples of distinct points in J^2 . In particular, we see that $\mathcal{C}_2(2)$ is homotopy equivalent to the circle.

Let Y be a topological space with a base point $*$. Define its second loop space $\Omega^2 Y$ to be the space of all continuous maps $f : [0, 1]^2 \rightarrow Y$ which send the boundary of the unit square to the base point $*$. We see that $\Omega^2 Y$ carries a natural structure of \mathcal{C}_2 -algebra. Conversely, if X is a connected topological space which is a \mathcal{C}_2 -algebra, then X is weakly equivalent to $\Omega^2 Y$ for some topological space with base point Y , cf. [75].

Theorem 4.2.1 (Cohen [18]). — *Let k be a field of characteristic 0. The homology operad $H_*(\mathcal{C}_2, k)$ of the little squares operad is isomorphic to the Gerstenhaber operad Gerst .*

As a simple instance of the theorem, note that

$$H_*(\mathcal{C}_2(2), k) \xrightarrow{\sim} H_*(S^1, k) \xrightarrow{\sim} k\mu \oplus k\lambda = \text{Gerst}(2),$$

where μ corresponds to the commutative multiplication (of degree 0) and λ to the bracket (of cohomological degree -1) of a Gerstenhaber algebra.

4.3. Application in Tamarkin's proof

4.3.1. Homotopy action of the Gerstenhaber operad on Hochschild cochains. — Let O, P be dg operads. By definition, a *quasi-isomorphism* $O \rightarrow P$ is a morphism of dg operads such that $O(n) \rightarrow P(n)$ is a quasi-isomorphism of complexes for each $n \geq 1$. The *homotopy category of operads* $\text{Ho}(Op)$ is obtained from the category of dg operads by formally inverting all quasi-isomorphisms. A dg

operad O is *formal* if, in $\text{Ho}(Op)$, O is isomorphic to H^*O viewed as a dg operad with vanishing differential. An *homotopy action* of the dg operad O on a dg vector space C is by definition a morphism

$$\rho : O \rightarrow \text{Endop}(C)$$

in $\text{Ho}(Op)$. Such a morphism induces a structure of $H^*(O)$ -algebra on the homology $H^*(C)$. In general, it is a highly non trivial problem to determine whether a given $H^*(O)$ -action on $H^*(C)$ lifts to a homotopy action of O on C .

Let A be an associative algebra. We know that the Lie algebra structure on $HH^*(A, A)[1]$ extends to a Gerstenhaber algebra structure. This means that we have a commutative triangle of graded operads

$$\begin{array}{ccc} \Sigma \text{Lie} & \xrightarrow{\lambda} & \text{Endop}(HH^*(A, A)) \\ \downarrow & \nearrow \varphi & \\ \text{Gerst} & & \end{array}$$

We also know that the Lie structure on $HH^*(A, A)[1]$ comes from a Lie algebra structure on the shifted Hochschild complex $C(A, A)[1]$ itself. So we have a canonical lift of λ to morphism of operads

$$\Sigma \text{Lie} \xrightarrow{\Lambda} \text{Endop}(C(A, A)).$$

Theorem 4.3.1 (Tamarkin). — *The morphism φ lifts to a morphism*

$$\Phi : \text{Gerst} \rightarrow \text{Endop}(C(A, A))$$

of $\text{Ho}(Op)$ such that the triangle

$$\begin{array}{ccc} \Sigma \text{Lie} & \xrightarrow{\Lambda} & \text{Endop}(C(A, A)) \\ \downarrow & \nearrow \Phi & \\ \text{Gerst} & & \end{array}$$

commutes in $\text{Ho}(Op)$.

Theorem 4.1.6 follows from this theorem: for \tilde{G} , one takes the ‘restriction along Φ ’ of the algebra $C(A, A)$. Since Φ is not a morphism of operads but only a morphism in $\text{Ho}(Op)$, some work is required to define the restriction. The main point is the

Theorem 4.3.2 (Hinich [47, 4.7.4]). — *If $\alpha : O \rightarrow O'$ is a quasi-isomorphism of dg operads over a field of characteristic 0, then the restriction along α is an equivalence*

$$\text{Ho}(\text{Alg}(O')) \rightarrow \text{Ho}(\text{Alg}(O)) ,$$

where $\text{Ho}(\text{Alg}(O))$ is the localization of the category of dg O -algebras with respect to the class of quasi-isomorphisms.

4.3.2. On the proof of Theorem 4.3.1. — Let us fix an isomorphism

$$\alpha : \text{Gerst} \rightarrow H^*(\mathcal{C}_2, k)$$

as in Cohen's theorem 4.2.1. The morphism Φ of theorem 4.3.1 is constructed as a composition of two morphisms of $\text{Ho}(\text{Op})$:

$$\text{Gerst} \xrightarrow{\Phi_1} N_*(\mathcal{C}_2, k) \xrightarrow{\Phi_2} \text{Endop}(C(A, A)).$$

Here, N_* denotes the normalized singular cochain complex defined at the end of section 4.2.2 and Φ_1 is an isomorphism inducing α in homology. The existence of Φ_1 is immediate from the

Theorem 4.3.3. — *If k is a field of characteristic 0, the normalized singular chains operad $N_*(\mathcal{C}_2, k)$ of the little squares operad is formal.*

The existence of a suitable morphism Φ_2 follows from the

Theorem 4.3.4. — *There is a morphism $\Phi_2 : N_*(\mathcal{C}_2, k) \rightarrow \text{Endop}(C(A, A))$ of $\text{Ho}(\text{Op})$ such that the triangle*

$$\begin{array}{ccc} & \text{Endop}(HH^*(A, A)) & \\ \nearrow \varphi & \uparrow H^*(\Phi_2) & \\ \text{Gerst} & \xrightarrow{\alpha} & N_*(\mathcal{C}_2) \end{array}$$

commutes.

The morphism Φ is defined as the composition $\Phi_2 \circ \Phi_1$. It is then clear that Φ lifts φ and it only remains to prove the commutativity of the triangle in theorem 4.3.1. For this, one needs a slightly more precise version of theorem 4.3.4: Let P be the dg suboperad of $\text{Endop} C(A, A)$ generated by the cup-product and by the brace operations: If f_0, \dots, f_p are Hochschild cochains, the brace operation is given by an expression of the form (cf. [76])

$$f_0\{f_1, \dots, f_p\} = \sum \pm f_0 \circ (\text{id}^{\otimes i_0} \otimes f_1 \otimes \text{id}^{\otimes i_1} \otimes \dots \otimes \text{id}^{\otimes i_{p-1}} \otimes f_p \otimes \text{id}^{\otimes i_p})$$

where the sequence i_0, \dots, i_p ranges over all possibilities such that the composition with f_0 makes sense. Note that if f_i is of degree r_i , then the degree r of the resulting cochain satisfies

$$r - 1 = \sum_i (r_i - 1).$$

It follows that the complexes $(\Sigma^{-1}P)(n)$ are all concentrated in degrees ≥ 0 . The Gerstenhaber bracket is expressed in terms of brace operations so that Λ factors as the composition of the inclusion of P with a morphism $\bar{\Lambda}$. Similarly, φ factors as the composition of the map $H^*(P) \rightarrow \text{Endop}(HH^*(A, A))$ with a morphism $\bar{\varphi}$. And even a very superficial inspection of the proofs [76] [13] [65] of theorem 4.3.4 yields the

Porism 4.3.5. — *The morphism Φ_2 of Theorem 4.3.4 factors through the morphism $P \rightarrow \text{Endop}(C(A, A))$.*

It remains to prove that the following diagram is commutative

$$\begin{array}{ccccc}
 \text{Lie} & \xrightarrow{\bar{\Lambda}} & \Sigma^{-1}P & \longrightarrow & \Sigma^{-1}\text{Endop}(C(A, A)) \\
 \downarrow & & \nearrow \bar{\Phi} & & \\
 \Sigma^{-1}\text{Gerst} & & & &
 \end{array}$$

Clearly, it induces a commutative diagram in homology. Since the $\Sigma^{-1}P(n)$ are all concentrated in degrees ≥ 0 , the proof is completed by the following easy

Lemma 4.3.6. — *Let O and O' be dg operads such that $O(n)$ is concentrated in degree 0 and $O'(n)$ in degrees ≥ 0 for all $n \geq 1$. Then the map*

$$\text{Hom}_{\text{Ho}(Op)}(O, O') \rightarrow \text{Hom}_{Op}(H^*O, H^*O')$$

is bijective.

4.4. Notes

Tamarkin's proof [93] of his theorem relies on Kazhdan-Etingof's biquantization theory [30], and thus, ultimately, on the existence of a rational Drinfeld associator [27]. The proof was streamlined in [91] and presented with more details in [49]. We have essentially followed Kontsevich's interpretation [62], where the use of Drinfeld associators becomes more transparent: they appear naturally in the construction of the formality isomorphism Φ_1 in [94].

A comprehensive reference on operads is [73]. The first sections of [44] also offer a nice introduction to the subject.

Theorem 4.3.3 on the formality of the little squares operad was first announced by Getzler-Jones in [42]. However, the proof contained an error. A correct proof was given by Tamarkin in [94] using work by Fiedorowicz [35] and the existence of a rational Drinfeld associator [27]. Later, Kontsevich [62] gave a different proof and also proved that, more generally, the little d -cubes operad is formal for all $d \geq 0$.

Theorem 4.3.4 on the homotopy action of the little squares operad on the Hochschild complex goes back to a question formulated by Deligne in [21]. It was proved in [100] by correcting a method proposed in [42]. It also results by combining [93] with [94]. A geometric proof was given in [65] and purely topologico-combinatorial proofs in [76] and [13]. In [59], a conceptual approach was proposed and a closely related statement proved in a 'non-linear' context. A proof involving an operad related to Connes-Kreimer's renormalization Hopf algebra was given in [57]. A generalization of the theorem to little cubes was announced in [62] and proved in [52].

The brace operations on two arguments were introduced in [39] and on an arbitrary number of arguments in [54] and [41]. Their action on the Hochschild complex was systematized in [40].

PARTIE II

APPLICATION À LA THÉORIE DE LIE (PAR C. TOROSSIAN)

CHAPITRE 5

INTRODUCTION

Dans cette introduction nous allons décrire certains apports de la quantification par déformation à la théorie de Lie. Nous faisons notamment le lien entre la méthode de Kashiwara-Vergne et l'isomorphisme de Duflo.

5.1. La structure de Poisson sur \mathfrak{g}^*

Si \mathfrak{g} est une algèbre de Lie réelle de dimension finie, alors on sait depuis Kostant, Souriau, Kirillov que \mathfrak{g}^* admet une structure de Poisson linéaire. En d'autres termes l'algèbre symétrique $S(\mathfrak{g})$ admet une structure d'algèbre de Poisson. Pour P, Q deux éléments de $S(\mathfrak{g})$ vus comme fonctions polynomiales sur \mathfrak{g}^* et pour $\xi \in \mathfrak{g}^*$ on a :

$$(5.1.1) \quad \{P, Q\}(\xi) = \langle \xi, [dP_\xi, dQ_\xi] \rangle.$$

Les différentielles de P et Q en ξ sont identifiées à des éléments de $\mathfrak{g} = \mathfrak{g}^{**}$, ce qui donne un sens au crochet de Lie. En particulier on a $\{X, Y\} = [X, Y]$ pour X, Y des formes linéaires sur \mathfrak{g}^* .

Question : Les problématiques liées aux structures de Poisson sont-elles pertinentes pour les structures de Lie ? Notamment peut-on obtenir un éclairage nouveau sur \mathfrak{g} grâce au point de vue de Poisson ?

5.2. La formule de Campbell-Hausdorff

Le théorème de Lie nous dit qu'il existe un groupe de Lie G , connexe et simplement connexe, et une application exponentielle notée $\exp_{\mathfrak{g}}$ qui définit un difféomorphisme local de \mathfrak{g} sur G . Il résulte alors que l'on peut lire la loi de groupe de G en coordonnées exponentielles. C'est la fameuse formule de Baker-Campbell-Hausdorff (BCH). En

d'autres termes, pour X, Y proches de 0 dans \mathfrak{g} , il existe une série $Z(X, Y)$ en de polynômes des Lie, convergente et à valeurs dans \mathfrak{g} , telle que l'on ait

$$\exp_{\mathfrak{g}}(X) \cdot_G \exp_{\mathfrak{g}}(Y) = \exp_{\mathfrak{g}}(Z(X, Y)).$$

Les premiers termes de la série BCH sont bien connus

$$(5.2.1) \quad Z(X, Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \\ \frac{1}{48}[Y, [X, [Y, X]]] - \frac{1}{48}[X, [Y, [X, Y]]] + \dots$$

Il existe de nombreuses façons d'écrire la formule BCH. On peut notamment écrire les développements que l'on obtient en calculant la dérivée de l'application exponentielle puis en intégrant à nouveau. Nous rappelons dans cette Introduction, la formule due à Dynkin. Pour simplifier, on note comme ci-dessous les crochets successifs normalisés :

$$[X_1, \dots, X_n]_* = \frac{1}{n}[X_1, [X_2, \dots [X_{n-1}, X_n]] \dots]$$

et

$$[X_1^{r_1}, \dots, X_n^{r_n}]_* = \underbrace{[X_1, \dots, X_1]}_{r_1}, \dots, \underbrace{[X_n, \dots, X_n]}_{r_n}]_*$$

On obtient alors la célèbre formule suivante

$$(5.2.2) \quad Z(X, Y) = X + Y + \sum_{m \geq 2} \frac{(-1)^{m-1}}{m} \sum_{p_i + q_i > 0} \frac{[X^{p_1}, Y^{q_1}, \dots, X^{p_m}, Y^{q_m}]_*}{p_1! q_1! \dots p_m! q_m!}$$

Rappelons comment on obtient cette formule : on se place dans l'algèbre enveloppante de l'algèbre de Lie libre engendrée par X, Y . On calcule formellement

$$e^X e^Y = \sum_{p, q \geq 0} \frac{X^p Y^q}{p! q!}$$

puis en utilisant le développement de $\log z = \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} (z-1)^m$ on trouve formellement alors

$$(5.2.3) \quad Z(X, Y) = \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} \sum_{p_i + q_i \geq 1} \frac{X^{p_1} Y^{q_1} \dots X^{p_m} Y^{q_m}}{p_1! q_1! \dots p_m! q_m!}.$$

On utilise alors une caractérisation des éléments de type Lie dans une algèbre enveloppante : un élément $a = \sum X_1 \dots X_n$ d'ordre n est de type Lie, si et seulement si $a = \sum [X_1, \dots, X_n]_*$. Par exemple $a = XY - YX$ est bien de type Lie et on a $a = [X, Y]_* - [Y, X]_* = \frac{1}{2}[X, Y] - \frac{1}{2}[Y, X] = [X, Y]$. On en déduit la formule annoncée.

A partir de cette formule on peut mener un calcul explicite (voir [84], page 103) qui permet de retrouver les nombres de Bernoulli b_n . Rappelons que la série de Bernoulli

est donnée par

$$(5.2.4) \quad \sum_{n \geq 0} \frac{b_n x^n}{n!} = \frac{x}{\exp(x) - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} \cdots$$

En calculant la série $Z(X, Y)$ à l'ordre 1 en Y , on déduit la formule bien connue suivante :

$$(5.2.5) \quad \begin{aligned} Z(X, Y) &\equiv X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \cdots \pmod{Y^2} \\ &\equiv X + \frac{\operatorname{ad} X}{1 - \exp(-\operatorname{ad} X)} \cdot Y \pmod{Y^2}. \end{aligned}$$

De la formule

$$(5.2.6) \quad \exp_{\mathfrak{g}}(X) \exp_{\mathfrak{g}}\left(\frac{1 - \exp(-\operatorname{ad} X)}{\operatorname{ad} X} \cdot Y\right) \equiv \exp_{\mathfrak{g}}(X + Y \pmod{Y^2}),$$

on conclut que la différentielle de l'application exponentielle s'identifie à l'endomorphisme $\frac{1 - \exp(-\operatorname{ad} X)}{\operatorname{ad} X}$.

Une difficulté avec la formule BCH c'est qu'il n'existe pas de base de l'algèbre de Lie libre qui soit particulièrement commode pour les calculs⁽¹⁾. On verra que la quantification de Kontsevich donne une autre façon d'écrire la formule BCH. On ne sait pas aujourd'hui tous les coefficients de cette écriture est à coefficients rationnels.

Dans cette partie on notera par $J(X)$ le déterminant jacobien de la fonction $\exp_{\mathfrak{g}}$, à savoir la fonction définie par

$$(5.2.7) \quad J_{\mathfrak{g}}(X) = \det_{\mathfrak{g}} \left(\frac{1 - \exp(-\operatorname{ad} X)}{\operatorname{ad} X} \right) = \exp(-\operatorname{tr}_{\mathfrak{g}} \frac{\operatorname{ad} X}{2}) \det_{\mathfrak{g}} \left(\frac{\sinh(\operatorname{ad} \frac{X}{2})}{\frac{\operatorname{ad} X}{2}} \right).$$

Cette fonction va intervenir de manière cruciale dans la suite.

5.3. Le centre de l'algèbre enveloppante

La symétrisation notée ici β réalise un isomorphisme d'espaces vectoriels entre $S(\mathfrak{g})$ l'algèbre symétrique de \mathfrak{g} et $U(\mathfrak{g})$ l'algèbre enveloppante de \mathfrak{g} qui commute à l'action adjointe (resp. aux dérivations $\operatorname{ad} X$) : c'est une version du théorème de Poincaré-Birkhoff-Witt. On en déduit que β réalise un isomorphisme d'espaces vectoriels de $S(\mathfrak{g})^{\mathfrak{g}}$ sur $U(\mathfrak{g})^{\mathfrak{g}}$ les invariants sous l'action adjointe de $S(\mathfrak{g})$ et $U(\mathfrak{g})$ respectivement. L'algèbre $S(\mathfrak{g})$ est une algèbre de Poisson et $S(\mathfrak{g})^{\mathfrak{g}}$ en est le centre, de même $U(\mathfrak{g})^{\mathfrak{g}}$ est le centre de $U(\mathfrak{g})$.

En fait il existe un isomorphisme d'algèbre de $S(\mathfrak{g})^{\mathfrak{g}}$ sur $U(\mathfrak{g})^{\mathfrak{g}}$: c'est l'isomorphisme de Duflo.

⁽¹⁾On sait par exemple qu'il existe une base qui permet d'écrire BCH en utilisant des combinaisons à coefficients complexes [58].

Soit π est une représentation de \mathfrak{g} , irréductible, alors cette représentation s'étend en un représentation de $U(\mathfrak{g})$ l'algèbre enveloppante de \mathfrak{g} .

Le lemme de Schur nous assure que la représentation est scalaire sur le centre de $U(\mathfrak{g})$ c'est à dire $U(\mathfrak{g})^{\mathfrak{g}}$. On en déduit alors que chaque représentation irréductible de $U(\mathfrak{g})$ fournit un caractère du centre. On espère ainsi séparer les représentations : c'est le début de la méthodes des orbites. Connaissant un caractère du centre de l'algèbre enveloppante, peut-on construire une représentation associée ?

Une autre conséquence de la méthode des orbites est la construction de l'isomorphisme de Duflo [29]. Rappelons ce qu'est la formule de Duflo. La série formelle $J^{\frac{1}{2}}$ est dans $S[[\mathfrak{g}^*]]$. C'est donc un opérateur différentiel d'ordre infini sur $S[\mathfrak{g}]$ que l'on note $J^{\frac{1}{2}}(\partial)$. La formule de Duflo s'écrit alors pour $P \in S(\mathfrak{g})$:

(5.3.1)

$$\gamma(P) = \beta(J^{\frac{1}{2}}(\partial)P).$$

C'est clairement un isomorphisme d'espace vectoriel de $S(\mathfrak{g})$ sur $U(\mathfrak{g})$, qui commute aux dérivations $\text{ad}X$. Le théorème de Duflo [29] affirme alors que γ est un isomorphisme d'algèbres de $S[\mathfrak{g}]^{\mathfrak{g}}$ sur $U(\mathfrak{g})^{\mathfrak{g}}$. Ce théorème est remarquable et non trivial.

Question : L'isomorphisme de Duflo peut-il se lire sur la formule BCH? Une réponse positive à cette question se trouve dans l'article de Kashiwara-Vergne [55]. Les méthodes de cet article sont connues aujourd'hui sous le vocable "méthode de Kashiwara-Vergne".

5.4. La déformation de Kashiwara-Vergne

On espère qu'il existe des déformations remarquables de la formule de BCH, qui permettent de comprendre le transport des distributions invariantes par l'application \exp_G . Introduisons une fonction dite de densité

(5.4.1)

$$D(X, Y) = \frac{J^{1/2}(X)J^{1/2}(Y)}{J^{1/2}(Z(X, Y))}.$$

Grâce à (5.2.7), on peut remplacer dans cette formule la fonction $J^{1/2}$ par la fonction paire suivante :

$$(5.4.2) \quad q(X) = \left(\frac{\sinh(\text{ad} \frac{X}{2})}{\frac{\text{ad} X}{2}} \right)^{\frac{1}{2}}.$$

Dans l'article [55], Kashiwara et Vergne conjecturent (et démontrent dans le cas résoluble) le résultat suivant démontré dans [95] suivant les idées de [4], [3], [60].

Une approche utilisant la géométrie symplectique donne un résultat analogue dans le cas quadratique [99], [1]. On montrera grâce à la quantification de Kontsevich au chapitre 8.6 le théorème :

Théorème 5.4.1. — *Il existe des séries de Lie en deux variables $F_t(X, Y)$ et $G_t(X, Y)$ à valeurs dans \mathfrak{g} , convergentes dans un voisinage de $(0, 0)$, sans termes constants et il existe une déformation de la fonction de densité notée $D_t(X, Y)$ et une déformation de la formule de Campbell-Hausdorff notée $Z_t(X, Y)$ telles que*

$$(5.4.3) \quad \begin{array}{ll} D_0(X, Y) = D(X, Y) & \text{et} \quad D_\infty(X, Y) = 1, \\ Z_0(X, Y) = Z(X, Y) & \text{et} \quad Z_\infty(X, Y) = X + Y. \end{array}$$

De plus les équations suivantes sont vérifiées :

$$(5.4.4) \quad \frac{\partial}{\partial t} Z_t(X, Y) = \left([X, F_t(X, Y)] \cdot \partial_X + [Y, G_t(X, Y)] \cdot \partial_Y \right) Z_t(X, Y)$$

$$\begin{aligned} \frac{\partial}{\partial t} D_t(X, Y) &= \left([X, F_t(X, Y)] \cdot \partial_X + [Y, G_t(X, Y)] \cdot \partial_Y \right) D_t(X, Y) \\ &\quad + \text{tr}_{\mathfrak{g}} \left(\text{ad}(X) D_X F_t + \text{ad}(Y) D_Y G_t \right) D_t(X, Y). \end{aligned}$$

Si $X = \sum_{i=1}^d x_i e_i$ avec $(e_i)_{i=1, \dots, d}$ une base de \mathfrak{g} et $(e_i^*)_{i=1, \dots, d}$ la base duale, on a noté

$$[X, F_t(X, Y)] \cdot \partial_X = \sum_{i=1}^d \langle e_i^*, [X, F_t(X, Y)] \rangle \frac{\partial}{\partial x_i}$$

le champ de vecteurs adjoints sur \mathfrak{g} .

Expliquons dans cette introduction pourquoi la déformation de la formule BCH du théorème ci-dessus redonne l'isomorphisme de Duflo.

Prenons u et v ⁽²⁾ sont deux distributions invariantes au voisinage de 0 dans \mathfrak{g} vérifiant une certaine condition de support afin d'assurer un sens à la convolution (par exemple si l'une des deux distributions est à support 0). Pour f une fonction C^∞ dans un voisinage de 0, on cherche à montrer que l'on a :

$$(5.4.5) \quad \langle u(x) \otimes v(y), D(x, y) f(Z(x, y)) \rangle = \langle u(x) \otimes v(y), f(x + y) \rangle.$$

Cette égalité exprime que la convolution sur le groupe écrite en coordonnées exponentielles (membre de gauche) vaut la convolution sur l'espace tangent (membre de droite). En remplaçant $D(x, y)$ par sa valeur on va retrouver l'isomorphisme de Duflo comme on le fera en détail dans la section 8.7.

⁽²⁾Dans cette partie on adopte la convention suivante : on met les variables en petite caroline quand on fait de l'analyse et en majuscule quand on fait de l'algèbre.

La stratégie initiée dans [55] est de démontrer que cette égalité est encore vraie pour la déformation en le paramètre t , c'est à dire que l'on a pour tout t :

$$(5.4.6) \quad \langle u(x) \otimes v(y), D_t(x, y)f(Z_t(x, y)) \rangle = \langle u(x) \otimes v(y), f(x + y) \rangle$$

L'idée est maintenant simple, il suffit de vérifier que le membre de gauche de (5.4.6) ne dépend pas de t , c'est à dire que la dérivée par rapport à t est nulle. Or d'après le théorème 5.4.1 la dérivée de la fonction $D_t(x, y)f(Z_t(x, y))$ s'écrit

$$(5.4.7) \quad \begin{aligned} \frac{\partial}{\partial t} D_t(x, y)f(Z_t(x, y)) &= \left([x, F_t(x, y)] \cdot \partial_x + [y, G_t(x, y)] \cdot \partial_y \right) (D_t(x, y)f(Z_t(x, y))) \\ &\quad + \text{tr}_{\mathfrak{g}} \left(\text{adx} \circ D_x F_t + \text{ady} \circ D_y G_t \right) D_t(x, y)f(Z_t(x, y)). \end{aligned}$$

On utilise ensuite définition d'une distribution invariante, c'est à dire que l'on a

$$(5.4.8) \quad \begin{aligned} u(x)v(y) \left([x, F_t(x, y)] \cdot \partial_x + [y, G_t(x, y)] \cdot \partial_y \right) = \\ - u(x)v(y) \text{tr}_{\mathfrak{g}} \left(\text{adx} \circ D_x F_t + \text{ady} \circ D_y G_t \right) \end{aligned}$$

En combinant les équations (5.4.8) et (5.4.7) dans la dérivée de l'équation (5.4.6), on en déduit que l'expression (5.4.6) est indépendante de t . En tenant compte des conditions aux limites on retrouve la formule (5.4.5).

CHAPITRE 6

LA FORMULE DE KONTSEVICH POUR \mathbb{R}^n

Dans sa partie, Bernhard Keller a développé les notions de déformation d'algèbre et a introduit la formule de quantification de Kontsevich. Dans son chapitre Alain (chapitre A) Bruguières a décrit la stratification des variétés de configurations et expliqué un résultat important de Kontsevich sur l'annulation des intégrales d'une n -formes d'angles pour $n \geq 3$.

On reprend dans cette introduction quelques éléments utiles et on précise certaines définitions qui nous seront nécessaires.

La formule de Kontsevich associe à toute structure de Poisson régulière sur \mathbb{R}^d un étoile-produit formel associatif. Ce n'est qu'un cas particulier du théorème de formalité démontré par Kontsevich dans [60] §6.4 et que l'on abordera en section 9

Lorsque f et g sont deux fonctions régulières sur \mathbb{R}^d et α un 2-vecteur de Poisson régulier sur \mathbb{R}^d , Kontsevich montre dans [60] §2 la proposition spectaculaire suivante :

Théorème 6.0.2. — *La formule suivante définit un étoile-produit associatif*

$$(6.0.9) \quad f \star g = fg + \sum_{n=1}^{\infty} \frac{h^n}{n!} \sum_{\substack{\Gamma \in G_{n,2} \\ \Gamma \text{ admissible}}} w_{\Gamma} B_{\Gamma, \alpha}(f, g).$$

Cette formule munit l'espace des fonctions régulières sur \mathbb{R}^d d'une structure associative formelle. Dans cette formule h est un paramètre formel, $G_{n,2}$ désigne l'ensemble des graphes étiquetés avec n points de première espèce et 2 points de seconde espèce, Γ est un graphe dit admissible pour les 2-vecteurs parmi les graphes de $G_{n,2}$, w_{Γ} est un coefficient obtenu par intégration sur un espace de configurations d'une forme différentielle dépendant de Γ et $B_{\Gamma, \alpha}$ est un opérateur bidifférentiel construit à partir de Γ et de α .

Nous allons préciser maintenant chacun des termes de cette formule et appliquer dans un second temps cette formule dans le cas des structures de Poisson linéaires c'est-à-dire dans le cas du dual des algèbres de Lie.

6.1. Espaces de configurations

On note par $\text{Conf}_{n,m}$ l'espace des configurations de n points distincts dans le demi-plan de Poincaré (ce sont les points aériens) et de m points distincts sur la droite réelle (ce sont les points terrestres). Le groupe :

$$G^{(1)} = \{z \mapsto az + b \text{ avec } (a, b) \in \mathbb{R}_+^* \times \mathbb{R}\}$$

agit librement sur $\text{Conf}_{n,m}$. Le quotient :

$$C_{n,m} = \text{Conf}_{n,m}/G^{(1)}$$

est une variété de dimension $2n + m - 2$. Dans [60] §5.1, Kontsevich construit des compactifications de ces variétés notées $\overline{C}_{n,m}$.⁽¹⁾ Ce sont des variétés à coins de dimension $2n + m - 2$. Ces variétés ne sont pas connexes pour $m \geq 2$. La composante connexe qui contient les configurations où les points terrestres sont ordonnés dans l'ordre croissant (i.e. on a $\overline{1} < \overline{2} < \dots < \overline{m}$) sera notée $\overline{C}_{n,m}^+$. On introduit de manière analogue Conf_n les variétés de configurations de n points dans le plan complexe. Le groupe

$$G^{(2)} = \{z \mapsto az + b \text{ avec } (a, b) \in \mathbb{R}_+^* \times \mathbb{C}\},$$

agit librement sur Conf_n . Le quotient

$$C_n = \text{Conf}_n/G^{(2)}$$

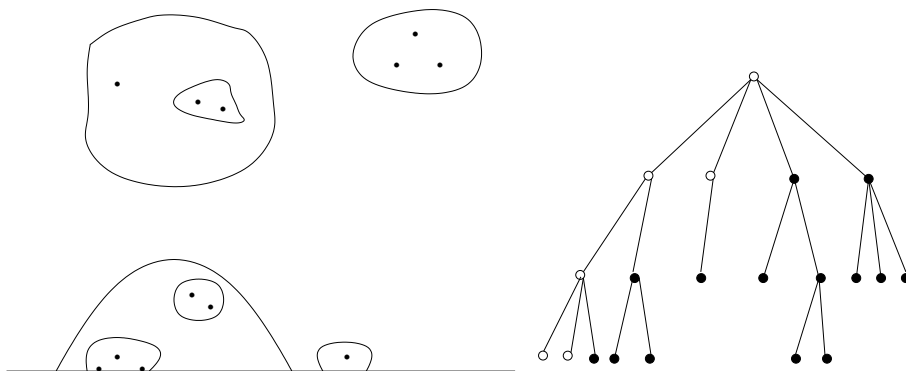
est une variété de dimension $2n - 3$. On note \overline{C}_n les compactifications associées ([60] §5.1). L'orientation du demi-plan de Poincaré et de \mathbb{R} définissent une orientation naturelle de $\overline{C}_{n,m}^+$

Ces variétés sont stratifiées et chaque strate est décrite par un arbre⁽²⁾. En termes géométriques, les strates sont obtenues par concentrations itérées de points en des amas ([60] §5.2). Les strates sont donc des produits de variétés de configurations.

Les strates de codimension 1, sont en particulier obtenues par concentration d'un seul amas de points. On note C_{int} le paquet que l'on concentre et C_{ext} la variété où on a remplacé C_{int} par un point. Ce paquet concentré "vu de loin" est soit concentré sur un point aérien (on dira qu'il y a eu concentration en un point aérien) soit sur un point terrestre (on dira qu'il y a eu concentration sur l'axe réel). La strate de codimension 1 a donc pour structure le produit de la variété $C_{int} \times C_{ext}$. Nous allons voir dans le paragraphe suivant l'orientation qu'il faut attribuer à cette strate.

⁽¹⁾Voir le sections §A.2, §A.3 rédigées par A. Bruguières à la fin de cette partie

⁽²⁾id est

FIGURE 1. Un exemple de strate dans la configuration $\overline{C}_{10,2}^+$

6.2. Orientation des strates de codimension 1

Elles sont délicates à traiter, mais ceci a été expliqué dans [6]. Chaque strate de codimension 1 correspond à un concentration. Les règles des orientations sont les suivantes en fonction du type de concentration :

1. Soit on contracte un amas de points aériens noté C_{int} en un point p (qui reste aérien) et que l'on met en première position dans la configuration C_{ext} .
2. Soit on contracte un amas, noté C_{int} , en un point terrestre, comportant des points aériens et m point terrestres. On suppose que cet amas se concentre sur la position terrestre $l + 1$ de C_{ext} .

Définition 6.2.1. — L'orientation des strates est selon les types précédents donnée par les formes suivantes selon [6] :

1. $-\Omega_{int} \wedge \Omega_{ext}$,
2. $(-1)^{l+m+lm} \Omega_{int} \wedge \Omega_{ext}$,

où $\Omega_{ext}, \Omega_{int}$ sont les formes d'orientation standards sur les variétés de configurations. Plus précisément sur les variétés $Conf_{n,m}^+$ on considère la forme d'orientation

$$dx_1 \wedge dy_1 \dots dx_n \wedge dy_n \wedge dq_{\overline{1}} \wedge dq_{\overline{2}} \dots \wedge dq_{\overline{m}}.$$

L'orientation ne change pas sous l'action du groupe $G^{(1)}$, ce qui définit une orientation sur $C_{n,m}^+$.

6.3. Graphes admissibles

Un graphe est dit étiqueté⁽³⁾ et orienté si les sommets sont numérotés et les arêtes orientées. On va partitionner l'ensemble des sommets en deux parties (sommets de

⁽³⁾Voir la partie de B. Keller section §2.2

première espèce et sommets de deuxième espèce). On désignera par $G_{n,m}$ l'ensemble des graphes ordonnés, étiquetés et orientés ayant n sommets du première espèce (numérotés $1, 2, \dots, n$) et m sommets du deuxième espèce (numérotés $\bar{1}, \bar{2}, \dots, \bar{m}$). Par graphe ordonné on entend un graphe Γ muni d'un ordre total sur l'ensemble E_Γ de ses arêtes, compatible avec l'ordre des sommets ($\bar{1} < \dots < \bar{m} < 1 < \dots < n$) : les arêtes issues du sommet 1 sont ordonnées avant celles issues du sommet 2 et ainsi de suite. La numérotation des sommets permet d'ordonner les arêtes selon l'ordre lexicographique, on parlera alors d'ordre privilégié.

Les graphes qui vont intervenir dans la formule de Kontsevich sont dans $G_{n,2}$ et vérifient des conditions supplémentaires.

Définition 6.3.1. — On dira qu'un graphe de $G_{n,m}$ est admissible ([60] §6.1 et §2) si

1. Les arêtes partent toutes des sommets de première espèce.
2. Le but d'une arête est différent de sa source (il n'y a pas de boucle).
3. Il n'y a pas d'arêtes multiples (même source, même but).

Définition 6.3.2. — On dira qu'un graphe admissible est admissible pour les 2-vecteurs s'il part deux arêtes de chaque sommet de première espèce.

Cette dernière définition n'intervient que pour la définition de l'opérateur $B_{\Gamma,\alpha}$. On aura besoin en particulier de la définition suivante :

Définition 6.3.3. — On dira qu'un graphe de $G_{n,m}$ est linéaire s'il vérifie les conditions 1, 2, 3 de la définition 6.3.1 et si les sommets de première espèce ne reçoivent qu'au plus une arête.

Cette catégorie de graphes possède des structures riches (voir par exemple les travaux de Connes-Kreimer [19] ou la thèse de Foissy [37]).

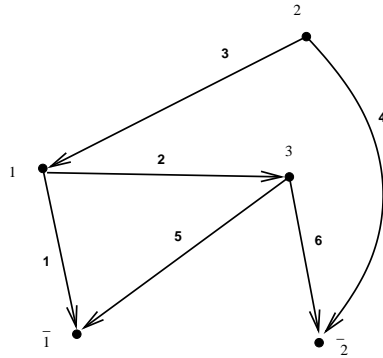


FIGURE 2. Graphe admissible (de type Lie)

6.4. Opérateur différentiel associé à un graphe

Soit α un 2-vecteurs régulier de \mathbb{R}^d . A tout graphe Γ admissible pour les 2-vecteurs de $G_{n,m}$ on peut associer un opérateur bi-différentiel $B_{\Gamma,\alpha}$ ⁽⁴⁾ agissant sur les fonctions $f_{\overline{1}}, \dots, f_{\overline{m}}$.

La formule est la suivante : de tout point aérien i partent deux arêtes numérotées (e_i^1, e_i^2) , on note I l'ensemble des applications de l'ensemble des arêtes E_Γ dans l'ensemble des indices de coordonnées $\{1, \dots, d\}$ et ∂_i désigne la dérivée partielle par rapport à x_i .

$$(6.4.1) \quad B_{\Gamma,\alpha}(f_{\overline{1}}, f_{\overline{2}}) = \sum_I \left[\prod_{k=1}^n \left(\prod_{\substack{e \in E_\Gamma \\ e=(\bullet, k)}} \partial_{I(e)} \right) \alpha^{I(e_k^1)I(e_k^2)} \right] \prod_{1 \leq j \leq m} \left(\prod_{\substack{e \in E_\Gamma \\ b(e)=(\bullet, j)}} \partial_{I(e)} \right) f_{\overline{j}}$$

Exemple : L'opérateur différentiel associé au graphe de la figure 2 pour l'ordre indiqué est

$$\sum_{\substack{i_1, i_2, i_3=1 \dots d \\ j_1, j_2, j_3=1 \dots d}} \alpha^{i_2, j_2} \frac{\partial \alpha^{i_1, j_1}}{\partial x_{i_2}} \frac{\partial \alpha^{j_3, j_3}}{\partial x_{j_1}} \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_3}} \frac{\partial^2 g}{\partial x_{j_3} \partial x_{j_2}}.$$

6.5. Forme d'angles

Soient deux points distincts (p, q) dans le demi-plan de Poincaré muni de la métrique de Lobachevsky. On définit la fonction d'angle

$$(6.5.1) \quad \phi(p, q) = \frac{1}{2i} \log \left(\frac{(q-p)(\overline{q}-p)}{(q-\overline{p})(\overline{q}-\overline{p})} \right) = \text{Arg} \left(\frac{q-p}{q-\overline{p}} \right).$$

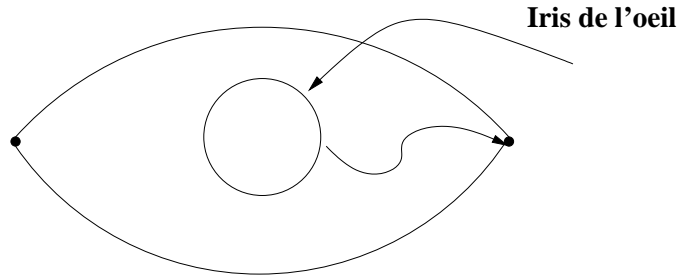
C'est l'angle entre la géodésique (p, ∞) et (p, q) où l'infini peut être vu comme l'infini sur la droite réelle.

La fonction d'angle s'étend à la compactification $\overline{\mathcal{C}}_{2,0}$ en une fonction d'angle régulière. La variété $\overline{\mathcal{C}}_{2,0}$ est précisément décrite dans l'article de Kontsevich ([60] §5.2), c'est le fameux œil (voir figure 3). On remarquera, mais c'est tautologique vu la construction des compactifications, que lorsque les points p, q s'approchent selon un angle θ , la fonction d'angle vaut précisément cet angle.⁽⁵⁾ Lorsque p s'approche de l'axe réel la fonction d'angle est nulle et lorsque q s'approche de l'axe réel on obtient deux fois l'angle de demi-droite avec l'axe réel.

Comme la fonction d'angle est régulière sur la compactification, on peut considérer sa différentielle qui est alors une 1-forme sur $\overline{\mathcal{C}}_{2,0}$.

⁽⁴⁾Voir la partie de B. Keller §2.2

⁽⁵⁾Pour plus de détails voir les sections A.2 et A.3

FIGURE 3. La variété $\overline{\mathcal{C}}_{2,0}$, dite œil de Kontsevich

6.6. Poids associé à un graphe

Soit Γ un graphe sans boucle de $G_{n,m}$ avec p arêtes. On ne suppose pas ce graphe admissible au sens de la définition 6.3.1. On le dessine dans le demi-plan de Poincaré en mettant les n points de première espèce en position aérienne, q points de deuxième espèce en position aérienne et $m - q$ points de deuxième espèce en position terrestre. On dira que le graphe est dessiné dans $\overline{\mathcal{C}}_{n+q,m-q}^+$. Toute arête e définit par restriction une fonction d'angle notée ϕ_e sur la variété $\overline{\mathcal{C}}_{n+q,m-q}^+$.

Définition 6.6.1. — Soit Γ un graphe sans boucle de $G_{n,m}$, dessiné dans $\overline{\mathcal{C}}_{n+q,m-q}^+$, alors le produit ordonné

$$(6.6.1) \quad \Omega_\Gamma = \bigwedge_{e \in E_\Gamma} \frac{d\phi_e}{2\pi}$$

est une p -forme sur $\overline{\mathcal{C}}_{n+q,m-q}^+$.

On dessine maintenant Γ graphe de $G_{n,m}$ dans $\overline{\mathcal{C}}_{n,m}^+$ en positionnant les points de seconde espèce en position terrestre, alors Ω_Γ est une p -forme sur $\overline{\mathcal{C}}_{n,m}^+$. La formule suivante définit le coefficient de Kontsevich.

Définition 6.6.2. — Soit Γ un graphe de $G_{n,m}$ sans boucle et possédant p arêtes. On définit le coefficient de Kontsevich par

$$(6.6.2) \quad w_\Gamma = \int_{\overline{\mathcal{C}}_{n,m}^+} \Omega_\Gamma.$$

Ce coefficient est nul si $p \neq 2n + m - 2$ la dimension de la variété de configurations $\overline{\mathcal{C}}_{n,m}^+$.

Remarque 6.6.3. — Si Γ est un graphe de $G_{n,2}$ admissible pour les 2-vecteurs alors le nombre d'arêtes correspond bien à la dimension de $\overline{C}_{n,2}^+$.

6.7. Comportement multiplicatif du coefficient

La fonction d'angle et par suite les coefficients de Kontsevich se comportent bien vis à vis des compactifications. Plus précisément notons Γ un graphe admissible de $G_{n,m}$ dessiné dans un espace de configurations et soit Ω_Γ la forme associée que l'on veut intégrer sur un bord de codimension 1. Ce bord est un produit de configurations $C_{int} \times C_{ext}$ muni d'une orientation comme dans la définition 6.2.1.

Notons Γ_{int} le graphe dit intérieur dont les sommets sont ceux de C_{int} (points intérieurs) et les arêtes sont les arêtes de Γ joignant deux sommets intérieurs muni de l'ordre induit sur les arêtes. Le graphe Γ_{ext} est obtenu à partir de Γ par concentration de Γ_{int} en un sommet supplémentaire (aérien en première position ou terrestre en position $\ell + 1$ selon les cas) et muni de l'ordre induit des arêtes rendu compatible avec l'ordre des sommets quitte à effectuer un battage (les arêtes issues du point supplémentaire sont ordonnées entre-elles selon l'ordre induit, mais ordonnées avant celles issues des autres sommets de Γ_{ext}).

Pour que l'intégrale $\int_{C_{int} \times C_{ext}} \Omega_\Gamma$ soit non nulle, il faut que le graphe Γ_{ext} soit admissible et que la dimension de C_{int} (resp. C_{ext}) corresponde au nombre d'arêtes dans Ω_{int} (resp. Ω_{ext}).

Dans ce cas on a

$$\int_{C_{int} \times C_{ext}} \Omega_\Gamma = \pm w_{\Gamma_{int}} w_{\Gamma_{ext}},$$

où le signe dépend de l'orientation de la strate décrite dans la définition 6.2.1 et du signe de la permutation (battage) qu'il faut pour ramener les arêtes de Γ correspondant à Γ_{int} en première position et les arêtes de Γ_{ext} issues du point supplémentaire en première position dans Ω_{ext} .

6.8. Permutation des arêtes

Soit Γ un graphe admissible pour les 2-vecteurs dans $G_{n,2}$ et α un 2-vecteur régulier sur \mathbb{R}^d . Le groupe $\underbrace{S_2 \times \cdots \times S_2}_n$, produit des groupes de permutations des arêtes attachés à chaque sommet, agit naturellement sur Γ par permutation de l'étiquetage des arêtes. On vérifie que l'on a ($\epsilon(\sigma)$ désigne la signature de la permutation σ) :

$$(6.8.1) \quad B_{\sigma, \Gamma, \alpha} = \epsilon(\sigma) B_{\Gamma, \alpha} \quad \text{et} \quad w_{\sigma, \Gamma} = \epsilon(\sigma) w_\Gamma,$$

de sorte que le produit $w_\Gamma \cdot B_{\Gamma, \alpha}$ ne dépend pas de l'étiquetage. On remarque aussi que le coefficient w_Γ et l'opérateur B_Γ ne dépendent pas de l'ordre des sommets.

6.9. Formule de Stokes

Les formes Ω_Γ sont clairement fermées. On pourra donc appliquer la formule de Stokes. En intégrant la différentielle $d\Omega_\Gamma$ (qui vaut 0) sur une variété de configurations adéquate, on obtient par la formule de Stokes une équation quadratique concernant des coefficients associés à des sous-graphes obtenus par concentration de points correspondant aux strates de codimension 1 associées.

Pour appliquer la formule de Stokes il faut d'abord numéroter les arêtes, de manière compatible avec l'ordre sur les sommets comme sont supposés l'être les graphes de $G_{n,m}$. Il faut penser à ramener toutes les arêtes que l'on contracte en première position, pour pouvoir ensuite intégrer selon la règle de Fubini. L'orientation de la strate décrite au paragraphe 6.2.1 ajoute un signe.

L'associativité du star-produit et plus généralement la propriétés du L_∞ -quasi-isomorphisme (chapitre 9) résulteront de ce principe .

6.10. Déformation deux dimensionnelle

Soit Γ un graphe admissible pour les 2-vecteurs. On va déformer le coefficient w_Γ en plaçant les deux points de seconde en position générale. C'est cette déformation qui servira à déformer la formule BCH. Notons $p : \overline{C_{n+2,0}} \mapsto \overline{C_{2,0}}$ l'application naturelle définie par

$$p(z_{\overline{1}}, z_{\overline{2}}, z_1, \dots, z_n) = (z_{\overline{1}}, z_{\overline{2}}) = \xi.$$

Définition 6.10.1. — Soit Γ un graphe de $G_{n,2}$ admissible pour les 2-vecteurs que l'on dessine dans $\overline{C_{n+2,0}}$ en plaçant les points de seconde espèce en position aérienne fixe notée $\xi \in \overline{C_{2,0}}$. On note

(6.10.1)

$$w_\Gamma(\xi) = \int_{p^{-1}(\xi)} \Omega_\Gamma$$

l'intégrale de la $2n$ -forme Ω_Γ (6.6.1) sur la sous-variété de dimension $2n$ des positions de première espèce munie de l'orientation standard.

On va expliquer le calcul de la différentielle dw_Γ en utilisant la formule de Stokes.

On choisit un chemin ξ_t dans $\overline{C_{2,0}}$ de ξ_0 à ξ_1 . On écrit la formule de Stokes sur $Y = \bigcup_{0 \leq t \leq 1} p^{-1}(\xi_t)$ la sous-variété de dimension $2n + 1$ image inverse du chemin. Il vient

$$0 = \int_Y d\Omega_\Gamma = \int_{\partial Y} \Omega_\Gamma = w_\Gamma(\xi_1) - w_\Gamma(\xi_0) + \int_Z \Omega_\Gamma,$$

où on a noté $Z = \bigcup_{0 \leq t \leq 1} \partial(p^{-1}(\xi_t))$ le réunion des bords des fibres le long du chemin. Lorsqu'on prend un chemin infinitésimal, on trouve que la différentielle $d\Omega_\Gamma$ en ξ se

calculé comme l'opposée de l'intégration de Ω_Γ sur le bord de $p^{-1}(\xi)$. On obtient donc le lemme suivant :

Lemme 6.10.2. — *En tant que 1-forme sur $\overline{C_{2,0}}$ on a*

$$dw_\Gamma(\xi) = - \int_{\partial(p^{-1}(\xi))} \Omega_\Gamma.$$

Seules importent dans le calcul de la différentielle les strates de codimension 1, c'est à dire il ne faut faire qu'un seul amas d'arêtes.

Lemme 6.10.3. — *Lorsque Γ est un graphe admissible pour les 2-vecteurs de $G_{n,2}$ le calcul de $dw_\Gamma(\xi)$ ne fait intervenir que les strates de codimension 1 où l'on a concentré uniquement deux points dont un est de première espèce.*

Démonstration. — Examinons les différentes strates qui vont intervenir sachant dans la forme Ω_Γ on doit conserver une différentielle contribuant dans le calcul du $d\xi$. Remarquons alors que cette contribution ne peut venir que du graphe externe car c'est le seul à contenir les deux points $(z_{\overline{1}}, z_{\overline{2}})$. La contribution du graphe interne sera alors scalaire.

Cas 1 : Si l'amas C_{int} de concentration se fait au bord terrestre alors cet amas est une strate $C_{p,0}$ de dimension $2p - 2$. Par ailleurs le graphe externe ne doit pas avoir d'arêtes sortantes de l'axe réel sinon la forme externe Ω_{ext} serait nulle. Or l'intégrale $\int_{C_{int}} \Omega_{int}$ est nulle pour des raisons de dimension, car Ω_{int} est une $2p$ forme à intégrer sur une variété de dimension $2p - 2$.

Remarquons que si le graphe Γ n'était pas admissible pour les 2-vecteurs, il se pourrait que les concentrations d'amas en un point terrestre interviennent.

Cas 2 : Supposons maintenant que l'on ait effectué une concentration aérienne entre p points aériens. On trouve donc une strate C_p . Il faut donc $2p - 3$ arêtes dans le graphe interne pour pouvoir intégrer cette forme. Dans le cas de graphes linéaires (définition 6.3.3), le nombre d'arêtes dans l'amas est majoré par p . On a donc $2p - 3 \leq p$ et il vient donc $p \leq 3$. Il est facile de voir que si $p = 3$ alors le coefficient est nul (voir [72] lemme 3.1). Dans le cas non linéaire il faut invoquer le lemme de Kontsevich [60] §6.6 qui assure que l'intégration de plus de 3 formes d'angles sur une variété C_p est nulle⁽⁶⁾. En conséquence la concentration se fait entre deux points avec une seule arête joignant les points. Comme l'amas n'a que deux points, un des points est forcément de première espèce car les points de seconde espèce sont en ξ et ne peuvent donc pas se rapprocher. \checkmark

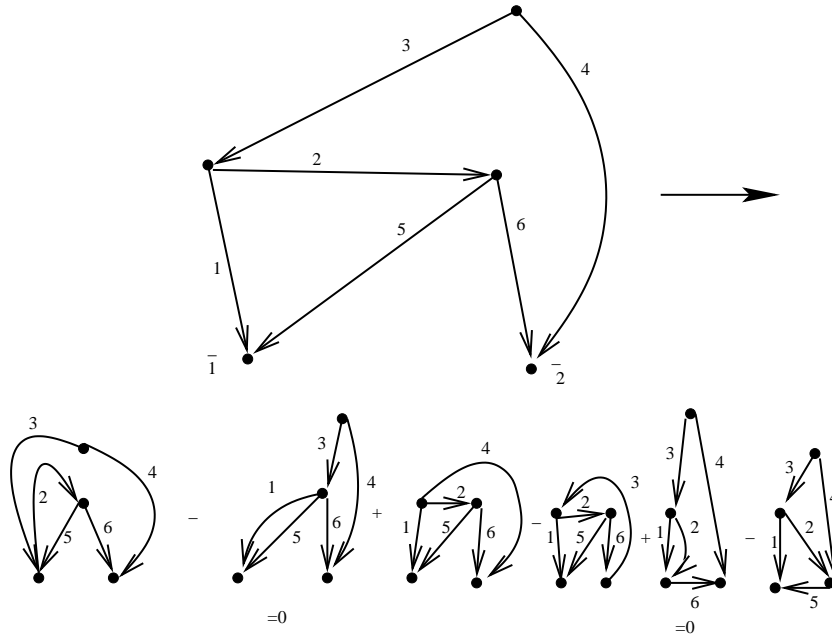


FIGURE 4. Calcul de la différentielle

Comme le graphe est admissible pour les 2-vecteurs, on peut sans perte de généralité ou d'ajout de signe supposer que l'on a $e = (1, 2)$ ou $e = (1, \bar{1})$ ou $e = (1, \bar{2})$. Notons $\Gamma(e)$ le graphe extérieur où on a contracté l'arête e et fait disparaître la position 1. Ce graphe a $2n - 1$ arêtes et est dans $G_{n-1,2}$ car il est muni de l'ordre induit sur les arêtes compatible avec la numérotation. Selon les cas, $\Gamma(e)$ a un point de première espèce avec 3 arêtes sortantes ou un point de seconde espèce avec 1 arête sortante (de $\bar{1}$ ou $\bar{2}$). Ce graphe n'est pas admissible au sens de la définition 6.3, mais il n'a pas de boucles ce qui permet, en dessinant ce graphe contracté $\Gamma(e)$ dans $\overline{C_{n+1,0}}$, de définir une $2n - 1$ forme $\Omega_{\Gamma(e)}$.

Le graphe intérieur (l'amas que l'on a contracté) se réduit à une arête. La valeur du coefficient associé est donc $\int_{C_2} \frac{d\Phi_e}{2\pi} = 1$.

Maintenant lorsqu'on l'on intègre $\Omega_{\Gamma(e)}$ sur la fibre $p^{-1}(\xi)$ on obtient une 1-forme sur $\overline{C_{2,0}}$ que l'on note $w'_{\Gamma(e)} d\xi$.

La différentielle est maintenant la somme des 1-formes précédentes. Compte tenu de la convention de signe sur l'orientation des strates (définition 6.2.1) les deux signes moins se compensent. On obtient finalement :

⁽⁶⁾Ce résultat est démontré dans le chapitre rédigé par A. Bruguières §A.4

Proposition 6.10.4. — Soit Γ un graphe admissible de $G_{n,2}$ et $w_\Gamma(\xi)$ la déformation naturelle du coefficient définie plus haut, on a

$$(6.10.2) \quad dw_\Gamma(\xi) = \sum_{e \text{ arête de } \Gamma} \epsilon(e) w'_{\Gamma(e)} d\xi$$

où $\epsilon(e)$ est le signe qu'il faut pour mettre l'arête e en première position (voir figure 4 pour un exemple de calcul).

Remarque 6.10.5. — Si le graphe Γ n'est pas admissible, le calcul de la différentielle peut faire intervenir les strates où l'on a concentré l'amas sur une position terrestre.

6.11. Démonstration de l'associativité de l'étoile-produit

Donnons une preuve élémentaire de l'associativité du star-produit.

Pour Γ graphe de $G_{n,3}$ admissible pour les 2-vecteurs, on définit comme en (6.4.1) un opérateur tri-différentiel $B_{\Gamma,\alpha}$ en plaçant les trois fonctions arguments aux sommets de second espèce (c.f. aussi (9.4.1)).

On considère d'autre part la $2n$ -forme de $\overline{C_{n,3}^+}$ notée Ω_Γ (6.6.1). On place les points de deuxième espèce aux positions terrestres $0, s, 1$ avec $s \in]0, 1[$ (figure 5). Notons

$$(6.11.1) \quad w_\Gamma(s)$$

la valeur du coefficient obtenu par intégration de la $2n$ -forme Ω_Γ de $\overline{C_{n,3}^+}$ sur la sous-variété de dimension $2n$ des positions de première espèce $1, \dots, n$ à s fixe.

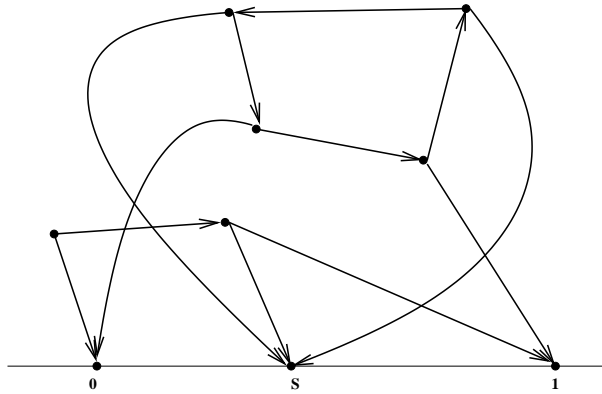


FIGURE 5. Graphe avec trois points terrestres

Lemme 6.11.1. — Pour f, g, k trois fonctions on a

$$(6.11.2) \quad f \star g \star k = fgk + \sum_{n=1}^{\infty} \frac{h^n}{n!} \sum_{\substack{\Gamma \in G_{n,3} \\ \Gamma \text{ admissible}}} w_{\Gamma}(s) B_{\Gamma, \alpha}(f, g, k).$$

Démonstration. — Montrons que l'expression de droite est indépendante de s . Pour cela on calcule la dérivée en s de $w_{\Gamma}(s)$ de manière analogue au calcul de la section 6.10.

Si l'on effectue une concentration terrestre, compte tenu du fait que les points $0, s, 1$ restent sur l'axe réel, on aura une strate de la forme $C_{p,1}$ ou $C_{p,0}$ qui sont de dimension respectives $2p-1$ ou $2p-2$. Or l'amas doit contenir $2p$ arêtes pour ne pas avoir d'arêtes sortantes⁽⁷⁾ qui contribuerait pour 0 car le point de concentration reste terrestre. Par conséquent ce type de concentration ne contribue pas dans la somme.

On doit donc effectuer des concentrations de points aériens et on a donc comme dans la proposition 6.10.4

$$dw_{\Gamma}(s) = \sum_{\substack{e \text{ arête} \\ \text{aérienne de } \Gamma}} \epsilon(e) w'_{\Gamma(e)}(s) ds.$$

Pour Γ' fixé, on regroupe les graphes Γ pour lesquels on peut trouver e tel que $\Gamma(e) = \Gamma'$. Grace à l'identité de Jacobi du 2-vecteurs α on trouve

$$\sum_{\Gamma, \text{ tel qu'il existe } e, \Gamma(e)=\Gamma'} \epsilon(e) w'_{\Gamma'}(s) B_{\Gamma, \alpha}(f, g, k) = 0.$$

L'expression de droite dans le lemme ne dépend donc pas de s . On va montrer qu'elle vaut bien l'expression de gauche.

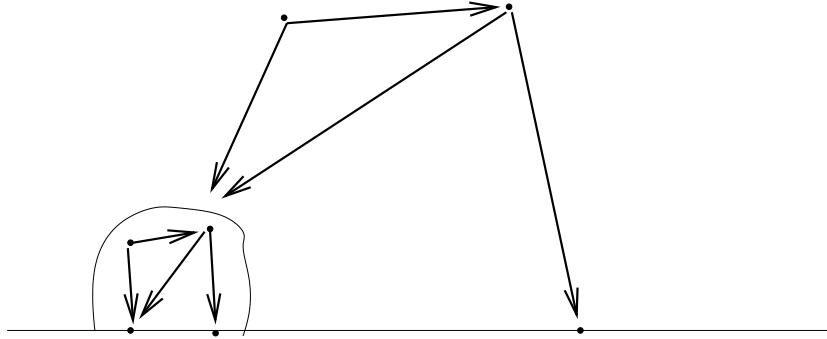


FIGURE 6. Le point s tend vers 0.

⁽⁷⁾Il n'est pas difficile de voir que dans le cas linéaire cet argument se résume au cas d'une arête dans $C_{1,1}$

Lorsque s tend vers 0, il faut tenir compte du fait que $w_\Gamma(s)$ est défini comme intégrale fibre et par conséquent il faut faire intervenir les différentes strates de la compactifications dans la fibre au dessus de 0 comme à la figure 6. On obtient toutes les concentrations de p points aériens avec leurs $2p$ arêtes sur un des deux points terrestres 0, s . Ce qui fait apparaître des bords de la forme $C_{p,2}$ qui auront une contribution non triviale : le résultat vaut la factorisation (en tenant compte de la numérotation des graphes)

$$(f \star g) \star k.$$

Lorsque s tend vers 1 on trouve $f \star (g \star k)$. On en déduit l'associativité de l'étoile-produit. √

CHAPITRE 7

EXEMPLES DE CALCULS DE GRAPHES

On présente dans cette section des exemples de calculs de coefficients de Kontsevich (6.6.2).

La méthode que l'on emploie est d'utiliser les équations données par la formule de Stokes, la linéarité de l'intégrale et les symétries éventuelles. En général les équations que l'on obtient sont quadratiques.⁽¹⁾

7.1. Graphes élémentaires

Le premier graphe à calculer est le graphe de $G_{1,n}$ (voir section 6.3) non admissible suivant, avec un sommet de aérien et n sommets terrestres comme à la figure 1. Le calcul va donner $\frac{1}{n!}$ et se fait à la main.

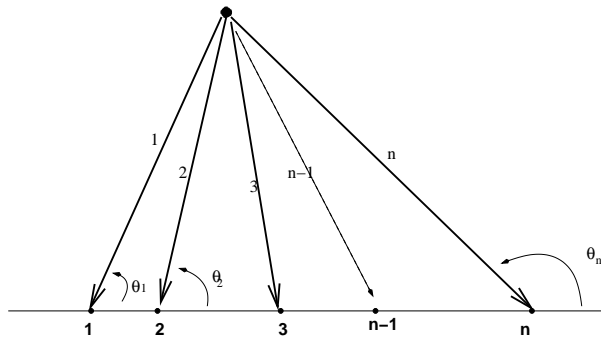


FIGURE 1. Graphe élémentaire $D_{1,n}$

⁽¹⁾Dans un preprint récent [83] prétend que les coefficients associés aux graphes linéaires de $G_{n,2}$ sont rationnels. Nous n'avons pas pu savoir si cette affirmation était vraie.

Par définition de la forme Ω_Γ on a

$$(7.1.1) \quad \frac{1}{(2\pi)^n} \int d\Phi_1 \wedge \cdots \wedge d\Phi_n = \frac{1}{\pi^n} \int_{0 \leq \theta_1 < \cdots < \theta_n \leq \pi} d\theta_1 \wedge \cdots \wedge d\theta_n = \frac{1}{n!}$$

7.2. Graphes de Bernoulli

On va calculer le coefficient de Kontsevich associé au graphe comme à la figure 2 . C'est un graphe admissible de $G_{n,2}$ muni de l'ordre lexicographique. On l'appelle graphe de Bernoulli. On notera de la même manière le graphe et le coefficient associé à savoir w_n . Ce graphe à $2n$ arêtes, n arêtes arrivent sur le point $\bar{1}$ et une seule arrive sur $\bar{2}$.

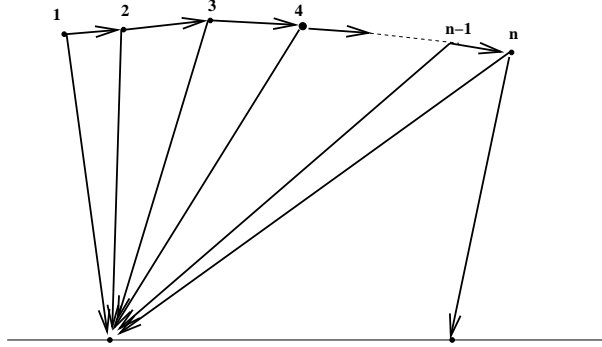


FIGURE 2. Graphe de Bernoulli w_n

Utilisons la formule de Stokes en tenant compte des orientations (voir définition 6.2.1). Pour l'exemple de la figure 3 on écrit

$$\int_{C_{2,3}^+} d\Omega_\Gamma = \int_{\partial C_{2,3}^+} \Omega_\Gamma = 0$$

et on obtient une équation quadratique (par convention on prend $w_0 = 1$)

$$w_2 - \frac{w_1}{2!} + \frac{w_0}{3!} = 0.$$

Plus généralement notons $D_{n,p}$ le graphe avec n points aériens et p points terrestres : le premier point terrestre reçoit n arêtes, les $p - 1$ autres reçoivent 1 arête, comme dans la figure 4. On remarque que l'on a $D_{n,2} = w_n$ et $D_{1,n} = \frac{1}{n!}$.

L'exemple des figures 5 et 6 montrent que l'on a les équations

$$\begin{cases} D_{3,2} + D_{2,3} - D_{1,2}D_{2,2} = w_3 - \frac{w_2}{2!} + D_{2,3} = 0 \\ D_{2,3} + D_{1,4} - D_{1,2}D_{1,3} = 0. \end{cases}$$

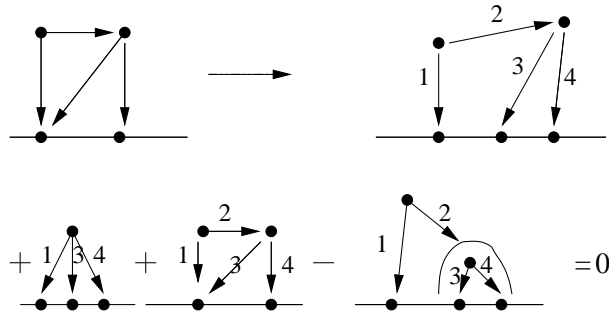


FIGURE 3. Calcul des équations de Stokes pour Bernoulli

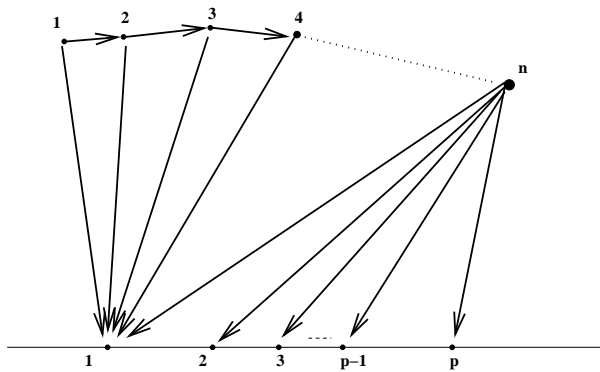


FIGURE 4. Graphe $D_{n,p}$

De $D_{1,n} = \frac{1}{n!}$, on en déduit l'équation

$$w_3 - \frac{w_2}{2!} + \frac{w_1}{3!} - \frac{w_0}{4!} = 0.$$

Plus généralement pour le graphe $D_{n,p}$ en déplaçant la dernière arête qui arrive sur le premier point terrestre vers un nouveau point terrestre et en utilisant Stokes on montre que l'on a l'équation

$$(7.2.1) \quad D_{n,p} + D_{n-1,p+1} - \frac{w_{n-1}}{p!} = 0$$

qui se spécialise pour $p = 2$ en

$$w_n - \frac{w_{n-1}}{2!} + D_{n-1,3} = 0.$$

On obtient facilement par récurrence alors la relation

$$(7.2.2) \quad w_n - \frac{w_{n-1}}{2!} + \frac{w_{n-2}}{3!} - \dots + (-1)^n \frac{w_0}{(n+1)!} = 0.$$

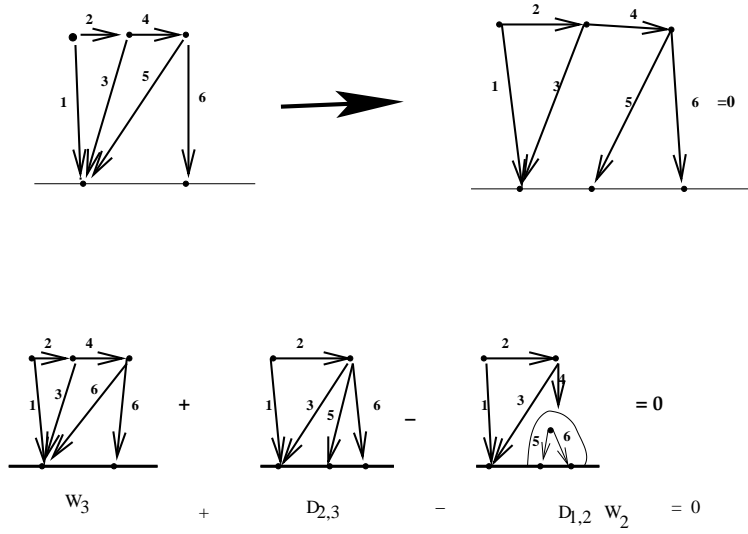


FIGURE 5. Calcul de w_3

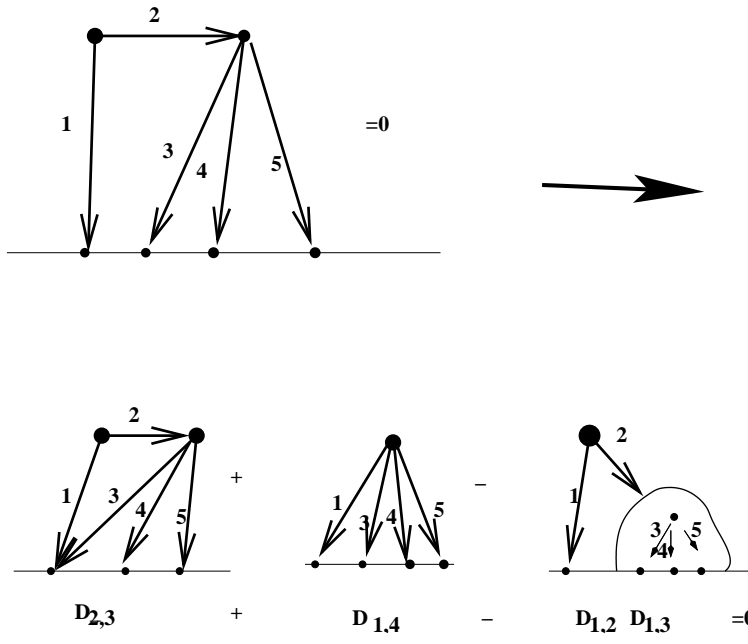


FIGURE 6. Calcul de $D_{2,3}$

On reconnaît les formules définissant les nombres de Bernoulli modifié

(7.2.3)

$$w_n = (-1)^n \frac{b_n}{n!}.$$

La série génératrice

$$(7.2.4) \quad \frac{x}{\exp(x) - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} \cdots = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}.$$

définit les nombres de Bernoulli. On peut écrire formellement pour $n \geq 1$,

$$(b + 1)^{[n+1]} = b_{n+1},$$

c'est à dire

$$\sum_{k=0}^{n+1} \frac{1}{(n+1-k)!} \frac{b_k}{k!} = \frac{b_{n+1}}{(n+1)!}.$$

Notons $S(x, t)$ la série génératrice

$$S(x, t) = \sum_{n \geq 1, p \geq 1} D_{n,p} x^{p+n} t^p = \sum_{n \geq 1} \frac{x^{n+1}}{n!} D_n(t).$$

Comme on a $D_{1,1} = 1$, $D_{n,1} = 0$ pour $n > 1$ et $D_{1,n} = \frac{1}{n!}$ on en déduit que

$$D_1(t) = t$$

et $D_n(t)$ (pour $n \geq 2$) est un polynôme de degré n et de valuation 2.

Rappelons que les polynômes $B_n(t)$ sont de termes constant b_n et sont définis par la série génératrice

$$\sum_{n \geq 0} B_n(t) \frac{x^n}{n!} = \frac{x e^{xt}}{e^x - 1}.$$

Un petit calcul utilisant l'équation (7.2.1) montre que l'on a

$$(7.2.5) \quad (1+t)S(x, t) = tx \left(\frac{x e^{tx}}{1 - e^{-x}} - \frac{x}{e^x - 1} \right)$$

$$(7.2.6) \quad S(x, t) = \frac{tx}{t+1} \left(\sum_{n \geq 0} \frac{B(-t)(-1)^n x^n}{n!} - \sum_{n \geq 0} \frac{b_n x^n}{n!} \right).$$

On en déduit que l'on a

$$(7.2.7) \quad \boxed{D_n(t) = \frac{t}{t+1} ((-1)^n B_n(-t) - b_n)}.$$

Comme $B_1(t) = t - \frac{1}{2}$ on trouve $D_1(t) = t$. Par ailleurs pour $n > 1$ on a $B_n(1) = b_n$ et $b_n = 0$ pour n impair, ce qui montre que D_n est bien un polynôme.

7.3. Polynômes de Bernoulli

On va maintenant illustrer le principe de déformation des coefficients pour les graphes de Bernoulli, en déplaçant les points terrestres en position aérienne.

On place les points $(\bar{1}, \bar{2})$ en position générale $\xi = (z_{n+1}, z_{n+2})$. On note, comme à la définition 6.10.1, $w_n(\xi)$ le coefficient déformé du graphe de Bernoulli.

Lorsque z_{n+1} est sur l'axe réel placé en 0 et z_{n+2} sur le cercle unité, le coefficient vérifie une équation différentielle très simple. On a alors $w_n(\xi) = w_n(\theta)$ où θ est l'angle entre $\bar{0}$ et z_{n+2} .

D'après la proposition 6.10.4 on a

$$dw_n(\theta) = - \int_{\partial(p^{-1}(\theta))} \Omega_\Gamma = \sum_{e \text{ arête de } \Gamma} \epsilon(e) w'_{\Gamma(e)} d\theta.$$

La seule concentration non triviale le long de la paupière concerne l'arête qui joint z_n à z_{n+2} . La différentielle est alors représentée par le graphe de la figure 7.

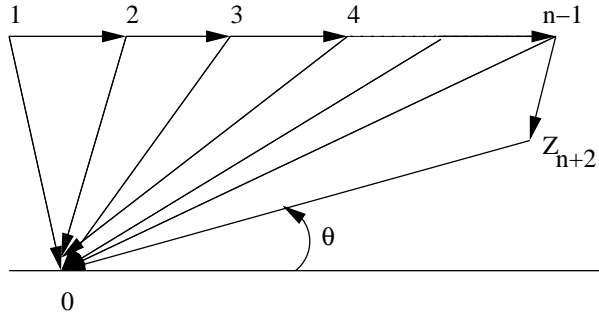


FIGURE 7. Graphe calculant $w'_n(\theta)d\theta$

La partie différentielle en $d\theta$ provient uniquement de l'arête joignant $\bar{0}$ et z_{n+2} et qui représente clairement $\frac{d\theta}{\pi}$. Le reste des intégrations se fait à θ fixe, il vient après trois signes moins :

$$\frac{d}{d\theta} w_n(\theta) = -w_{n-1}(\theta) \frac{d\theta}{\pi}.$$

Il est bien connu que les polynômes de Bernoulli $B_n(\theta)$, vérifient les deux conditions suivantes

$$(7.3.1) \quad \begin{cases} B_0(\theta) = 1, \\ B'_n(\theta) = nB_{n-1}(\theta) \quad n \geq 1 \\ \int_0^1 B_n(\theta) d\theta = 0 \quad n \geq 1 \end{cases}$$

On a deux manières de conclure :

1. Comme on sait que l'on a $w_n(0) = (-1)^n b_n/n!$, on en déduit que l'on a

$$(7.3.2) \quad \boxed{w_n(\theta) = \frac{(-1)^n}{n!} B_n\left(\frac{\theta}{\pi}\right).}$$

Ceci est conforme au résultat de [56], mais dont les calculs n'utilisaient pas la formule de Stokes et étaient bien plus compliqués.

2. Il suffirait de montrer que l'on a bien $\int_0^\pi w_n(\theta)d\theta = 0$ pour $n \geq 1$. Ceci se montre très facilement grâce encore une fois à la formule de Stokes comme à la figure 8. En effet l'intégrale à calculer revient à intégrer en θ un graphe comme à la figure 7. Or en déplaçant la dernière arête, on remarque que l'équation de Stokes nous donne exactement ce que l'on cherche. En effet, il y a deux contractions successives d'arêtes aériennes, mais de contributions opposées (il faut ramener la 1-forme en première position, donc un signe apparaît), il ne reste donc qu'un seul terme (figure 8).

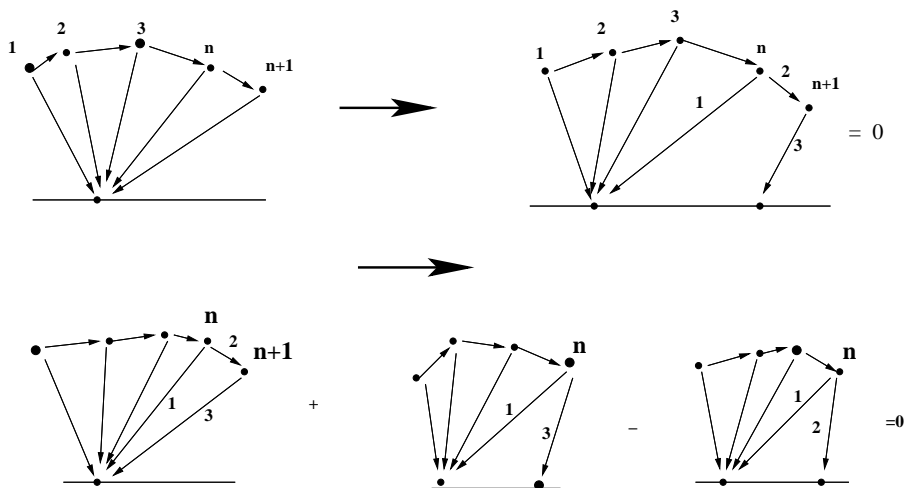


FIGURE 8. Graphe calculant $\int_0^\pi w_n(\theta)d\theta$

Remarque 7.3.1. — La déformation le long de la paupière supérieure est tout simplement la déformation des nombres de Bernoulli en les polynômes de Bernoulli, tandis que la déformation le long de la paupière inférieure reste encore mystérieuse (c'est le point z_{n+1} qui reçoit les n arêtes que l'on déplace dans le demi-plan supérieur).

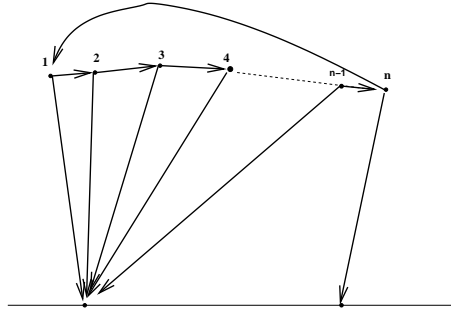
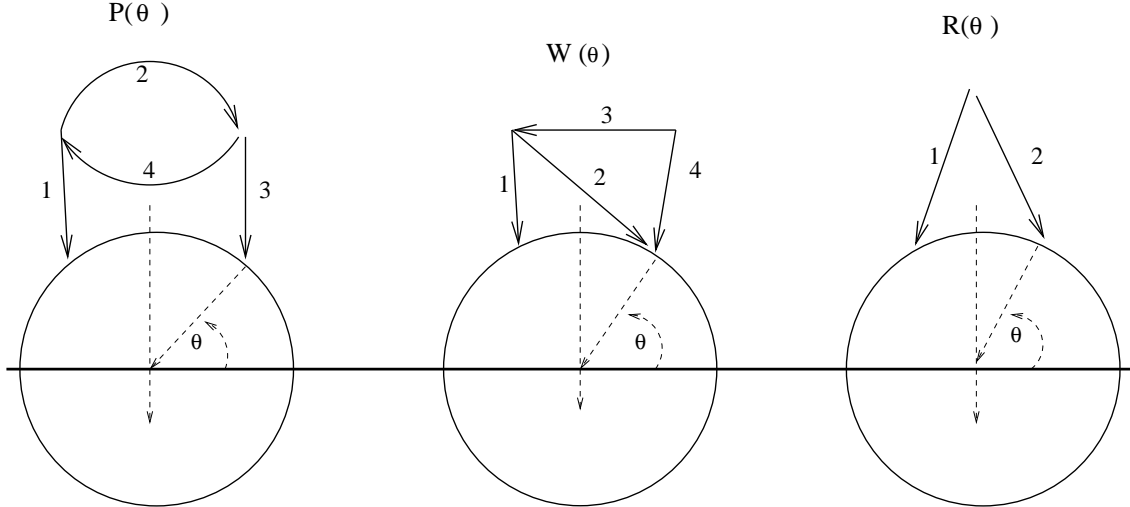


FIGURE 9. Graphe de Bernoulli Fermé.

7.4. Graphes de Bernoulli Fermé

Les graphes de la figure 9 sont munis de l'ordre lexicographique induit par la numérotation des sommets et sont appelés Bernoulli-fermé. Ils vont intervenir de manière cruciale dans le calcul à l'ordre 1 de la fonction de densité dans le cas linéaire, dans l'entrelacement de Dito et celui de Kontsevich pour le cas linéaire.

On peut montrer de manière indirecte en utilisant l'universalité de l'isomorphisme de Duflo que le coefficient de Kontsevich du graphe de Bernoulli fermé vaut $-\frac{1}{2}w_n$, avec w_n la valeur du graphe de Bernoulli (7.2.3).

FIGURE 10. Graphe de Bernoulli Fermé pour $n = 2$.

Nous détaillons le calcul du coefficient pour $n = 2$.

On note $P(\theta)$, $W(\theta)$ et $R(\theta)$ les valeurs des coefficients de la figure 10. La valeur cherchée est $P(0)$. Les valeurs en $\pi/2$ des différentes fonctions sont nulles. C'est clair pour W, R car les graphes ont des arêtes doubles en $\pi/2$. Pour $P(\pi/2)$ cela résulte de l'annulation des coefficients associés aux roues pures (voir §8.8). On a $W(0) = \frac{1}{12}$ et $R(0) = \frac{1}{2}$. On montre par le calcul des dérivées que l'on a

$$(7.4.1) \quad -\frac{P'(\theta)}{2} = W'(\theta) - R(\theta)\frac{d\theta}{2\pi}.$$

Par ailleurs en dérivant $R(\theta)$, on trouve $R'(\theta) = -\frac{d\theta}{\pi}$ puis $R(\theta) = \frac{1}{2} - \frac{\theta}{\pi}$. En intégrant l'équation (7.4.1) entre 0 et $\pi/2$, on trouve $P(0) = -\frac{1}{24}$.

CHAPITRE 8

APPLICATION AU CAS DES ALGÈBRES DE LIE

Soit \mathfrak{g} une algèbre de Lie de dimension finie sur \mathbb{R} . L'espace vectoriel dual \mathfrak{g}^* possède une structure de Poisson linéaire donnée par la *moitié du crochet de Lie* (c.f. chapitre 5), donnée par le 2-vecteur $\alpha = \frac{1}{2} \sum_{i,j} [e_i, e_j] \partial_i \otimes \partial_j$ où e_1, \dots, e_d désigne une base de \mathfrak{g} , e_1^*, \dots, e_d^* la base duale et ∂_i la dérivée dans la direction du vecteur e_i^* . En particulier on a $\{X, Y\} = \frac{1}{2}[X, Y]$ pour X, Y des formes linéaires sur \mathfrak{g}^* .

8.1. Graphes simples dans le cas linéaire

Les graphes qui vont intervenir dans la formule de Kontsevich dans le cas linéaire seront linéaires au sens de la définition 6.3.3.

La formule de Kontsevich est une somme finie quand on l'applique à deux fonctions polynomiales. On a pour X, Y dans \mathfrak{g} la relation

$$X \star Y - Y \star X = h[X, Y].$$

En localisant en $h = 1$, l'étoile-produit de Kontsevich définit donc sur l'algèbre symétrique $S(\mathfrak{g})$ une structure d'algèbre isomorphe à la structure d'algèbre enveloppante de \mathfrak{g} ([60] §8.3.1). On notera

(8.1.1)

$$I_{alg} : U(\mathfrak{g}) \rightarrow S(\mathfrak{g})$$

l'isomorphisme de $U(\mathfrak{g})$ sur $(S(\mathfrak{g}), \star_{h=1})$.

Tout graphe admissible linéaire se décompose en produit de graphes simples. Un graphe est simple si le graphe auquel on enlève les arêtes qui arrivent sur les sommets de seconde espèce est connexe. Il y a deux types de graphes simples :

1. Les graphes **de type Lie** (figure 2 page 64) c'est à dire des graphes dont le sous-graphe plein sur les sommets aérien est un arbre (avec une seule racine).

2. Les graphes **de type roue** (figure 1 ci-dessous) c'est à dire des roues orientées avec une arborescence tentaculaire.

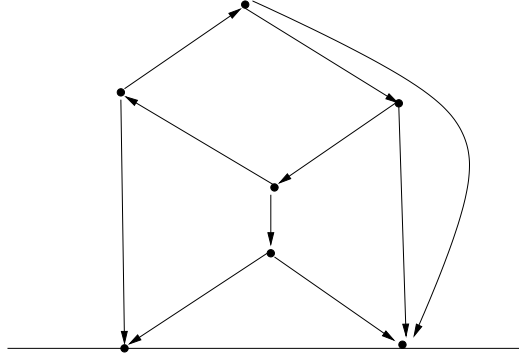


FIGURE 1. Graphe de Type roue de symbole $\Gamma(X, Y) = \text{tr}(\text{ad}X \text{ad}[X, Y] \text{ad}Y \text{ad}Y)$.

8.2. Symbole de $B_{\Gamma, \alpha}$ et symbole formel de l'étoile-produit de Kontsevich

Dans [3] §3 on a associé à chaque graphe admissible Γ un symbole. Pour simplifier on note B_{Γ} l'opérateur $B_{\Gamma, \alpha}$ introduit en (6.4.1).

Définition 8.2.1. — Pour $\Gamma \in G_{n,2}$ graphe admissible pour les 2-vecteurs, on définit le symbole géométrique de l'opérateur B_{Γ} par la formule

$$(8.2.1) \quad (X, Y) \mapsto 2^n B_{\Gamma}(e^X, e^Y) e^{-X-Y}.$$

L'expression e^X est la fonction exponentielle sur \mathfrak{g}^* . Le symbole est donc une fonction polynomiale sur $\mathfrak{g} \times \mathfrak{g}$ à valeurs dans $S(\mathfrak{g})$. On notera abusivement $\Gamma(X, Y) = 2^n B_{\Gamma}(e^X, e^Y) e^{-X-Y}$. Par exemple le symbole associé au graphe des figures 2 page 64 et 1 ci-dessus sont respectivement $\Gamma(X, Y) = [[X, [X, Y]], Y]$ et $\Gamma(X, Y) = \text{tr}_{\mathfrak{g}}(\text{ad}[X, Y] \text{ad}X \text{ad}Y \text{ad}Y)$.

Lorsque le graphe est simple de type Lie, alors $\Gamma(X, Y)$ est naturellement un élément de l'algèbre de Lie engendrée par X et Y . Lorsque Γ est produit de graphes simples, le symbole est le produit des symboles associés ([3], lemme 3.6). Le symbole ne dépend pas de la réalisation du graphe Γ dans le demi-plan de Poincaré.

Vu la propriété multiplicative du coefficient de Kontsevich et du symbole, le symbole formel du produit de Kontsevich est l'exponentielle de la contributions des graphes simples de type Lie et les graphes simples de type roue.

Les graphes simples de type Lie qui contribuent de manière non triviale dans la formule de Kontsevich n'ont pas de symétries. Par conséquent les graphes de $G_{n,2}$

étiquetés associés à un graphe géométrique de type Lie (graphe orienté associé pour lequel on oublie l'étiquetage) sont au nombre de $n!2^n$. Les graphes simples de type roue peuvent admettre des symétries⁽¹⁾, on notera m_Γ le cardinal du groupe de symétries du graphe Γ .

Notons alors

$$(8.2.2) \quad Z_*(X, Y) = X + Y + \sum_{n=1}^{\infty} h^n \sum_{\substack{\Gamma \text{ simple} \\ \text{géométrique} \\ \text{de type Lie (n,2)}}} w_\Gamma \Gamma(X, Y)$$

et

$$(8.2.3) \quad D_*(X, Y) = \exp \left(\sum_{n=1}^{\infty} h^n \sum_{\substack{\Gamma \text{ simple} \\ \text{géométrique} \\ \text{de type Roue (n,2)}}} \frac{w_\Gamma}{m_\Gamma} \Gamma(X, Y) \right)$$

Lemme 8.2.2 ([3]). — Pour $h = 1$, les séries $Z_*(X, Y)$ et $D_*(X, Y)$ sont convergentes pour X, Y dans un voisinage de 0.

On peut donc écrire maintenant le symbole formel de l'étoile produit de Kontsevich.

Théorème 8.2.3 ([3]). — On a la formule pour l'étoile-produit de Kontsevich

$$(8.2.4) \quad e^X \star e^Y = D_*(X, Y) \exp(Z_*(X, Y)).$$

Remarque 8.2.4. — On remarquera que le symbole $\Gamma(X, Y)$ est mal défini si le graphe n'est pas étiqueté. Pour résoudre ce problème il suffit de remarquer que c'est aussi le cas pour le coefficient w_Γ et que ces deux difficultés se compensent grâce aux équations (6.8.1).

8.3. Formule de Campbell-Hausdorff en termes de diagrammes

On note comme dans l'introduction $Z(X, Y)$ la série de Campbell-Hausdorff définie pour X et Y dans \mathfrak{g} par

$$Z(X, Y) = \log(\exp(X) \exp(Y)).$$

L'expression ci-dessous de la série de Campbell-Hausdorff est obtenue dans [56] théorème 5.1.

⁽¹⁾Dans [95, 96] il faut modifier l'écriture de la fonction de densité comme ci-dessous pour tenir compte des symétries des graphes géométriques de type roue

Théorème 8.3.1. — *Comme séries convergentes dans un voisinage de $(0, 0)$ on a $Z(X, Y) = Z_*(X, Y)_{h=1}$.*

Démonstration. — L'étoile produit de Kontsevich est associatif; il suffit de vérifier que les deux formules coïncident à l'ordre 1 en Y , à savoir :

$$Z(X, Y) \equiv X + \frac{\text{ad}X}{1 - \exp(-\text{ad}X)} \cdot Y \pmod{Y^2}.$$

En effet deux star-produits \star_1 et \star_2 qui coïncident à l'ordre 1 en Y coïncident à tout ordre. En dérivant en X la formule $e^X \star_1 e^{tY} = e^X \star_2 e^{tY} \pmod{t^2}$, on déduit que pour tout $p \in S[\mathfrak{g}]$ et pour tout $Y \in \mathfrak{g}$ on a

$$p \star_1 Y = p \star_2 Y.$$

Une récurrence immédiate utilisant l'associativité montre que pour tout $p, q \in S[\mathfrak{g}]$ on a $p \star_1 q = p \star_2 q$.

Or les graphes simples de type Lie qui interviennent dans la formule de Kontsevich à l'ordre 1 en Y sont ceux qui sont associés aux graphes de type Bernoulli. La formule (7.2.3) montre que ce sont bien les nombres de Bernoulli modifiés. \checkmark

8.4. Étoile-produit de Gutt

La symétrisation β est un isomorphisme d'espaces vectoriels entre $S(\mathfrak{g})$ (l'algèbre symétrique de \mathfrak{g}) et $U(\mathfrak{g})$ (l'algèbre enveloppante de \mathfrak{g}), vérifiant la condition pour $X \in \mathfrak{g}$,

$$\beta(X^n) = X^n.$$

On en déduit la formule bien connue

$$(8.4.1) \quad \beta(X_1 \cdots X_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} X_{\sigma(1)} \cdots X_{\sigma(n)}.$$

En général β n'est pas un homomorphisme d'algèbres, et a priori il n'y a pas d'évidence que $\beta \circ J^{1/2}(\partial)$ soit un homomorphisme d'algèbres sur les invariants.

On peut ramener le produit dans $U(\mathfrak{g})$, via l'application β en un produit associatif dans $S(\mathfrak{g})$. C'est l'étoile produit de Gutt [46]. On a donc pour u, v dans $S(\mathfrak{g})$

$$(8.4.2) \quad \boxed{u \star_{Gutt} v = \beta^{-1}(\beta(u)\beta(v)).}$$

Pour comprendre le symbole formel du produit de Gutt il faut comprendre la symétrisation au niveau des cogèbres $S(\mathfrak{g})$ et $U(\mathfrak{g})$. Considérons alors $S(\mathfrak{g})$ (resp. $U(\mathfrak{g})$) comme l'algèbre des distributions de support 0 (resp. 1) dans \mathfrak{g} (resp. G), munie de la convolution des distributions, notée $\star_{\mathfrak{g}}$ (resp. \star_G).

Lemme 8.4.1. — Pour u, v des éléments de $S(\mathfrak{g})$, $\beta^{-1}(\beta(u)\beta(v))$ correspond à la distribution de support $0 \in \mathfrak{g}$ définie pour f , fonction test, par la formule :

$$\langle u(x) \otimes v(y), f(Z(x, y)) \rangle.$$

La formule de Taylor énonce que l'on a l'égalité de distributions formelles $e^X = \delta_X$, avec δ_X la masse de Dirac au point X . On a donc dans une complétion adéquate de $U(\mathfrak{g})$:

$$(8.4.3) \quad \boxed{\beta(\delta_X) = \beta(e^X) = \sum_{n \geq 0} \frac{X^n}{n!} = \delta_{\exp_{\mathfrak{g}}(X)}}$$

où $\delta_{\exp_{\mathfrak{g}}(X)}$ est la distribution ponctuelle au point $\exp_{\mathfrak{g}}(X)$ dans G .

La symétrisation envoie donc la distribution δ_X sur la distribution $\delta_{\exp_{\mathfrak{g}}(X)}$.

Démonstration. — On vérifie la formule du lemme pour les distributions génériques $u = e^X = \delta_X$ et $v = e^Y = \delta_Y$:

$$(8.4.4) \quad \begin{aligned} \langle \beta^{-1}(\beta(\delta_X) \star_G \beta(\delta_Y)), f \rangle &= \langle \beta^{-1}(\delta_{\exp_{\mathfrak{g}}(X)} \star_G \delta_{\exp_{\mathfrak{g}}(Y)}), f \rangle \\ &= \langle \beta^{-1}(\delta_{\exp_{\mathfrak{g}}(X) \exp_{\mathfrak{g}}(Y)}), f \rangle = \langle \beta^{-1}(\delta_{\exp_{\mathfrak{g}}(Z(X, Y))}), f \rangle = \langle \delta_{Z(X, Y)}, f \rangle \\ &= f(Z(X, Y)) = \langle u(x) \otimes v(y), f(Z(x, y)) \rangle. \end{aligned}$$

✓

Le symbole formel de l'étoile produit de Gutt peut donc s'écrire

$$(8.4.5) \quad \boxed{e^X \star_{Gutt} e^Y = e^{Z(X, Y)}}.$$

8.5. Fonction de densité en termes de diagrammes

On explicite dans ce paragraphe la fonction de densité (8.2.3) et on relie les star-produits de Kontsevich et de Gutt.

Factorisation de la fonction de densité. —

Proposition 8.5.1. — Il existe une série convergente dans un voisinage de $0 \in \mathfrak{g}$ de la forme

$$a(X) := \exp \left(\sum_{n \geq 1} \frac{c_{2n}}{2n} \operatorname{tr}_{\mathfrak{g}}(\operatorname{ad}(X)^{2n}) \right)$$

telle que l'on ait

$$(8.5.1) \quad D_*(X, Y) = \frac{a^{-1}(hX)a^{-1}(hY)}{a^{-1}(Z(hX, hY))}.$$

Démonstration. — Par un argument astucieux Kontsevich montre [60] §8.3.3.2 que l'isomorphisme I_{alg} (8.1.1) admet une présentation simple quand on le lit à travers la symétrisation. Plus précisément on montre qu'en terme de diagrammes on a

$$e^{*X} := 1 + hX + \frac{h^2}{2!}X \star X + \cdots + \frac{h^n}{n!}X^{*n} + \cdots = a(hX)e^{hX}$$

où $a(X)$ une série de la forme indiquée avec c_{2n} est une somme des $2n!$ coefficients obtenu par intégration sur les positions aériennes de la forme associée à une roue de taille $2n$ attachée directement à $2n$ points sur l'axe réel ([60] §8.3.3.2). Comme on considère les $2n!$ façons d'attacher la roue, cette somme ne dépend pas des positions terrestres. On déduit alors de la formule donnant l'exponentielle

$$(8.5.2) \quad I_{alg} \circ \beta = a(\partial)$$

avec $a(\partial)$ l'opérateur différentiel formel à coefficients constants sur \mathfrak{g}^* de symbole a . En effet on a (l'exponentielle intermédiaire étant calculer dans $U(\mathfrak{g})$)

$$I_{alg} \circ \beta(e^X) = I_{alg}(e^{\cdot U(\mathfrak{g})X}) = e^{*h=1X} = a(\partial)(e^X).$$

En appliquant l'isomorphisme I_{alg} à $\beta(e^X) \cdot_{U(\mathfrak{g})} \beta(e^Y) = \beta(e^{Z(X, Y)})$, on trouve :

$$(8.5.3) \quad e^X \star_{h=1} e^Y = \frac{a(Z(X, Y))}{a(X)a(Y)} e^{Z(X, Y)}.$$

On conclut par le théorème 8.2.3. ✓

Remarque 8.5.2. — La fonction de densité telle qu'elle s'exprime à l'aide de graphes de Kontsevich, n'est pas a priori factorisée. Voici un argument élégant qui montre que c'est le cas. On remarque d'abord pour des raisons de symétries que l'on a $D(X, Y) = D(-Y, -X)$ et $Z(-Y, -X) = -Z(X, Y)$.

On calcule ensuite

$$e^X \star e^Y \star e^{-Y} \star e^{-X}$$

en plaçant sur l'axe réel aux 4 positions symétriques $0, t, 1-t, 1$ les points $\exp(X), \exp(Y), \exp(-Y), \exp(-X)$. Comme le produit est associatif on doit trouver

$$(8.5.4) \quad (e^X \star e^Y) \star (e^{-Y} \star e^{-X}) = D(X, Y)e^{Z(X, Y)} \star D(-Y, -X)e^{Z(-Y, -X)} \\ = c(Z(X, Y))D(X, Y)^2 = e^X \star (e^Y \star e^{-Y}) \star e^{-X} = c(X)c(Y),$$

où $c(X)$ désigne la contributions des roues de type $(n, 2)$ attachées aux points $\exp(X), \exp(-X)$. Ce sont des roues sans arborescence. On obtient donc sans peine une autre factorisation de la fonction de densité, ou plutôt de son carré

$$(8.5.5) \quad \boxed{D(X, Y)^2 = \frac{c(X)c(Y)}{c(Z(X, Y))}.}$$

On montrera par un argument indirect résultant de l'universalité de l'isomorphisme de Duflo (§8.7) que l'on a le théorème suivant :

Théorème 8.5.3. — On a $a^{-1}(X) = q(X) = \left(\frac{\sinh(\frac{\text{ad} X}{2})}{\frac{\text{ad} X}{2}} \right)^{\frac{1}{2}} = \exp \left(\sum_{n \geq 1} \frac{b_{2n}}{2n!4n} x^{2n} \right)$.

Grâce à ce théorème la fonction de densité $D(X, Y) = \frac{q(X)q(Y)}{q(Z(X, Y))}$ (5.4.1) vaut $D_*(X, Y)$ et s'exprime en termes de diagrammes (8.2.3).

Entrelacement des produits de Gutt et de Kontsevich. — On peut maintenant énoncer l'entrelacement entre le produit de Gutt et celui de Kontsevich :

Lemme 8.5.4. — L'application $q(\partial)$ (resp. $J^{1/2}(\partial)$) entrelace les deux étoile-produits, c'est à dire on a pour $u, v \in S(\mathfrak{g})$

$$q(\partial)(u \star_{Kont} v) = q(\partial)u \star_{Gutt} q(\partial)v.$$

Démonstration. — Il suffit de le faire pour les éléments génériques de $S(\mathfrak{g})$, à savoir e^X et e^Y . Par la théorème 8.5.3 on a $e^X \star_{Kont} e^Y = \frac{q(X)q(Y)}{q(Z(X, Y))} e^{Z(X, Y)}$. On en déduit que l'on a

$$(8.5.6) \quad q(\partial)(e^X \star_{Kont} e^Y) = q(\partial)(e^X) \star_{Gutt} q(\partial)(e^Y).$$

✓

Remarque 8.5.5. — Dans le cas linéaire on peut montrer ([7], [5]) que tous les étoile-produits de la forme de celui de Kontsevich (mais avec d'autres coefficients) sont entrelacés avec l'étoile-produit de Gutt par un opérateur à coefficient constant.

Entrelacement de Dito. — Dans [24], Dito construit par récurrence un entrelacement entre le produit de Gutt et celui de Kontsevich et montre l'existence d'une fonction $b(X)$ telle que

$$e^{\star X} = b(hX)e^X.$$

On a nécessairement $a(X) = b(X)$.

La formule pour b fait intervenir les poids de graphes comme à la figure 9 page 82. Si on note \widehat{w}_n le poids d'un tel graphe, alors la fonction b est donnée par la formule

$$(8.5.7) \quad b(X) = \exp\left(\sum_{n \geq 1} \frac{\widehat{w}_{2n}}{2n} \operatorname{tr}((\operatorname{ad} X)^{2n})\right).$$

Par comparaison avec l'entrelacement de Kontsevich⁽²⁾ on a donc

$$\widehat{w}_{2n} = -\frac{1}{2}w_{2n} = -\frac{b_{2n}}{2(2n)!}.$$

Question : Est-il possible de démontrer cette formule directement, comme on l'a fait dans 7.4 dans le cas le plus simple, ce qui assurerait que l'isomorphisme de Duflo s'obtient indépendamment de la théorie des représentations et de la méthode des orbites.

8.6. Déformation de la formule de Campbell-Hausdorff

8.6.1. Déformation de la formule BCH. — Utilisons la déformation $w_\Gamma(\xi)$ définie en (6.10.1) des coefficients w_Γ pour définir une déformation 2-dimensionnelle de la formule BCH.

Définition 8.6.1. — On définit une déformation régulière de la formule BCH en le paramètre $\xi \in \overline{C}_{2,0}$ en posant :

$$Z_\xi(X, Y) = X + Y + \sum_{\substack{\Gamma \text{ simple} \\ \text{géométrique} \\ \text{de type Lie}}} w_\Gamma(\xi) \Gamma(X, Y).$$

Vérifions les conditions aux limites. Pour $\xi = (0, 1)$ on retrouve clairement le coefficient de Kontsevich car les concentrations sur le point 0 donnent des contributions triviales (on a des arêtes doubles). Par conséquent on a

$$Z_{\xi=(0,1)}(X, Y) = Z(X, Y).$$

Pour $\xi \in \text{Iris}$, c'est à dire lorsque les deux points $z_{\overline{1}}, z_{\overline{2}}$ se rapprochent selon un angle donné, tous les graphes de type Lie ont une contribution triviale dans le coefficient car les graphes ont des arêtes doubles. On aura alors

$$Z_{\xi \in \text{Iris}}(X, Y) = X + Y.$$

On dispose donc d'une déformation régulière qui transforme la formule de Campbell-Hausdorff en la somme $X + Y$, c'est à dire que l'on a interpolé les lois de composition entre le groupe et son algèbre de Lie.

⁽²⁾On fera attention que l'on ne peut conclure en général sur l'identification des poids, même si on a deux formules globales, car les écritures ne font pas intervenir une base de l'algèbre de Lie libre engendrée par X, Y . En particulier les arguments de la fin de l'article de [12] sont en défaut.

8.6.2. Calcul de la dérivée de la formule BCH déformée. — Le calcul de la dérivée en $Z_\xi(X, Y)$ se fait grâce à (6.10.2). D'après la démonstration du lemme 6.11.1 (voir aussi [60] §6.4.1.1) seules importent pour le calcul de la différentielle les contractions d'arêtes où un des sommets de seconde espèce est en jeu.

On introduit maintenant des définitions pour énoncer le résultat principal de cette section.

Définition 8.6.2. — Soit A un graphe dans $G_{n,2}$ admissible simple de type Lie, avec points de deuxième espèce numérotés $\bar{1}, \bar{2}$. On définit le graphe étendu $\widehat{A}^{(1)}$ (resp. $\widehat{A}^{(2)}$) de $G_{n+1,1}$ en ajoutant une arête du sommet $\bar{1}$ (resp. $\bar{2}$) vers la racine du graphe A .

Le graphe $\widehat{A}^{(1)}$ n'a pas de boucles et a $2n + 1$ arêtes ordonnées selon l'ordre induit de A de manière compatible avec l'ordre des sommets $\bar{1} < \bar{2} < 1 < \dots < n$. Dessinons le graphe $\widehat{A}^{(1)}$ dans $C_{n+2,0}^+$ et considérons la forme $\Omega_{\widehat{A}^{(1)}}$ (6.6.1).

Définition 8.6.3. — On définit une 1-forme $w_{\widehat{A}^{(1)}}(\xi)d\xi$ sur $C_{2,0}^+$ par

$$w_{\widehat{A}^{(1)}}(\xi)d\xi = \int_{p^{-1}(\xi)} \Omega_{\widehat{A}^{(1)}}.$$

(On intègre sur les positions de première espèce de A)

Soit Γ un graphe de $G_{n,2}$ simple de type Lie. Ce graphe a $2n$ arêtes. Soit $e_1 = (i, \bar{1})$ une arête arrivant sur $\bar{1}$. On note $e_2 = (i, j)$ ($j \neq i, \bar{1}$) l'autre arête qui a pour source i .

Soit $\Gamma(e_1)$ le graphe obtenu après contraction de l'arête e_1 (figure 2). Les sommets i et $\bar{1}$ sont confondus en un sommet noté encore $\bar{1}$. Ce graphe $\Gamma(e_1)$ est dans $G_{n-1,2}$ il a $2n - 1$ arêtes ordonnées selon l'ordre induit de Γ (mais il n'est pas admissible au sens de la définition 2).

Il part maintenant du sommet $\bar{1}$ une arête vers le sommet j (c'est l'arête e_2). Le but de cette arête est donc la racine d'un sous-graphe noté A qui est soit \emptyset soit un graphe simple de type Lie. Lorsque A est vide, on notera abusivement $\widehat{\emptyset}^{(1)}$ pour désigner le graphe constitué de l'arête $e_2 = (\bar{1}, \bar{2})$.

On note B , le complément de $\widehat{A}^{(1)}$ dans $\Gamma(e_1)$. Il est facile de voir que ce graphe est soit vide, soit simple de type Lie. Les graphes $\widehat{A}^{(1)}$ et B sont indépendants (figure 3).

On dira que le graphe Γ a dégénéré suivant l'arête e_1 en $(\widehat{A}^{(1)}, B)$. On notera $\Gamma \xrightarrow{e_1} (\widehat{A}^{(1)}, B)$. On fait de même avec le sommet $\bar{2}$.

Le lemme de factorisation suivant est le point clé pour l'obtention de l'équation d'évolution de la déformation de BCH.

Lemme 8.6.4 ([95] proposition 4.1). — Soit $\Gamma \xrightarrow{e_1} (\widehat{A}^{(1)}, B)$. On a avec les notations (6.10.1) et la définition 8.6.3

$$\int_{p^{-1}(\xi)} \Omega_{\Gamma(e_1)} = w_B(\xi)w_{\widehat{A}^{(1)}}(\xi)d\xi.$$

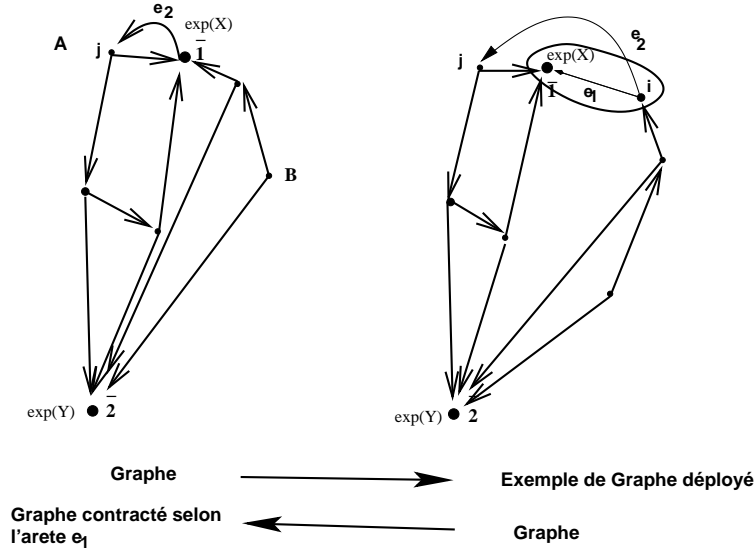


FIGURE 2. Contraction d'une arête

Désignons par $A(X, Y)$ et $B(X, Y)$ les symboles géométriques des graphes A, B . Lorsque $A = \emptyset$ il faut prendre $A(X, Y) = X$ ou $A(X, Y) = Y$ selon l'indice de $\hat{\emptyset}$ et lorsque $B = \emptyset$ il faut prendre $B(X, Y) = X + Y$.

Définition 8.6.5. — Notons $\Omega_F(X, Y)$ (resp. $\Omega_G(X, Y)$) la 1-forme sur $\overline{C_{2,0}}$ définie par

$$\Omega_F(X, Y) = \sum_{\substack{A \text{ simple géométrique} \\ \text{de type Lie}}} w_{\hat{A}(1)}(\xi) d\xi A(X, Y)$$

(resp. $\sum_A w_{\hat{A}(2)}(\xi) d\xi A(X, Y)$) où la somme porte sur les graphes simples géométriques de type Lie ou vide, avec deux points de deuxième espèce placés en ξ . Ces séries n'ont donc pas de termes constants et sont convergentes dans un voisinage de $(0, 0)$.

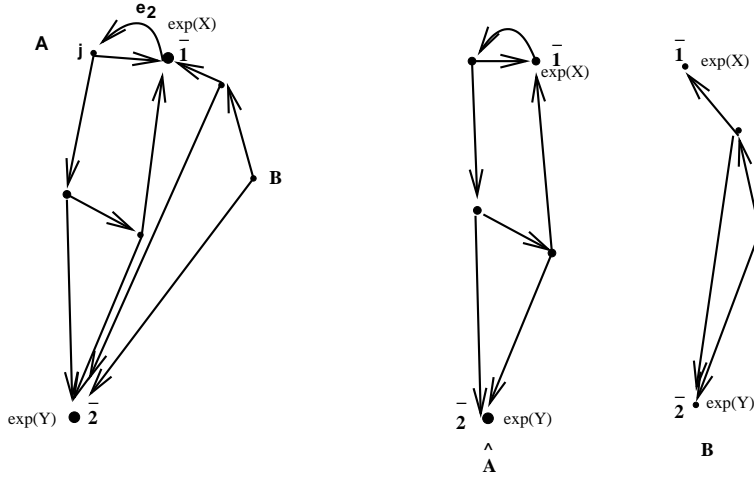
La convergence de ces séries résulte comme dans [4] ou [3] de l'estimation des coefficients. Introduisons le champ de vecteurs sur \mathfrak{g} ,

$$[X, \Omega_F(X, Y)] \cdot \partial_X = \sum_{i=1}^d \langle e_i^*, [X, \Omega_F(X, Y)] \rangle \frac{\partial}{\partial x_i}.$$

On peut maintenant énoncé le résultat principal de cette section.

Théorème 8.6.6 ([95]). — La déformation de BCH donnée par $Z_\xi(X, Y)$ vérifie l'équation suivante

$$dZ_\xi(x, y) = [X, \Omega_F(X, Y)] \cdot \partial_X Z_\xi(X, Y) + [Y, \Omega_G(X, Y)] \cdot \partial_Y Z_\xi(X, Y)$$

FIGURE 3. Graphe $\widehat{A}^{(1)}$ et graphe B

Démonstration. — On peut travailler sur les graphes géométriques car on ne considère que des produits coefficients-symboles. On peut alors considérer que l'arête contractée est en première position. Il vient alors en utilisant (6.10.2) et le lemme 8.6.4 de factorisation

$$(8.6.1) \quad dZ_\xi(X, Y) = \sum_{\substack{\Gamma \text{ simple géométrique} \\ \text{de Type Lie}}} dw_\Gamma(\xi) \Gamma(X, Y) =$$

$$\sum_{\substack{\Gamma \text{ simple} \\ \text{géométrique} \\ \text{de Type Lie}}} \left(\sum_{\substack{e=(-, \bar{1}) \\ \Gamma \xrightarrow{e} (\widehat{A}^{(1)}, B)}} w_B(\xi) w_{\widehat{A}^{(1)}}(\xi) d\xi + \sum_{\substack{e=(-, \bar{2}) \\ \Gamma \xrightarrow{e} (\widehat{A}^{(2)}, B)}} w_B(\xi) w_{\widehat{A}^{(2)}}(\xi) d\xi \right) \Gamma(X, Y)$$

où les sommes portent sur les arêtes de Γ qui ont pour but $\bar{1}$ ou $\bar{2}$. La somme des symboles $\Gamma(X, Y)$ des graphes Γ qui dégénèrent selon une arête en $(\widehat{A}^{(1)}, B)$ (resp. $(\widehat{A}^{(2)}, B)$) vaut la différentielle de $B(X, Y)$ dans la première variable évaluée sur le vecteur $[X, A(X, Y)]$ c'est à dire

$$[X, A(X, Y)] \cdot \partial_X B(X, Y)$$

(resp. $[Y, A(X, Y)] \cdot \partial_Y B(X, Y)$). En inversant les sommes de (8.6.1) on fait apparaître les termes $\Omega_F(X, Y)$ (resp. $\Omega_G(X, Y)$) de la définition 8.6.5. On en déduit alors l'équation du théorème 8.6.6 :

$$(8.6.2) \quad dZ_\xi(x, y) = [X, \Omega_F(X, Y)] \cdot \partial_X Z_\xi(X, Y) + [Y, \Omega_G(X, Y)] \cdot \partial_Y Z_\xi(X, Y).$$

✓

8.6.3. Déformation de la fonction de densité et calcul de la différentielle.

— On déforme la fonction de densité $D_*(X, Y)$ en faisant intervenir les conditions aux limites.

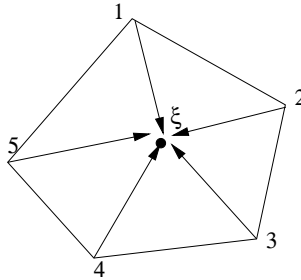


FIGURE 4. Roue pure

Définition 8.6.7. — Avec les notations de (6.10.1), on définit la déformation de la fonction de densité $D_*(X, Y)$ par

$$(8.6.3) \quad D_\xi(X, Y) = \exp \left(\sum_{\substack{\Gamma \text{ simple géométrique} \\ \text{de type Roue}}} \frac{w_\Gamma(\xi)}{m_\Gamma} \Gamma(X, Y) \right).$$

Regardons les conditions au limites. Lorsque ξ est sur l'iris de l'œil (les points $z_{\overline{1}}$ et $z_{\overline{2}}$ se rapprochent) un petit raisonnement sur les graphes linéaires montre que les graphes simples dont la contribution est a priori non triviale sont des roues pures (figure 4). Notons W_n le poids associé à ces roues pures de taille n et introduisons la série correspondante pour $T \in \mathfrak{g}$

$$(8.6.4) \quad A(T) = \exp \left(\sum_{n \geq 1} \frac{W_n}{n} \text{tr} ad(T)^n \right).$$

. On a donc

$$D_{\xi \in \text{Iris}}(X, Y) = A(X + Y).$$

Lorsque ξ tend vers $(0, 1)$, outre le terme $D_*(X, Y)$, on doit aussi tenir compte des strates correspondant aux concentrations d'amas autour des points 0 et 1. Ces amas sont encore du même type. On a donc

$$D_{\xi=(0,1)}(X, Y) = D_*(X, Y)A(X)A(Y).$$

En effectuant un calcul analogue au théorème 8.6.6 on montre le résultat suivant (avec notation de la définition 8.6.5) :⁽³⁾

Théorème 8.6.8 ([95]). — *La déformation de la fonction de densité $D_*(X, Y)$ donnée par $D_\xi(X, Y)$ vérifie l'équation suivante :*

$$(8.6.5) \quad dD_\xi(X, Y) = \left(\text{tr}_{\mathfrak{g}} \left((\partial_X \Omega_F) \circ \text{ad}X + (\partial_Y \Omega_G) \circ \text{ad}Y \right) D_\xi(X, Y) + \right. \\ \left. \left([X, \Omega_F(X, Y)] \cdot \partial_X + [Y, \Omega_G(X, Y)] \cdot \partial_Y \right) D_\xi(X, Y) \right).$$

8.7. Déformation et isomorphisme de Duflo

Nous faisons maintenant le lien entre la déformation de la formule BCH, l'isomorphisme de Duflo et celui de Kontsevich.

Comme expliqué en §Introduction l'existence d'une déformation de la formule $Z(X, Y)$ et de la fonction $D_*(X, Y)$ montre que l'on a pour tout f , fonction C^∞ dans un voisinage de 0, u et v dans $S(\mathfrak{g})^{\mathfrak{g}}$ considéré comme des distributions invariantes de support 0 (cf Introduction) :

$$(8.7.1) \quad \langle u(x) \otimes v(y), A(x)A(y)D_*(x, y)f(Z(x, y)) \rangle = \langle u(x) \otimes v(y), A(x+y)f(x+y) \rangle.$$

Comme on a

$$D_*(x, y) = \frac{a^{-1}(x)a^{-1}(y)}{a^{-1}(Z(x, y))},$$

on en déduit :

$$(8.7.2) \quad \langle u(x)(Aa^{-1})(x) \otimes v(y)(Aa^{-1})(y), f(Z(x, y)) \rangle = \\ \langle u(x) \otimes v(y), (Aa^{-1})(x+y)f(x+y) \rangle$$

On peut maintenant prouver le théorème 8.5.3 et la formule de Kontsevich sur le centre de $S(\mathfrak{g})$.

Théorème 8.7.1. — *L'application $I_{alg}^{-1} = \beta \circ (a^{-1})(\partial)$ est un isomorphisme d'algèbres de $S(\mathfrak{g})^{\mathfrak{g}}$ sur $U(\mathfrak{g})^{\mathfrak{g}}$. Par unicité de l'isomorphisme de Duflo, on a $a^{-1} = q$.*

⁽³⁾Compte tenu de la correction sur l'expression de la fonction de densité en terme de diagrammes géométriques (facteur $1/m_\Gamma$ manquant dans [95]), le théorème ci-dessous est correctement énoncé dans [95]. En effet dans un graphe qui a une symétrie d'ordre m_Γ il y a m_Γ façons de contracter une arête qui arrive sur X . Le graphe de type roue obtenu est alors sans symétrie, et le terme m_Γ a bien disparu. On fait un raisonnement analogue dans l'autre sens.

Démonstration. — Rappelons que lorsqu'on identifie les distributions de support 0 aux éléments de l'algèbre symétrique, alors la multiplication par la fonction Aa^{-1} correspond à l'opérateur différentiel (d'ordre infini) $Aa^{-1}(\partial)$. D'après le lemme 8.4.1, le membre de gauche de l'équation (8.7.2) correspond à

$$\beta^{-1}\left(\beta(Aa^{-1}(\partial)u)\beta(Aa^{-1}(\partial)v)\right)$$

et le membre de droite correspond à

$$Aa^{-1}(\partial)(uv).$$

Par (8.5.2) et (8.1.1) on en déduit que l'on a pour tout u, v éléments $S(\mathfrak{g})^{\mathfrak{g}}$,

$$I_{alg}^{-1}(A(\partial)(uv)) = I_{alg}^{-1}(A(\partial)u)I_{alg}^{-1}(A(\partial)v) = I_{alg}^{-1}(A(\partial)u \star_{Kont} A(\partial)v)$$

On a donc pour tout $u, v \in S(\mathfrak{g})^{\mathfrak{g}}$,

$$(8.7.3) \quad \boxed{A(\partial)u \star_{Kont} A(\partial)v = A(\partial)(uv).}$$

D'après un résultat de Shoikhet [88] que l'on expliquera au §8.8 la fonction A vaut 1, ce qui montre la première assertion du théorème. Maintenant supposons $a \neq q^{-1}$, alors on aurait

$$a(X)q(X) = \exp(c \operatorname{tr}_{\mathfrak{g}}(\operatorname{ad}X)^{2n} + \dots)$$

avec $c \neq 0$. Nécessairement $X \mapsto \operatorname{tr}_{\mathfrak{g}} \operatorname{ad}(X)^{2n}$, vu comme élément de $S(\mathfrak{g}^*)$ agirait comme une dérivation sur $S(\mathfrak{g})^{\mathfrak{g}}$ (puisque l'exponentielle agit comme un homomorphisme d'algèbres) ce qui n'est pas vrai sur des exemples. En définitive l'isomorphisme de Kontsevich et celui de Duflo coïncident. \checkmark

8.8. Graphe associé au roues pures

Proposition 8.8.1. — [88] *Le coefficient de Kontsevich associée aux roues pures comme à la figure 4 page 96 est nul.*

Démonstration. — En utilisant la symétrie $z \mapsto -\bar{z}$ on constate que le nombre de sommets de la roue doit être pair pour ne pas avoir un cas de nullité triviale. On a donc $A(X) = \exp\left(\sum_{n \geq 1} \frac{W_{2n}}{2n} \operatorname{tr}_{\mathfrak{g}} \operatorname{ad}(X)^{2n}\right)$.

Dans [34], G. Felder et B. Shoikhet montrent que si Ω est une forme volume constante sur V espace vectoriel muni d'une structure de Poisson α , pour laquelle $\operatorname{div}_{\Omega}\alpha$, alors on a

$$(8.8.1) \quad \int_V f \cdot g \cdot \Omega = \int_V (f \star g) \cdot \Omega$$

où f, g sont des fonctions à support compact. On rappelle que la divergence correspond à l'opérateur de de Rham, une fois que l'on a identifié les formes différentielles et les polychamps de vecteurs, grâce à la forme volume.

L'argument d'homotopie de Kontsevich que l'on vient de voir montre alors que l'on a pour f, g invariants

$$A(\partial)(f \cdot g) = A(\partial)f \star A(\partial)g.$$

Le produit est en particulier commutatif sur les éléments invariants. En écrivant

$$f \star g = f \cdot g + \sum_{k \geq 1} h^k B_k(f, g),$$

on en déduit $B_{2k+1}(f, g) = 0$.

Notons W_{2N} le premier coefficient non nul et tr_{2N} pour la fonction $\text{tr}_{\mathfrak{g}}(\text{ad}(X)^{2N})$. On aurait alors

$$f \star g = f \cdot g + h^{2N} \frac{W_{2N}}{2N} \left(\text{tr}_{2N}(\partial)(f \cdot g) - \text{tr}_{2N}(\partial)(f) \cdot g - f \cdot \text{tr}_{2N}(\partial)(g) \right) + o(h^{2N}).$$

Considérons le cas des algèbres de Lie semi-simples, avec forme volume Ω canonique (on a bien $\text{div}_{\Omega} \alpha = 0$). Soient f, g des fonctions invariants à support compact (il en existe). On a alors

$$W_{2N} \int_{\mathfrak{g}^*} \left(\text{tr}_{2N}(\partial)(f \cdot g) - \text{tr}_{2N}(\partial)(f) \cdot g - f \cdot \text{tr}_{2N}(\partial)(g) \right) \Omega = 0.$$

Or on a $\int_{\mathfrak{g}^*} \text{tr}_{2N}(f \cdot g) \Omega = 0$ et $\int_{\mathfrak{g}^*} \text{tr}_{2N}(f) \cdot g \Omega = \int_{\mathfrak{g}^*} f \cdot \text{tr}_{2N}(g) \Omega$. Finalement on trouve

$$2W_{2N} \int_{\mathfrak{g}^*} \text{tr}_{2N}(\partial)(f) \cdot g \Omega = 0.$$

On vérifie sans peine sur un exemple que ceci n'est pas vérifié. ✓

Nous sommes en mesure maintenant de donner une preuve du théorème 5.4.1.

Compte tenu du résultat ci-dessus, la déformation 2-dimensionnelle $D_{\xi}(X, Y)$ de la fonction de densité vérifie les conditions aux limites du théorème 5.4.1. Les déformations obtenues par restriction des déformations 2-dimensionnelles $D_{\xi}(X, Y)$ et $Z_{\xi}(X, Y)$ à un chemin liant le point $(0, 1)$ à un point de l'iris de l'œil de Kontsevich vérifient alors les assertions du théorème 5.4.1.

CHAPITRE 9

FORMALITÉ DANS LE CAS \mathbb{R}^n

En fait la construction de l'étoile-produit de Kontsevich, n'est qu'un bout de la formalité, c'est à dire du L^∞ quasi-isomorphisme entre les polychamps de vecteurs et les opérateurs polydifférentiels. On peut alors penser que l'isomorphisme de Duflou n'est qu'un bout d'un résultat concernant toute la cohomologie.

On résume dans cette section (pour les experts) les résultats de Kontsevich sur la formalité en reprenant les présentations de [60, 72, 82, 6].

9.1. Algèbre des polychamps de vecteurs

Soit X une variété de classe C^∞ . On lui associe deux algèbres de Lie différentielles graduées. La première algèbre de Lie différentielle graduée $\mathfrak{g}_1 = T_{\text{poly}}(X)$ est l'algèbre graduée des polychamps de vecteurs sur X :

$$T_{\text{poly}}^n(X) := \Gamma(X, \Lambda^{n+1}TX), \quad n \geq -1,$$

munie du crochet de Schouten-Nijenhuis $[,]_{SN}$ et de la différentielle $d := 0$.

Rappelons tout d'abord que le crochet de Schouten-Nijenhuis est donné pour tous $k, l \geq 0$, $\xi_i, \eta_j \in \Gamma(X, TX)$ et $h \in \Gamma(X, \mathcal{O}_X)$ par ([66]) :

$$(9.1.1) \quad [\xi_0 \wedge \cdots \wedge \xi_k, \eta_0 \wedge \cdots \wedge \eta_l]_{SN} = \sum_{i=0}^k \sum_{j=0}^l (-1)^{i+j} [\xi_i, \eta_j] \wedge \xi_0 \wedge \cdots \wedge \xi_{i-1} \wedge \xi_{i+1} \wedge \cdots \wedge \xi_k \wedge \eta_0 \wedge \cdots \wedge \eta_{j-1} \wedge \eta_{j+1} \wedge \cdots \wedge \eta_l$$

$$[\xi_0 \wedge \cdots \wedge \xi_k, h]_{SN} = \sum_{i=0}^k (-1)^i \xi_i(h) \cdot (\xi_0 \wedge \cdots \wedge \xi_{i-1} \wedge \xi_{i+1} \wedge \cdots \wedge \xi_k).$$

Le symbole $[\xi_i, \eta_j]$ désigne le crochet standard des champs des vecteurs, *i.e.* la dérivée de Lie $L_{\xi_i}(\eta_j)$.⁽¹⁾

9.2. Algèbre des opérateurs polydifférentiels

La deuxième algèbre de Lie différentielle graduée associée à X est celle des opérateurs polydifférentiels $\mathfrak{g}_2 = D_{\text{poly}}(X)$ vue comme une sous-algèbre du complexe de Hochschild décalé de l'algèbre des fonctions sur X .

On définit sur $D_{\text{poly}}(X)$ une graduation donnée par $|A| = m - 1$ où $A \in D_{\text{poly}}(X)$ est un opérateur m -différentiel.

La composition de deux opérateurs $A_1 \in D_{\text{poly}}^{m_1}(X)$ et $A_2 \in D_{\text{poly}}^{m_2}(X)$ s'écrit pour $f_i \in \mathcal{O}_X$:

$$(9.2.1) \quad (A_1 \circ A_2)(f_1, \dots, f_{m_1+m_2-1}) = \sum_{j=1}^{m_1} (-1)^{(m_2-1)(j-1)} A_1(f_1, \dots, f_{j-1}, \\ A_2(f_j, \dots, f_{j+m_2-1}), f_{j+m_2}, \dots, f_{m_1+m_2-1}).$$

Cette opération de composition permet de définir le crochet de Gerstenhaber par :

$$[A_1, A_2]_G := A_1 \circ A_2 - (-1)^{|A_1||A_2|} A_2 \circ A_1.$$

La différentielle dans $D_{\text{poly}}(X)$ s'écrit alors $dA = -[\mu, A]_G$, où μ est l'opérateur bi-différentiel de multiplication des fonctions : $\mu(f_1, f_2) = f_1 f_2$.

Notons que cette différentielle est liée à celle de Hochschild d_H par la relation $d_H(A) = (-1)^{|A|+1} dA$. Ce choix de signe fait de $(D_{\text{poly}}(X), d, [,]_G)$ une algèbre de Lie différentielle graduée.

L'application $\mathcal{U}_1^{(0)} : T_{\text{poly}} \mapsto D_{\text{poly}}$ donnée par

$$(9.2.2) \quad \mathcal{U}_1^{(0)} : (\xi_0 \wedge \dots \wedge \xi_n) \mapsto \left(f_0 \otimes \dots \otimes f_n \rightarrow \sum_{\sigma \in S_{n+1}} \frac{\text{sgn}(\sigma)}{(n+1)!} \prod_{i=0}^n \xi_{\sigma(i)}(f_i) \right)$$

pour $n \geq 0$ et par $f \mapsto (1 \rightarrow f)$ pour $f \in \Gamma(X, \mathcal{O}_X)$ est un quasi-isomorphisme des complexes. C'est une version du théorème de Kostant-Hochschild-Rosenberg

⁽¹⁾Précisons ici que nous adoptons la convention de [72] l'identification des polychamps de vecteurs avec les tenseurs anti-symétriques. Les produits $\xi_1 \wedge \dots \wedge \xi_k$ s'identifient avec

$$\frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \xi_{\sigma_1} \otimes \dots \otimes \xi_{\sigma_k}.$$

Cette convention est différente de celle de [60] §6.3 mais sera compensée par une autre convention dans la définition des poids.

([60] §4.6.1.1). Il est illusoire de vouloir faire de $\mathcal{U}_1^{(0)}$ un quasi-isomorphisme d'algèbres de Lie différentielle gradué. C'est ce défaut qui motive la recherche d'un quasi-isomorphisme à homotopie près.⁽²⁾

9.3. Codérivation et coefficient de Taylor

Considérons les algèbres décalées $\mathfrak{g}_1[1]$ et $\mathfrak{g}_2[1]$ comme des variétés formelles graduées, pointées ([60], [6]§III.2). Chacune des cogèbres sans co-unité

$$S^+(\mathfrak{g}_i[1]) = \bigoplus_{n \geq 0} S^n(\mathfrak{g}_i[1]), \quad i = 1, 2$$

possède une codérivation $Q^{(i)}$ de degré 1 définie par la structure d'algèbre de Lie différentielle graduée de \mathfrak{g}_i et telle que $[Q^{(i)}, Q^{(i)}] = 0$. Ces codérivations sont définis par leurs coefficients de Taylor.

Cas $S^+(\mathfrak{g}_1[1])$:

D'après [6] on doit remplacer le crochet de Schouten par l'opposé du crochet pris dans l'ordre inverse. Ce crochet coïncide avec le crochet de Schouten modulo un signe moins lorsque deux éléments impairs sont en jeu. On notera alors $[\gamma_1, \gamma_2] = -[\gamma_2, \gamma_1]_{SN}$.

On rappelle ([6] § II.4, §IV.1 et §IV.2) que le deuxième coefficient de Taylor de la codérivation $Q^{(1)}$ de $S^+(\mathfrak{g}_1[1])$ vérifie :

$$(9.3.1) \quad Q_2^{(1)}(\gamma_1 \cdot \gamma_2) = (-1)^{(k_1-1)k_2} [\gamma_1, \gamma_2].$$

On a $Q_1^{(1)} = 0$ et le coefficient de Taylor $Q_2^{(1)}$ défini sur $S(\mathfrak{g}_1[1])$ vaut pour $\gamma_1 = \xi_1 \wedge \dots \wedge \xi_{k_1}$ et $\gamma_2 = \eta_1 \wedge \dots \wedge \eta_{k_2}$:

$$(9.3.2) \quad Q_2^{(1)}(\gamma_1 \cdot \gamma_2) = \sum_{\substack{1 \leq r \leq k_1 \\ 1 \leq s \leq k_2}} (-1)^{r+s+k_1-1} [\xi_r, \eta_s] \wedge \xi_1 \dots \wedge \widehat{\xi}_r \wedge \dots \wedge \xi_{k_1} \wedge \eta_1 \dots \wedge \widehat{\eta}_s \wedge \dots \wedge \eta_{k_2}$$

ce qui est conforme à la formule de [60] §4.6.1. Notons ψ_i les dérivées $\frac{\partial}{\partial x_i}$ vues comme variables impaires. Définissons alors comme dans [60] §4.6.1

$$\gamma_1 \bullet \gamma_2 = \sum_{i=1}^n \frac{\partial \gamma_1}{\partial \psi_i} \frac{\partial \gamma_2}{\partial x_i}.$$

On a alors $Q_2^{(1)}(\gamma_1 \cdot \gamma_2) = (-1)^{k_1 k_2} Q_2^{(1)}(\gamma_2 \cdot \gamma_1)$ et

$$(9.3.3) \quad Q_2^{(1)}(\gamma_1 \cdot \gamma_2) = \gamma_1 \bullet \gamma_2 + (-1)^{k_1 k_2} \gamma_2 \bullet \gamma_1.$$

Cette dernière formule est clairement symétrique (graduée) en γ_1, γ_2 .

⁽²⁾Pour toutes ces questions on lira le chapitre de B. Keller

Remarque 9.3.1. — Pour $\gamma_1 = (f\partial_{i_1}) \wedge \partial_{i_2} \dots \wedge \partial_{i_{k_1}}$ et $\gamma_2 = (g\partial_{j_1}) \wedge \partial_{j_2} \dots \wedge \partial_{j_{k_2}}$ on a alors, toujours d'après [6] § IV.2 :

$$(9.3.4) \quad \gamma_1 \bullet \gamma_2 = \sum_{r=1}^{k_1} (-1)^{r-1} f \partial_{i_r} g \partial_{i_1} \wedge \dots \wedge \widehat{\partial_{i_r}} \wedge \dots \wedge \partial_{i_{k_1}} \wedge \partial_{j_1} \wedge \dots \wedge \partial_{j_{k_2}}.$$

On déduit la formule pour un coefficient quelconque du multi-tenseur $\gamma_1 \bullet \gamma_2$:

$$(9.3.5) \quad (\gamma_1 \bullet \gamma_2)^{u_1 \dots u_{k_1+k_2-1}} = \frac{1}{(k_1+k_2-1)!} \sum_{\sigma \in S_{k_1+k_2-1}} \varepsilon(\sigma) \sum_{v=1}^d \sum_{r=1}^{k_1} (-1)^{r-1} \gamma_1^{u_{\sigma_1} \dots u_{\sigma_{r-1}} v u_{\sigma_r} \dots u_{\sigma_{k_1-1}}} \partial_v (\gamma_2)^{u_{\sigma_{k_1}} \dots u_{\sigma_{k_1+k_2-1}}}.$$

Cas $S^+(\mathfrak{g}_2[1])$:

On a $Q_1^{(2)} = -d_H$ l'opposé de la dérivée de Hochschild et

$$(9.3.6) \quad Q_2^{(2)}(A_1 \cdot A_2) = (-1)^{|A_1|(|A_2|-1)} [A_1, A_2]_G = (-1)^{|A_1|(|A_2|-1)} A_1 \circ A_2 - (-1)^{|A_1|} A_2 \circ A_1.$$

9.4. Théorème de formalité de M. Kontsevich

9.4.1. Énoncé du théorème. — Le théorème de la formalité peut être énoncé alors de la façon suivante.

Théorème 9.4.1. — *Il existe un L_∞ -quasi-isomorphisme entre les variétés formelles graduées pointées $\mathfrak{g}_1[1]$ et $\mathfrak{g}_2[1]$, i.e. un morphisme de cogèbres*

$$\mathcal{U} : S^+(\mathfrak{g}_1[1]) \rightarrow S^+(\mathfrak{g}_2[1])$$

tel que

$$\mathcal{U} \circ Q^{(1)} = Q^{(2)} \circ \mathcal{U}$$

et tel que la restriction de \mathcal{U} à $\mathfrak{g}_1[1] \simeq S^1(\mathfrak{g}_1[1])$ est le quasi-isomorphisme de complexe de cochaînes $\mathcal{U}_1^{(0)}$ donné par (9.2.2).

Cette construction est basée sur la réalisation explicite du L_∞ -quasi-isomorphisme \mathcal{U} dans le cas plat, dont nous allons à présent rappeler la construction.⁽³⁾

D'après la propriété universelle des cogèbres cocommutatives le L_∞ -quasi-isomorphisme \mathcal{U} est entièrement déterminé par ses "coefficients de Taylor"

$$\mathcal{U}_k : S^k(\mathfrak{g}_1[1]) \rightarrow \mathfrak{g}_2[1],$$

⁽³⁾Le passage de \mathbb{R}^n au cas général est traité dans le chapitre de B. Keller.

avec $k \geq 1$ obtenus en composant \mathcal{U} avec la projection canonique $\pi : S^+(\mathfrak{g}_2) \rightarrow \mathfrak{g}_2$. On notera $\bar{\mathcal{U}}$ cette composition.

Ces coefficients de Taylor \mathcal{U}_n sont définis à l'aide des graphes de Kontsevich et de leurs poids comme dans la définition du star-produit (6.0.9).

Soit $\Gamma \in G_{n,m}$ un graphe admissible (section 6.3), on note s_k le nombre d'arêtes partant du sommet du premier type ayant le numéro k . A tout n -uplet $(\alpha_1, \dots, \alpha_n)$ de polychamps de vecteurs sur X tels que pour tout $k = 1, \dots, n$ l'élément α_k soit un s_k -champ de vecteurs, on associe, suivant [60] §6.3 un opérateur m -différentiel

$$B_\Gamma(\alpha_1 \otimes \dots \otimes \alpha_n),$$

généralisant la construction (6.4.1). construit de la façon suivante : on désigne par $\{e_k^1, \dots, e_k^{s_k}\}$ le sous-ensemble ordonné de E_Γ des arêtes partant du sommet aérien k . A toute application $I : E_\Gamma \rightarrow \{1, \dots, d\}$ et à tout sommet x du graphe Γ (de type aérien ou terrestre) on associe l'opérateur différentiel à coefficient constant :

$$D_{I(x)} = \prod_{e=(-,x)} \partial_{I(e)},$$

où pour tout $i \in \{1, \dots, d\}$ on désigne par ∂_i l'opérateur de dérivation partielle par rapport à la i -ème variable. Le produit est pris pour toutes les arêtes qui arrivent au sommet x .

On désigne par α_k^I le coefficient (suivant la convention sur le produit extérieur) :

$$\begin{aligned} \alpha_k^I &= \alpha_k^{I(e_k^1) \dots I(e_k^{s_k})} = \langle \alpha_k, dx_{I(e_k^1)} \wedge \dots \wedge dx_{I(e_k^{s_k})} \rangle \\ &= \langle \alpha_k, dx_{I(e_k^1)} \otimes \dots \otimes dx_{I(e_k^{s_k})} \rangle. \end{aligned}$$

On pose alors :

$$(9.4.1) \quad B_\Gamma(\alpha_1 \otimes \dots \otimes \alpha_n)(f_1 \otimes \dots \otimes f_m) = \sum_{I: E_\Gamma \rightarrow \{1, \dots, d\}} \prod_{k=1}^n D_{I(k)} \alpha_k^I \prod_{l=1}^m D_{I(\bar{l})} f_l.$$

On rappelle que pour Γ graphe admissible de $G_{n,m}$ on a associé une forme Ω_Γ (6.6.1) et un coefficient de Kontsevich w_Γ par

$$(9.4.2) \quad w_\Gamma = \int_{\bar{\mathcal{C}}_{n,m}^+} \Omega_\Gamma.$$

Le coefficient de Taylor \mathcal{U}_n est donné par la formule :

$$\mathcal{U}_n(\alpha_1, \dots, \alpha_n) = \sum_{\Gamma \in G_{n,m}} w_\Gamma B_\Gamma(\alpha_1 \otimes \dots \otimes \alpha_n),$$

où la somme porte sur les graphes admissibles Γ pour lesquels l'opérateur $B_\Gamma(\alpha_1 \otimes \cdots \otimes \alpha_n)$ est bien défini et l'entier m est relié à n et aux α_j par la formule

$$(9.4.3) \quad m - 2 = \sum_{k=1}^n s_k - 2n.$$

Alors $\mathcal{U}_n(\alpha_1, \dots, \alpha_n)$ est un opérateur m -différentiel.

Remarque 9.4.2. — Ce poids est un peu différent du poids défini par M. Kontsevich dans [60] § 6.2 : nous ne multiplions pas l'intégrale par le facteur $(\prod_{k=1}^n \frac{1}{s_k!})$. Cette convention est compensée par celle sur les sur le produit extérieur.

Soit Γ un graphe admissible dans $G_{n,m}$. Le groupe $S_{s_1} \times \cdots \times S_{s_n}$, produit des groupes de permutations des arêtes attachés à chaque sommet, agit naturellement sur Γ par permutation de l'étiquetage des arêtes. Il est clair que l'on a :

$$\begin{aligned} B_{\sigma.\Gamma} &= \varepsilon(\sigma)B_\Gamma \\ w_{\sigma.\Gamma} &= \varepsilon(\sigma)w_\Gamma, \end{aligned}$$

de sorte que le produit $w_\Gamma \cdot B_\Gamma$ ne dépend pas de l'étiquetage.

Les coefficients de Taylor sont bien définis sur $S^k(\mathfrak{g}_1[1])$. En effet si les α_i sont des k_i -polychamps de vecteurs, alors ils sont de degré

$$k_i - 2 = k_i \pmod{2}$$

dans $\mathfrak{g}_1[1]$. On a alors

$$\mathcal{U}_n(\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_n) = (-1)^{k_i k_{i+1}} \mathcal{U}_n(\alpha_1, \dots, \alpha_{i+1}, \alpha_i, \dots, \alpha_n).$$

En effet la numérotation n'intervient pas dans la définition des opérateurs B_Γ , mais uniquement dans le coefficient w_Γ . Si on intervertit α_i et α_{i+1} alors le signe $(-1)^{k_i k_{i+1}}$ apparaît.

9.4.2. Principe de la démonstration. — L'équation de formalité $\mathcal{U} \circ Q^{(1)} = Q^{(2)} \circ \mathcal{U}$ se développe en une suite (E_n) d'équations exprimant le coefficient de Taylor \mathcal{U}_n en fonction des coefficients de Taylor $Q_2^{(1)}, Q_1^{(2)}, Q_2^{(2)}$ et les \mathcal{U}_k pour $k < n$.

Pour $i = 1, \dots, n$, soit α_i un k_i -polychamps de vecteurs.

Pour $I = \{i_1, \dots, i_p\}$ un sous-ensemble ordonné de $[1, n]$ de cardinal $|I| = p$ on note α_I le produit ordonné $\alpha_{i_1} \dots \alpha_{i_p}$ avec $i_1 < \dots < i_p$.

Pour une partition I, J de $[1, n]$ on introduit le signe de Quillen $\epsilon(I, J)$ défini par la signature impaire (c'est à dire on ne considère que l'effet de la permutation sur

les éléments α_i impairs) de la permutation-battement associée à la partition I, J . En d'autres termes on a dans $S(\mathfrak{g}_1[1])$

$$\alpha_{[1,n]} = \epsilon(I, J)\alpha_I \cdot \alpha_J.$$

On note $\{i, j\}^c$ le complémentaire de la paire $\{i, j\}$. On définit alors $\epsilon_{i,j}$ le signe par

$$\alpha_{[1,n]} = \epsilon_{(i,j)}\alpha_i \cdot \alpha_j \cdot \alpha_{\{i,j\}^c}.$$

On montre que l'équation (E_n) est donné par

$$(9.4.4) \quad Q_1^{(2)}\left(\mathcal{U}_n(\alpha_1 \dots \alpha_n)\right) + \frac{1}{2} \sum_{\substack{[1,n]=I \cup J \\ \text{partition non triviale}}} \epsilon(I, J) Q_2^{(2)}\left(\mathcal{U}_{|I|}(\alpha_I) \cdot \mathcal{U}_{|J|}(\alpha_J)\right) \\ = \frac{1}{2} \sum_{i \neq j} \epsilon_{(i,j)} \mathcal{U}_{n-1}(Q_2^{(1)}(\alpha_i \cdot \alpha_j) \cdot \alpha_{\{i,j\}^c}).$$

Chaque terme de cette formule est un opérateur m -différentiel avec

$$m = \sum k_i - 2n + 2.$$

En remplaçant $Q_2^{(1)}$, Q_1^2 et Q_2^2 par les expressions correspondantes et en notant que l'on a

$$\epsilon(I, J) = (-1)^{k_I |k_J|} \epsilon(J, I) \quad \text{et} \quad \epsilon_{(i,j)} = (-1)^{k_i k_j} \epsilon_{(j,i)}$$

on trouve

$$(9.4.5) \quad -d_H\left(\mathcal{U}_n(\alpha_1 \dots \alpha_n)\right) + \sum_{\substack{[1,n]=I \cup J \\ \text{partition non triviale}}} \epsilon(I, J) (-1)^{k_I |k_J|} \mathcal{U}_{|J|}(\alpha_J) \circ \mathcal{U}_{|I|}(\alpha_I) \\ - \sum_{i \neq j} \epsilon_{(i,j)} \mathcal{U}_{n-1}((\alpha_i \bullet \alpha_j) \cdot \alpha_{\{i,j\}^c}) = 0.$$

Considérons maintenant tous les graphes admissibles $\Gamma \in G_{n,m+1}$ et appliquons la formule de Stokes aux différentiels $d\Omega_\Gamma$ avec $m = \sum k_i - 2n + 2$.

On a donc pour chaque graphe Γ la formule suivante :

$$\sum_{\Gamma \in G_{n,m+1}} \left(\int_{\overline{\mathcal{C}}_{n,m+1}^+} d\Omega_\Gamma \right) B_\Gamma(\alpha_1, \dots, \alpha_n) = 0.$$

L'opérateur $B_\Gamma(\alpha_1, \dots, \alpha_n)$ est un opérateur $m+1$ -différentiel et pour qu'il soit bien défini il faut que le nombre d'arêtes de Γ soit $\sum k_i$. Par ailleurs pour que la formule de Stokes donne un résultat non trivial il faut que $\sum k_i = 2n + m - 2$, ce que nous avons supposé. On écrit maintenant

$$(9.4.6) \quad 0 = \int_{\overline{C_{n,m+1}^+}} d\Omega_\Gamma = \int_{\partial \overline{C_{n,m+1}^+}} \Omega_\Gamma.$$

On exprime alors que l'intégrale sur le bord est la somme des intégrales sur les strates de codimension 1 convenablement orientées comme dans le paragraphe 6.2.1. Pour chaque type de strate, en regroupant alors les termes $B_\Gamma(\alpha_1, \dots, \alpha_n)$ pour les différents Γ qui apparaissent on va retrouver l'équation E_n de formalité. Détaillons tous ceci dans le paragraphe suivant.

9.4.3. Examen des différentes strates. — Les strates de codimension 1 se présentent d'après le paragraphe 6.1 comme un produit $C_{cont} \times C_{ext}$. Si Γ est un graphe on appellera Γ_{int} le sous-graphe (étiqueté) de Γ dont les sommets sont ceux de C_{cont} et les arêtes sont celles qui joignent deux sommets de C_{cont} . On appellera Γ_{ext} le graphe (étiqueté) induit de Γ par en concentrant Γ_{int} (sommets et arêtes) par un sommet (aérien ou terrestre en fonction de la nature de C_{cont}). On notera $\Gamma \mapsto (\Gamma_{int}, \Gamma_{ext})$ cette opération (voir figure 2).

Strate de Type I

Si on a l'amas C_{cont} est aérien et contient plus de trois sommets alors sa dimension est au moins 3. Or d'après le lemme d'annulation rappelé dans la partie Bruguières, l'intégrale de plus de trois formes d'angles est nulles, par conséquent ces strates n'interviennent pas.

Strate de Type II

Si l'amas C_{cont} ne contient que 2 points aériens, alors on est dans la situation C_2 , variété de dimension 1. Pour avoir une contribution non nulle dans le coefficient il faut que le graphe interne soit réduit à une arête que l'on suppose être l'arête (1 \mapsto 2). On a $\Gamma_{int} = (1 \mapsto 2)$ et on note Γ_{ext} le graphe externe induit à partir de Γ en contractant l'arête (1 \mapsto 2).

C'est un graphe dans $G_{n-1,m+1}$ avec $(\sum k_i) - 1 = 2n + m - 3 = 2(n-1) + (m+1) - 2$ arêtes. En particulier l'opérateur $B_{\Gamma_{ext}}(\alpha_1 \bullet \alpha_2, \alpha_3, \dots, \alpha_n)$ est bien défini. C'est un opérateur $m + 1$ différentiel.

Le graphe Γ_{ext} hérite de Γ d'un ordre (étiquetage) sur l'ensemble de ses arêtes. Réciproquement étant donné Δ , graphe de $G_{n-1,m+1}$ il existe r graphes (étiquetés, ordonnés) Γ tel que $\Gamma_{ext} = \Delta$. La contribution de Ω_Γ sur ces strates de bord est donc

$$-(-1)^{r-1} w_\Delta,$$

où r est la position de l'arête (1 \mapsto 2) parmi celles issues du sommet 1.

Fixons $\Delta_o \in G_{n-1,m+1}$ avec un ordre privilégié par rapport au premier sommet, c'est à dire les arêtes issues de ce sommet sont ordonnées selon l'ordre lexicographique.

Un calcul attentif de [72] lemme 4.2⁽⁴⁾ montre que l'on a :

$$(9.4.7) \quad \begin{aligned} (k_1 + k_2 - 1)! B_{\Delta_0}((\alpha_1 \bullet \alpha_2), \alpha_3, \dots, \alpha_n) &= \sum_{\substack{\Delta \\ \Delta_{priv} = \Delta_0}} \varepsilon(\Delta) B_{\Delta}((\alpha_1 \bullet \alpha_2), \alpha_3, \dots, \alpha_n) \\ &= \sum_{\substack{\Delta \\ \Delta_{priv} = \Delta_0}} \varepsilon(\Delta) \sum_{r=1}^{k_1} \sum_{\substack{\Gamma_{ext} = \Delta \\ \Gamma_{int} = (1 \mapsto 2) \\ \text{en position } r}} (-1)^{r-1} B_{\Gamma}(\alpha_1, \dots, \alpha_n), \end{aligned}$$

où la somme interne porte sur les $\Gamma \in G_{n, m+1}$ qui après contraction de l'arête $(1 \mapsto 2)$ (ordonné en position r) donne Δ . On en déduit que l'on a

$$\sum_{\substack{\Delta \\ \Delta_{priv} = \Delta_0}} w_{\Delta} B_{\Delta}(\alpha_1 \bullet \alpha_2, \dots, \alpha_n) = \sum_{r=1}^{k_1} \sum_{\substack{\Delta \\ \Delta_{priv} = \Delta_0}} (-1)^{r-1} w_{\Delta} \sum_{\substack{\Gamma_{ext} = \Delta \\ \Gamma_{int} = (1 \mapsto 2) \\ \text{en position } r}} B_{\Gamma}(\alpha_1, \alpha_2, \dots, \alpha_n).$$

En sommant sur Δ on en déduit que la contribution de ces strates de type II avec contraction de l'arête $(1 \mapsto 2)$ vaut donc

$$-U_{n-1}((\alpha_1 \bullet \alpha_2), \dots, \alpha_n)$$

Lorsque la contraction concerne une arête $(i \mapsto j)$, alors on se ramène au cas précédent en ramenant les sommets i, j aux positions 1, 2, alors le signe $\epsilon_{(i,j)}$ apparaît. Les contributions de ces strates de type II donnent donc le terme

$$-\sum_{i \neq j} \epsilon_{(i,j)} \mathcal{U}_{n-1}((\alpha_i \bullet \alpha_j) \cdot \alpha_{\{i,j\}^c})$$

Strate de Type III

L'amas C_{cont} est terrestre, réduit à deux sommets consécutifs $(l, l+1)$ comme à la figure 1 et ne contient aucun point aérien. Donc C_{ext} contient tous les points aériens. On suppose avoir mis les fonctions f_1, \dots, f_{m+1} aux points terrestres. En tenant compte de l'orientation des strates on trouve facilement

$$(9.4.8) \quad (-1)^{(l-1)} \mathcal{U}_n(\alpha_1, \dots, \alpha_n)(f_1, \dots, f_l f_{l+1}, f_{l+2}, \dots, f_{m+1}).$$

L'amas C_{ext} est réduit à deux sommets terrestres et ne contient pas de points aériens. Donc C_{cont} contient m points terrestres et tous les points aériens. On trouve alors les deux contributions suivantes

$$(9.4.9) \quad -f_1 \left(\mathcal{U}_n(\alpha_1, \dots, \alpha_n)(f_2, \dots, f_{m+1}) \right) + (-1)^m \left(\mathcal{U}_n(\alpha_1, \dots, \alpha_n)(f_1, \dots, f_m) \right) f_m.$$

⁽⁴⁾Il faut corriger le lemme en question comme ci-dessous en tenant compte de la position de l'arête déployée dans parmi celle issues du sommet 1

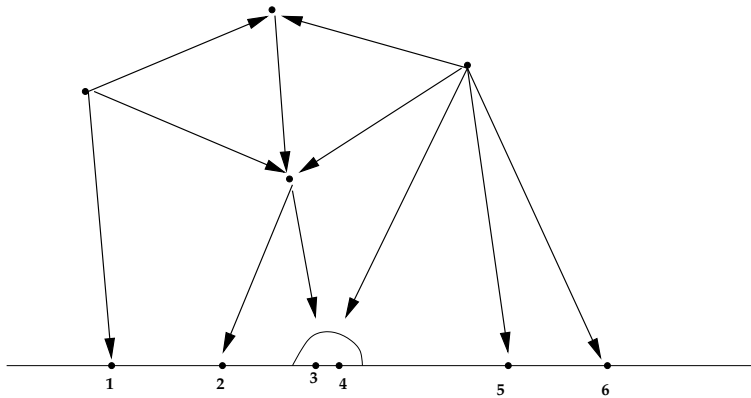


FIGURE 1. Strate de type III

La somme de ces deux contributions (III+III-bis) donne le terme

$$-d_H(\mathcal{U}_n(\alpha_1, \dots, \alpha_n))$$

Strate de Type IV

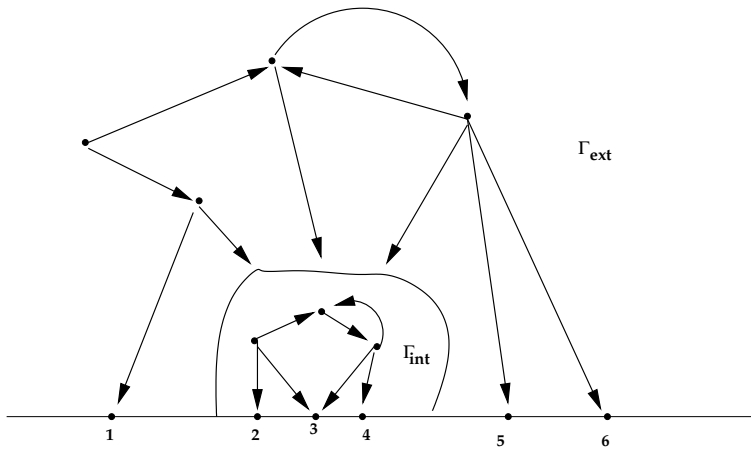


FIGURE 2. Strate de type IV

L'amas C_{cont} est terrestre et contient m_1 points terrestres consécutifs et $n_1 < n$ points aériens numérotés i_1, \dots, i_{n_1} . On suppose que la concentration se fait en position $l + 1$ figure 2. Pour avoir une contribution non nulle de Ω_Γ sur cette strate il faut que le graphe Γ_{ext} n'ait pas d'arêtes issues de l'axe réel. Donc le graphe Γ_{int}

contienne toutes ses arêtes. Notons $J = \{j_1, \dots, j_{n-n_1}\}$ les sommets aériens du graphe Γ_{ext} . Il contient $m+1 - m_1 + 1$ points terrestres. Pour que cette contribution de Ω_Γ sur la strate soit non nulle il faut que

$$\begin{aligned} k_{|I|} &:= k_{i_1} + \dots + k_{i_{n_1}} = 2n_1 + m_1 - 2 && \text{condition sur } \Gamma_{int} \\ k_{|J|} &:= k_{j_1} + \dots + k_{j_{n-n_1}} = 2(n - n_1) + (m + 1 - m_1 + 1) - 2 && \text{condition sur } \Gamma_{ext} \end{aligned}$$

Par conséquent le contribution de Ω_Γ sur cette strate est le produit

$$(-1)^{l+m_1+lm_1} \epsilon(I, J) w_{\Gamma_{int}} w_{\Gamma_{ext}}.$$

En regroupant les contributions des graphes Γ tels que $\Gamma \mapsto (\Gamma_{ext}, \Gamma_{int})$ et en utilisant la formule de Leibniz on voit facilement que l'on a

$$(9.4.10) \quad \sum_{\Gamma \mapsto (\Gamma_{ext}, \Gamma_{int})} B_\Gamma(\alpha_1, \dots, \alpha_n) = B_{\Gamma_{ext}}(\alpha_J) \left(f_1, \dots, f_l, B_{\Gamma_{int}}(\alpha_I)(f_{l+1}, \dots, f_{l+m_1}), f_{l+m_1+1}, \dots, f_{m+1} \right).$$

En tenant compte des différentes position l et de la définition (9.2.1) la contribution des strates de ce type donne le terme

$$(-1)^{m_1} \epsilon(I, J) \left(w_{\Gamma_{ext}} B_{\Gamma_{ext}}(\alpha_J) \right) \circ \left(w_{\Gamma_{int}} B_{\Gamma_{int}}(\alpha_I) \right).$$

Comme $(-1)^{m_1} = (-1)^{k_{|I|}}$ on trouve finalement

$$\boxed{\sum_{\substack{[1, n] = I \cup J \\ \text{partition non triviale}}} \epsilon(I, J) (-1)^{k_{|I|}} \mathcal{U}_{|J|}(\alpha_J) \circ \mathcal{U}_{|I|}(\alpha_I).$$

Au final on a démontré la formule (9.4.5) et le théorème de Kontsevich du L^∞ -quasi-isomorphisme.

9.5. Lien avec la quantification par déformation

Le théorème de formalité est relié à la quantification par déformation de la manière suivante.⁽⁵⁾ Soit $\mathfrak{m} = \hbar \mathbb{R}[[\hbar]]$ la limite projective des algèbres nilpotentes de dimension finie $\mathfrak{m}_r = \hbar \mathbb{R}[[\hbar]] / \hbar^r \mathbb{R}[[\hbar]]$. Soit $\hbar\gamma$ un 2-vecteur de Poisson.

C'est une solution dans $\mathfrak{g}_1 \widehat{\otimes} \mathfrak{m}$ de l'équation de Maurer-Cartan :

$$Q^{(1)}(\hbar\gamma) = \hbar d\gamma - \frac{1}{2} [\hbar\gamma, \hbar\gamma] = 0.$$

⁽⁵⁾Voir aussi théorème 3.6.2 la partie de B. Keller

Soit $\hbar\hat{\gamma} = \overline{U}(\hbar\gamma) = \sum_{k \geq 1} \frac{\hbar^k}{k!} \mathcal{U}_k(\gamma \cdot^k)$, l'image par \overline{U} du 2-vecteur de Poisson $\hbar\gamma$. Le théorème de formalité dit que l'élément $\hbar\hat{\gamma}$ vérifie l'équation de Maurer-Cartan dans $\mathfrak{g}_2 \widehat{\otimes} \mathfrak{m}$

$$Q^{(2)}(\hbar\hat{\gamma}) = \hbar d\hat{\gamma} - \frac{1}{2}[\hbar\hat{\gamma}, \hbar\hat{\gamma}]_G = 0.$$

Soit m l'opérateur bidifférentiel de multiplication :

$$f \otimes g \mapsto fg,$$

alors $*_{\hbar\hat{\gamma}} = m + \hbar\hat{\gamma}$ est un produit associatif car il vérifie l'équation $[\ast, \ast]_G = 0$. On a donc, grâce au théorème de Formalité, associé à tout crochet de Poisson un étoile-produit.

9.6. Quasi-isomorphisme tangent et la formule d'homotopie

9.6.1. Rappels des résultats de Mochizuki. — Nous avons rappelé en au paragraphe 6.1 la définition des compactifications des espaces de configurations $\overline{C}_{n,m}$. Comme une variation de ces objets Mochizuki considère les espaces $X_{n,m}^\ell$ définis pour tout entier positif ℓ de façon suivante.

Soit $C_{\ell+1,0}^{\mathbb{R}}$ l'ensemble des configurations de $\ell + 1$ points ordonnés sur l'axe réel, modulo l'action du groupe $x \mapsto ax + b$, $a > 0, b \in \mathbb{R}$. C'est une variété de dimension $\ell - 1$ (si $\ell > 0$). Soit Δ^ℓ le simplexe de dimension ℓ défini par

$$\{(y_0, \dots, y_\ell) \in \mathbb{R}^{\ell+1}, y_j \geq 0, \sum y_j = 1\}.$$

Pour $\ell > 0$ l'intérieur de $\Delta^{\ell-1}$ s'identifie à $C_{\ell+1,0}^{\mathbb{R}}$, et on a $\overset{\circ}{\Delta}^\ell \simeq \overset{\circ}{\Delta}^{\ell-1} \times \mathbb{R}_+^*$.

On obtient ainsi l'application $\psi : \overset{\circ}{\Delta}^\ell \simeq C_{\ell+1,0}^{\mathbb{R}} \times \mathbb{R}_+ \rightarrow C_{\ell+1,0}$ définie par $\psi(x, t) = (x_0 + it, \dots, x_\ell + it)$.

Notons alors $\overset{\circ}{X}_{n,m}^\ell = C_{n,m}^+ \times_{C_{\ell+1,0}} \overset{\circ}{\Delta}^\ell$ et $X_{n,m}^\ell$ son adhérence dans $\overline{C}_{n,m}$. On obtient ainsi une variété à coins de dimension $2n + m - 2 - \ell$. Cela revient heuristiquement à imposer que les $\ell + 1$ premiers points aériens se déplacent à la même hauteur et avec abscisses ordonnés.

Un graphe $\Gamma \in G_{n,m}$ est dit ℓ -admissible ($\ell \in \mathbb{N}$) si Γ est admissible de type (n, m) avec n sommets de premier type, m sommets du second type, $n \geq \ell + 1$ et $2n + m - 2 - \ell$ arêtes.

On définit alors les poids de Mochizuki des graphes ℓ -admissibles par (notations (6.6.1))

$$(9.6.1) \quad \boxed{w_\Gamma^\ell := \int_{X_{n,m}^\ell} \Omega_\Gamma.}$$

Suivant [77] considérons les applications $\mathcal{U}_{n,m}^{\ell,\gamma}$ définies par

$$(9.6.2) \quad \begin{aligned} \mathcal{U}_{n,m}^{\ell,\gamma} : \bigotimes^{\ell+1} (T_{\text{poly}}(\mathbb{R}^d)[1]) &\rightarrow D_{\text{poly}}(\mathbb{R}^d)[1 + \ell] \\ \mathcal{U}_{n,m}^{\ell,\gamma}(\alpha_1 \otimes \cdots \otimes \alpha_{\ell+1}) &:= \sum_{\Gamma \in G_{n,m}^\ell} w_\Gamma^\ell \cdot B_\Gamma(\alpha_1 \otimes \cdots \otimes \alpha_{\ell+1} \otimes \gamma \otimes \cdots \otimes \gamma). \end{aligned}$$

9.6.2. Complexes Tangents. — Nous nous intéressons maintenant à la différentielle $d\bar{\mathcal{U}}_{\hbar\gamma}$ du morphisme de variétés formelles \mathcal{U} au point $\hbar\gamma$. L'espace tangent à $\mathfrak{g}_1[1]$ en $\hbar\gamma$ s'identifie à $\mathfrak{g}_1[1] \widehat{\otimes} \mathfrak{m}$. La linéarisation du champ de vecteurs impair $Q^{(1)}$ qui s'annule au point $\hbar\gamma$ donne un champ de vecteurs impair $Q^{\hbar\gamma}$ sur cet espace tangent de carré nul et qui s'écrit :

$$(9.6.3) \quad Q^{\hbar\gamma}(\hbar\delta) = Q_2^{(1)}(\hbar\gamma.\hbar\delta) = \hbar^2[\delta, \gamma].$$

(Voir [6] § II.4 et IV.1). On linéarise de même le champ de vecteurs impair $Q^{(2)}$ qui s'annule en $\hbar\widehat{\gamma} = \bar{\mathcal{U}}(\hbar\gamma)$. On obtient ainsi un champ de vecteurs impair $Q^{\hbar\widehat{\gamma}}$ de carré nul sur l'espace tangent à $\mathfrak{g}_2[1]$ qui s'écrit :

$$(9.6.4) \quad Q^{\hbar\widehat{\gamma}}(\hbar\delta) = Q_1^{(2)}(\hbar\delta) + Q_2^{(2)}(\hbar\widehat{\gamma}.\hbar\delta) = \hbar[\delta, *].$$

Les deux espaces tangents ci-dessus sont donc ainsi munis d'une structure de complexe de cochaînes. La dérivée $d\bar{\mathcal{U}}_{\hbar\gamma}$ du L_∞ -morphisme \mathcal{U} est un quasi-isomorphisme de complexes du premier espace tangent vers le deuxième, et s'exprime pour $\delta \in \mathfrak{g}_1[1]$ en étendant par \mathfrak{m} -linéarité la formule :

$$(9.6.5) \quad d\bar{\mathcal{U}}_{\hbar\gamma}(\delta) = \sum_{n \geq 0} \frac{\hbar^n}{n!} \mathcal{U}_{n+1}(\delta.\gamma^n).$$

Plus généralement, suivant [77] on définit les applications

$$\begin{aligned} d\mathcal{U}^{\ell,\gamma} : \bigotimes^{\ell+1} T_{\text{poly}}[1] &\rightarrow D_{\text{poly}}[1 + \ell][[\hbar]] \\ d\mathcal{U}^{\ell,\gamma}(\alpha_1 \otimes \cdots \otimes \alpha_{\ell+1}) &= \sum_{n,m} \frac{\hbar^{n-\ell-1}}{(n-\ell-1)!} \mathcal{U}_{n,m}^{\ell,\gamma}(\alpha_1 \otimes \cdots \otimes \alpha_{\ell+1}). \end{aligned}$$

Notons que l'on a en particulier $d\overline{\mathcal{U}}_{\hbar\gamma} = d\mathcal{U}^{0,\gamma}$. Ces applications $(d\mathcal{U}^{\ell,\gamma})_{\ell \geq 0}$ définissent un A_∞ -quasi isomorphisme pour les complexes tangents ([77] §3.4).⁽⁶⁾

L'opérateur de cobord du complexe des cochaînes tangentielles de la première algèbre de Lie différentielle graduée $T_{\text{poly}}(X)[1]$ est donné par $Q^{\hbar\gamma} = -[\hbar\gamma, -]_{SN}$ qui est une dérivation graduée pour le produit extérieur \wedge des polychamps de vecteurs (graduation standard).

Ce produit extérieur induit donc un produit associatif et commutatif que l'on notera \cup sur l'espace de la cohomologie $H_{\hbar\gamma}$ du premier espace tangent.

Sur le deuxième espace tangent on introduit un produit associatif gradué donné par la formule suivante⁽⁷⁾ :

(9.6.6)

$$(A_1 \cup A_2)(f_1 \otimes \cdots \otimes f_{m_1+m_2}) = A_1(f_1 \otimes \cdots \otimes f_{m_1}) \star_{\hbar\gamma} A_2(f_{m_1+1} \otimes \cdots \otimes f_{m_2})$$

pour tout opérateur m_1 -différentiel A_1 et tout opérateur m_2 -différentiel A_2 . Cette opération est compatible avec le cobord $[-, \star]_G$ du deuxième complexe des cochaînes tangentielles et elle induit donc un cup-produit sur l'espace de la cohomologie $H_{\overline{\mathcal{U}}(\hbar\gamma)}$ du deuxième espace tangent⁽⁸⁾.

Le théorème suivant est démontré avec différents degrés de précision dans [60] §8, [72] Théorème 1.2.⁽⁹⁾ et [77] §3.4.

Théorème 9.6.1. — *Soit $X = \mathbb{R}^d$ et \mathcal{U} le L_∞ -quasi-isomorphisme donné explicitement dans [60] §6.4. La dérivée $d\overline{\mathcal{U}}_{\hbar\gamma}$ induit un isomorphisme d'algèbres de l'espace de la cohomologie $H_{\hbar\gamma}$ de l'espace tangent $T_{\hbar\gamma}(\mathfrak{g}_1[1])$ sur l'espace de la cohomologie $H_{\overline{\mathcal{U}}(\hbar\gamma)}$ de l'espace tangent $T_{\overline{\mathcal{U}}(\hbar\gamma)}(\mathfrak{g}_2[1])$.*

Plus précisément pour $\alpha \in T_{\text{poly}}^{|\alpha|}(X)$ et $\beta \in T_{\text{poly}}^{|\beta|}(X)$ on a

$$(9.6.7) \quad d\mathcal{U}^0(\alpha \cup \beta) - d\mathcal{U}^0(\alpha) \cup d\mathcal{U}^0(\beta) = \\ -[\star_\gamma, d\mathcal{U}^1(\alpha \otimes \beta)]_G + d\mathcal{U}^1(-[\hbar\gamma, \alpha]_{SN} \otimes \beta - (-1)^{|\alpha|+1} \alpha \otimes [\hbar\gamma, \beta]_{SN}),$$

⁽⁶⁾Il nous semble que dans la définition des graphes ℓ -admissible de Mochizuki, la restriction de ne pas avoir d'arêtes entre les sommets (i, j) pour $1 \leq i, j \leq \ell + 1$ est trop forte. En effet la formule de Stokes pour les strates de type I (dans l'article [77]), fait apparaître des graphes qui ne sont plus ℓ -admissible en ce sens. A notre avis, il faut considérer tous les graphes admissibles au sens de Kontsevich, et les différentes strates donnent bien le A_∞ -morphisme tangent

⁽⁷⁾Dans le complexe décalé $\mathfrak{g}[1]$ les fonctions sont de degré -2 , donc il n'y a pas de signe

⁽⁸⁾Par rapport à [60] §8, nous plaçons dans le cas simplifié où \mathfrak{m} est supposée concentrée en degré zéro.

⁽⁹⁾Dans [72] une erreur mineure de signe dans la proposition 4.1 et le théorème 4.6 due à une confusion dans l'utilisation du lemme 4.2 est corrigée dans un Erratum [70]

9.7. Isomorphisme de Duflo pour la cohomologie

On va maintenant appliquer le théorème de formalité tangente au cas des structures de Poisson linéaire.

Identifions les espaces qui interviennent : les applications préservent les fonctions polynomiales, donc on peut passer aux algèbres de polynômes.

On a d'un coté la cohomologie des polychamps de vecteurs pour le crochet de Poisson $[-, \pi]$. Cette cohomologie s'identifie donc à la cohomologie de Poisson de $S(\mathfrak{g})$. D'un autre coté on a la cohomologie de Hochschild de l'algèbre $S(\mathfrak{g})$ munie de l'étoile produit. Il s'agit donc de la cohomologie de Hochschild de $U(\mathfrak{g})$ qui se calcule aussi par le complexe de Koszul.

Le théorème de formalité tangente nous dit que ces deux algèbres sont isomorphes par l'isomorphisme explicite tangent.⁽¹⁰⁾ On a donc

$$H_{Poisson}(\mathfrak{g}, S(\mathfrak{g})) \xrightarrow{\sim} HH^*(U(\mathfrak{g}), U(\mathfrak{g})).$$

où $HH^*(U(\mathfrak{g}), U(\mathfrak{g}))$ désigne l'algèbre de cohomologie de Hochschild. Par ailleurs l'application naturelle d'antisymétrisation permet d'identifier cette dernière avec $H^*(\mathfrak{g}, U(\mathfrak{g}))$. On dispose donc d'un isomorphisme d'algèbres de $H^*(\mathfrak{g}, S(\mathfrak{g}))$ sur $H^*(\mathfrak{g}, U(\mathfrak{g}))$.

Ce théorème en degré zéro redonne l'existence de l'isomorphisme de Duflo entre $S(\mathfrak{g})^{\natural}$ et $U(\mathfrak{g})^{\natural} = (S(\mathfrak{g})^{\natural}, \star)$. Dans [82] on montre (conjecture de [90]) que c'est en fait l'identité quand on identifie $U(\mathfrak{g})$ avec $S(\mathfrak{g})$ muni de l'étoile-produit de Kontsevich, c'est à dire la formule de Duflo s'étend à toute la cohomologie en un isomorphisme d'algèbre.

Théorème 9.7.1. — *L'application de Duflo s'étend en un isomorphisme d'algèbres de $H^*(\mathfrak{g}, S(\mathfrak{g}))$ sur $H^*(\mathfrak{g}, U(\mathfrak{g}))$.*

⁽¹⁰⁾Voir aussi théorème 2.3.2 de la partie de B. Keller

PART III

DEFORMATION QUANTIZATION FROM
FUNCTIONAL INTEGRALS
(BY A. CATTANEO)

CHAPTER 10

INTRODUCTION

The aim of this Part is to explain how to obtain Kontsevich's formula [60] from the perturbative computation of the functional integral of a topological field theory known as the Poisson sigma model. We start with an introduction to the perturbative evaluation of functional integrals. We describe next how to do it in the presence of symmetries generated by the free action of a Lie algebra. This allows the full treatment of the Poisson sigma model for an affine Poisson structure. For the general case, we refer to [16].

Kontsevich proposed in [60] a formula for the deformation quantization of a Poisson structure on \mathbb{R}^n . This formula is given by a sum over certain graphs of a bidifferential operator and a real number associated to each graph. A typical graph is displayed in figure 1. The vertices are free to move in the upper half plane but have to avoid each

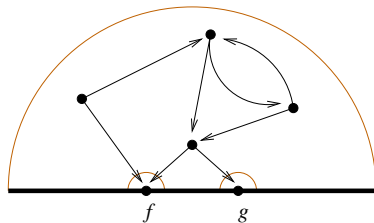


FIGURE 1. A term in the Kontsevich formula.

other and the two points on the boundary where one places the functions whose star product is wished. The number associated to the graph is obtained by integrating certain one-forms associated to the edges over the positions of the vertices. Deforming the integration domain as in figure 2, where the three boundary segments correspond to the half circles around the two boundary points and infinity in the previous picture, we see that Kontsevich formula is related to some sort of open string theory. The

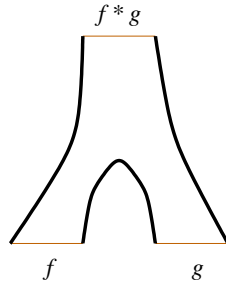


FIGURE 2. The Kontsevich formula as a string evolution.

associativity of the star product can then be expressed by saying that the two string evolutions in figure 3 are equivalent. Thus, the dynamics of this peculiar string theory

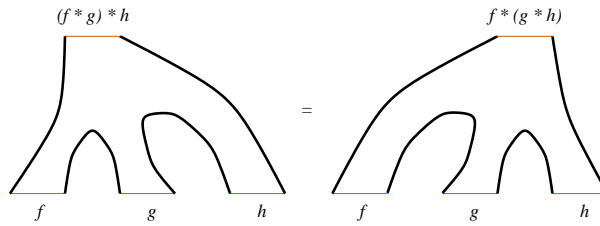


FIGURE 3. Associativity.

must be invariant under diffeomorphisms of the worldsheet, and this invariance has to persist at the quantum level. A string or field theory with this property is usually referred to as topological.

The aim of this chapter is to give a crash course on the perturbative quantization of field theories via path integrals, with special attention for the case of the topological string theory leading to the Kontsevich formula: the Poisson sigma model.

CHAPTER 11

FUNCTIONAL INTEGRALS

11.1. Functional integrals and expectation values

A field theory is defined by specifying a space of fields \mathcal{M} (usually a space of maps or, more generally, of sections) and an action function S (a function on \mathcal{M} which is defined by integrating over the source manifold a function of the fields and their jets). For a given function \mathcal{O} on \mathcal{M} , one considers the expectation value

$$(11.1.1) \quad \langle \mathcal{O} \rangle := \frac{\int_{\mathcal{M}} e^{\frac{i}{\hbar} S} \mathcal{O}}{\int_{\mathcal{M}} e^{\frac{i}{\hbar} S}}.$$

The integrals appearing in the above fraction are called **path integrals** or **functional integrals** (the latter term is more general, the former should be reserved for the case of quantum mechanics where the fields are actually paths) and are computed w.r.t. a fictitious measure on the space of fields. Functions on \mathcal{M} whose expectation value is well-defined are called **observables**.

The perturbative evaluation of such integrals consists in expanding S around a nondegenerate critical point and in defining the integral as a formal power series in the expansion parameter \hbar with coefficients given by Gaussian expectation values (see section 11.2 and section 11.4). As a simple application, we will recover in section 11.3 the Moyal formula from the path-integral formulation of quantum mechanics on $T^*\mathbb{R}^n$.

Often, however, (and in particular in the case of interest for us) the critical point is degenerate due to **symmetries**, viz., a distribution of vector fields on \mathcal{M} under which the action function S is invariant. When these symmetries are given by the infinitesimal free action of a Lie algebra, there is a nice algebraic way, the **BRST formalism** [11, 98], to describe the perturbative expansion of the functional integral on the quotient space (see Chapter 12). Observables in this setting will be understood as *invariant* functions whose expectation values may be defined. This method will allow us to compute Kontsevich's formula in the case of affine Poisson structures (see Chapter 14). A more general method, the **BV formalism** [9], allows one to deal with

more general distributions of vector fields describing the symmetries; this method (in the version described in [2]) is needed, e.g., to obtain Kontsevich's formula for general Poisson structures, see [16].

11.2. Gaussian integrals

Let A be a positive-definite symmetric matrix on \mathbb{R}^n , which we assume to be endowed with the Lebesgue measure $d^n x$ and the Euclidean inner product (\cdot, \cdot) . Then

$$I(\lambda) := \int_{\mathbb{R}^n} e^{-\frac{\lambda}{2}(x, Ax)} d^n x = \frac{(2\pi)^{\frac{n}{2}}}{\lambda^{\frac{n}{2}}} \frac{1}{\sqrt{\det A}}, \quad \lambda > 0.$$

We may continue I to the whole complex plane minus the negative real axis. Then we get, e.g.,

$$I(-i) = (2\pi)^{\frac{n}{2}} e^{\frac{i\pi n}{4}} \frac{1}{\sqrt{\det A}}, \quad I(i) = (2\pi)^{\frac{n}{2}} e^{-\frac{i\pi n}{4}} \frac{1}{\sqrt{\det A}}.$$

These formulae allow us to define the Gaussian integral when A is negative definite:

$$\int_{\mathbb{R}^n} e^{\frac{i}{2}(x, Ax)} d^n x = \int_{\mathbb{R}^n} e^{-\frac{i}{2}(-x, Ax)} d^n x = (2\pi)^{\frac{n}{2}} e^{-\frac{i\pi n}{4}} \frac{1}{\sqrt{|\det A|}}.$$

When A is nondegenerate—but not necessarily positive or negative definite—by combining the above we get

$$(11.2.1) \quad \int_{\mathbb{R}^n} e^{\frac{i}{2}(x, Ax)} d^n x = (2\pi)^{\frac{n}{2}} e^{\frac{i\pi \operatorname{sign} A}{4}} \frac{1}{\sqrt{|\det A|}},$$

where $\operatorname{sign} A$ denotes the signature of A .

Expectation values w.r.t. a Gaussian distribution, which we will denote by $\langle \cdot \rangle_0$, are easy to compute. The first step is to define the generating function

$$Z(J) = \int_{\mathbb{R}^n} e^{\frac{i}{2}(x, Ax) + (J, x)} d^n x = (2\pi)^{\frac{n}{2}} e^{\frac{i\pi \operatorname{sign} A}{4}} \frac{1}{\sqrt{|\det A|}} e^{\frac{i}{2}(J, A^{-1}J)}.$$

Then we can write

$$(11.2.2) \quad \begin{aligned} \langle x^{i_1} \dots x^{i_k} \rangle_0 &= \frac{\int_{\mathbb{R}^n} e^{\frac{i}{2}(x, Ax)} x^{i_1} \dots x^{i_k} d^n x}{\int_{\mathbb{R}^n} e^{\frac{i}{2}(x, Ax)} d^n x} = \\ &= \frac{\frac{\partial}{\partial J^{i_1}} \dots \frac{\partial}{\partial J^{i_k}} Z(J)|_{J=0}}{Z(0)} = \frac{\partial}{\partial J^{i_1}} \dots \frac{\partial}{\partial J^{i_k}} e^{\frac{i}{2}(J, A^{-1}J)}|_{J=0}. \end{aligned}$$

Observe then that $\langle x^{i_1} \dots x^{i_k} \rangle_0$ vanishes if k is odd and is a sum of products of matrix elements of the inverse of A if k is even. For example, for $k = 2$ we have $\langle x^i x^j \rangle_0 = i(A^{-1})^{ij}$, and for $k = 2s$

$$\langle x^{i_1} \dots x^{i_{2s}} \rangle_0 = i^s \sum_{\sigma \in \mathfrak{S}_{2s}} \frac{1}{2^s s!} (A^{-1})^{i_{\sigma(1)} i_{\sigma(2)}} \dots (A^{-1})^{i_{\sigma(2s-1)} i_{\sigma(2s)}},$$

where \mathfrak{S}_{2s} is the symmetric group on $2s$ elements. This formula may be simplified if we sum only over pairings, viz., permutations $\sigma \in \mathfrak{S}_{2s}$ with the property that $\sigma(2i-1) < \sigma(2i)$, $i = 1, \dots, s$, and $\sigma(1) < \sigma(3) < \dots < \sigma(2s-3) < \sigma(2s-1)$. Denoting by $P(s)$ the set of pairings of $2s$ elements, we then have the so-called Wick theorem:

$$(11.2.3) \quad \langle x^{i_1} \dots x^{i_{2s}} \rangle_0 = i^s \sum_{\sigma \in P(s)} (A^{-1})^{i_{\sigma(1)} i_{\sigma(2)}} \dots (A^{-1})^{i_{\sigma(2s-1)} i_{\sigma(2s)}}.$$

11.2.1. Infinite dimensions. — The extension to the infinite-dimensional case is done by taking formula (11.2.3) verbatim. This makes sense whenever the symmetric operator A is invertible.⁽¹⁾ Usually A is a differential operator and in this case $G = A^{-1}$ will denote the distributional kernel of its inverse, i.e., its Green function. Thus, for example, if A is a differential operator on functions on some manifold Σ , then

$$\begin{aligned} \langle \phi(x_1) \dots \phi(x_{2s}) \rangle_0 &= \frac{\int e^{\frac{i}{2\hbar} \int_{\Sigma} \phi A \phi} \phi(x_1) \dots \phi(x_{2s}) D\phi}{\int e^{\frac{i}{2\hbar} \int_{\Sigma} \phi A \phi} D\phi} := \\ &:= (i\hbar)^s \sum_{\sigma \in P(s)} G(x_{\sigma(1)}, x_{\sigma(2)}) \dots G(x_{\sigma(2s-1)}, x_{\sigma(2s)}), \end{aligned}$$

where ϕ denotes a function on Σ , \hbar is a parameter that we have introduced for further convenience, $D\phi$ denotes the “formal Lebesgue measure” on the space of functions, and the points x_1, \dots, x_{2s} are assumed to be all distinct.

Normal ordering. — The formula is usually extended also to the singular case when some points coincide by restricting the sum to pairings with the property that $x_{\sigma(2i-1)} \neq x_{\sigma(2i)} \forall i$. This prescription often goes under the name of **normal ordering** as it corresponds to the usual normal ordering in the operator formulation of Gaussian field theories.

The propagator. — As we have seen, in Gaussian integrals all expectation values are given in terms of the expectation value of quadratic monomials. These are usually called **two-point functions** or **propagators**. In the case of a Gaussian quantum field theory defined by a differential operator, propagator is thus just another name for the Green function.

⁽¹⁾When it is possible to define the determinant and the signature of the invertible operator A , one can also define the “normalized” Gaussian integral by (11.2.1) dropping the irrelevant constant “ $(2\pi)^{\frac{\infty}{2}}$ ”.

Derivatives. — By linearity one also extends the definition of expectation values to derivatives of the field ϕ . Namely, for multiindices I_1, \dots, I_{2s} one sets

$$(11.2.4) \quad \langle \partial_{I_1} \phi(x_1) \dots \partial_{I_{2s}} \phi(x_{2s}) \rangle_0 = \\ = (i\hbar)^s \frac{\partial^{|I_1|}}{\partial x_1^{I_1}} \dots \frac{\partial^{|I_{2s}|}}{\partial x_{2s}^{I_{2s}}} \sum_{\sigma \in P(s)} G(x_{\sigma(1)}, x_{\sigma(2)}) \dots G(x_{\sigma(2s-1)}, x_{\sigma(2s)}),$$

where the derivatives on the r.h.s. are in the distributional sense.

11.2.2. A special case. — Let $n = 2m$ and assume

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix},$$

where B is a nondegenerate $m \times m$ -matrix and B^T denotes its transpose. By regarding $x \in \mathbb{R}^{2m}$ as $y \oplus z$, with $y, z \in \mathbb{R}^m$, we may compute

$$\int_{\mathbb{R}^{2m}} e^{i(y, Bz)} d^m y d^m z = \int_{\mathbb{R}^{2m}} e^{\frac{i}{2}(x, Ax)} d^{2m} x = (2\pi)^m \frac{1}{|\det B|},$$

and, for $K, L \in \mathbb{R}^m$,

$$Z(K, L) = \int_{\mathbb{R}^{2m}} e^{i(y, Bz) + (K, y) + (L, z)} d^m y d^m z = (2\pi)^m \frac{1}{|\det B|} e^{i(L, B^{-1}K)}.$$

In this case, the expectation value of a monomial vanishes if the degree in y is different from the degree in z and is a sum of products of matrix elements of B^{-1} otherwise:

$$(11.2.5) \quad \langle y^{i_1} \dots y^{i_s} z^{j_1} \dots z^{j_s} \rangle_0 = i^s \sum_{\sigma \in \mathfrak{S}_s} (B^{-1})^{j_{\sigma(1)} i_1} \dots (B^{-1})^{j_{\sigma(s)} i_s}.$$

The infinite-dimensional generalization follows the same lines as before. For an example, see section 11.3.

11.2.3. Grassmann variables. — A theory of integration, and in particular of Gaussian integration, may be developed also for Grassmann variables. Let V be a vector space. Its algebra of functions is the completion of the symmetric algebra SV^* , i.e., the algebra of polynomials. Its odd counterpart is the exterior algebra ΛV^* which is regarded as the algebra of functions on the odd vector space ΠV . Choosing a basis and an orientation, one may identify $\Lambda^{\text{top}} V^*$ with \mathbb{R} . Composing this isomorphism with the projection $\Lambda V^* \rightarrow \Lambda^{\text{top}} V^*$ defines a linear map $\Lambda V^* \rightarrow \mathbb{R}$ that we denote by $\int_{\Pi V}$ and call the integral on ΠV .

Let now B be an endomorphism of V . We may regard B as an element of $V^* \otimes V$ and so as a function on $\Pi V^* \times \Pi V := \Pi(V^* \oplus V)$. Up to a sign, there is a natural identification of $\Lambda^{\text{top}}(V^* \oplus V)$ with \mathbb{R} . Then it is not difficult to see that (up to a sign, which we fix from now on to agree with this formula)

$$\int_{\Pi V^* \times \Pi V} e^B = \det B.$$

If \mathbf{B} is nondegenerate, one defines the expectation value of a function f on $\Pi V^* \times \Pi V$ by

$$\langle f \rangle_0 := \frac{\int_{\Pi V^* \times \Pi V} e^{\mathbf{B}} f}{\int_{\Pi V^* \times \Pi V} e^{\mathbf{B}}}.$$

Let us now choose a basis $\{e_i\}$ of V and denote by $\{\bar{e}^i\}$ its dual basis. Then $\Lambda(V^* \oplus V)$ may be identified with the Grassmann algebra generated by the anticommuting “coordinate functions” \bar{e}^i and e_j . Functions on $\Pi V^* \times \Pi V$ are then linear combinations of monomials $e_{j_1} \cdots e_{j_r} \bar{e}^{i_1} \cdots \bar{e}^{i_s}$. To the endomorphism \mathbf{B} we associate the function $\langle \bar{e}, \mathbf{B}e \rangle = \bar{e}^j B_j^i e_i$, where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between V^* and V . The above formulae are then often written as

$$\int e^{\langle \bar{e}, \mathbf{B}e \rangle} = \det \mathbf{B},$$

$$\langle e_{j_1} \cdots e_{j_r} \bar{e}^{i_1} \cdots \bar{e}^{i_s} \rangle_0 = \frac{\int e^{\langle \bar{e}, \mathbf{B}e \rangle} e_{j_1} \cdots e_{j_r} \bar{e}^{i_1} \cdots \bar{e}^{i_s}}{\int e^{\bar{e}^j B_j^i e_i}}.$$

One easily proves that $\langle e_{j_1} \cdots e_{j_r} \bar{e}^{i_1} \cdots \bar{e}^{i_s} \rangle_0$ is zero if $r \neq s$ and that

$$\langle e_{j_1} \cdots e_{j_s} \bar{e}^{i_s} \cdots \bar{e}^{i_1} \rangle_0 = \sum_{\sigma \in \mathfrak{S}_s} \text{sign } \sigma (B^{-1})_{j_1}^{i_{\sigma(1)}} \cdots (B^{-1})_{j_s}^{i_{\sigma(s)}}.$$

This formula is the odd counterpart to (11.2.5) and may also be extended to the infinite-dimensional case.

Vector fields. — A vector field on ΠV is by definition a graded derivation of the algebra ΛV^* . Namely, we say that an endomorphism X of ΛV^* is a vector field of degree $|X|$ if

$$X(fg) = X(f)g + (-1)^{|X|r} fX(g), \quad \forall f \in \Lambda^r V^*, \forall g \in \Lambda V^*, \forall r.$$

One may also define a right vector field X of degree $|X|$ to be an endomorphism $f \mapsto (f)X$ of ΛV^* that satisfies

$$(fg)X = f(g)X + (-1)^{|X|s} (f)Xg, \quad \forall f \in \Lambda V^*, \forall g \in \Lambda^s V^*, \forall s.$$

The vector space of all vector fields on ΠV may be identified with $\Lambda V^* \otimes V$, elements of V being constant vector fields. Integration has the natural property that

$$\int_{\Pi V} X(f) = 0, \quad \forall f,$$

if X is a constant vector field. In general, one defines the divergence $\text{div } X$ of X by the formula

$$\int_{\Pi V} X(f) = \int_{\Pi V} \text{div } X f, \quad \forall f.$$

Given the vector field $X = g \otimes v$, $g \in \Lambda^r V^*$, $v \in V$, one can show that

$$(11.2.6) \quad \text{div } X = (-1)^{r+1} \iota_v g,$$

where ι denotes contraction.

11.3. The Moyal star product from path integrals

We want to quantize $T^*\mathbb{R}^n$ using the techniques described in section 11.2. Let q denote a point in \mathbb{R}^n (in coordinates we will write q^1, \dots, q^n) and let p denote a covector (in coordinates we will write p_1, \dots, p_n). Classical mechanics is defined in terms of the canonical symplectic form $\omega = dp_i dq^i$ or, more precisely, of its potential $\theta = p_i dq^i$ (sums over repeated indices are from now on understood). Given a path $\gamma: I \rightarrow T^*\mathbb{R}^n$, where I is a one-manifold, the action function is defined by $S(\gamma) = \int_I \gamma^* \theta$. If we write $\gamma(t) = (Q(t), P(t))$, $t \in I$, we may also write

$$(11.3.1) \quad S(Q, P) = \int_{t \in I} P_i \frac{d}{dt} Q^i dt.$$

If a Hamiltonian function H is given, one then deforms the action function to

$$(11.3.2) \quad S_H(Q, P) = \int_{t \in I} \left(P_i \frac{d}{dt} Q^i + H(Q(t), P(t), t) \right) dt,$$

but we will not consider this case now (see section 11.3.6).

From now on we choose the one-manifold I to be S^1 . In order to make the quadratic form nondegenerate, we also choose a base point $\infty \in S^1$ and prescribe the value of the path at the base point. Setting

$$\mathcal{M} = \{(Q, P) \in C^\infty(S^1, T^*\mathbb{R}^n)\}$$

and

$$\mathcal{M}(q, p) = \{(Q, P) \in C^\infty(S^1, T^*\mathbb{R}^n) : Q(\infty) = q, P(\infty) = p\},$$

we may define the path integral by imposing Fubini's theorem in the form

$$(11.3.3) \quad \int_{\mathcal{M}} \cdots := \int_{(q,p) \in T^*\mathbb{R}^n} \mu(q, p) \int_{\mathcal{M}(q,p)} \cdots$$

where we have chosen a measure μ on $T^*\mathbb{R}^n$. Observe that the quadratic form in S is nondegenerate when restricted to $\mathcal{M}(q, p)$. Thus, we may compute

$$\langle \mathcal{O} \rangle_0(q, p) := \frac{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S} \mathcal{O}}{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S}}$$

where \mathcal{O} is some function on \mathcal{M} that is a polynomial or is a formal power series in Q and P . As the denominator, though infinite, is constant, we may improve (11.3.3) to

$$\langle \mathcal{O} \rangle_0 := \frac{\int_{(q,p) \in T^*\mathbb{R}^n} \mu(q, p) \langle \mathcal{O} \rangle_0(q, p)}{\int_{(q,p) \in T^*\mathbb{R}^n} \mu(q, p)}$$

in the case when the functions $\langle \mathcal{O} \rangle_0(q, p)$ and 1 are integrable. The latter condition prevents us from choosing μ to be the Liouville measure $\omega^n/n!$. However, at this point we may also forget this denominator and define the expectation value to be

$$\langle \mathcal{O} \rangle'_0 := \int_{(q,p) \in T^*\mathbb{R}^n} \mu(q, p) \langle \mathcal{O} \rangle_0(q, p)$$

so that the Liouville measure is allowed. Observe finally that we may also choose μ to be the delta measure peaked at a point $(q, p) \in T^*\mathbb{R}^n$. In this case we have $\langle \mathcal{O} \rangle_0 = \langle \mathcal{O} \rangle'_0 = \langle \mathcal{O} \rangle_0(q, p)$.

Once $(q, p) \in T^*\mathbb{R}^n$ has been fixed, we make the “change of variables” $Q = q + \tilde{Q}$ and $P = p + \tilde{P}$, where (\tilde{Q}, \tilde{P}) is a map from S^1 to $T^*\mathbb{R}^n$ that vanishes at ∞ , which is the same as a map $\mathbb{R} \rightarrow T^*\mathbb{R}^n$ that vanishes at infinity. The action function then reads

$$S(Q, P) = S(q + \tilde{Q}, p + \tilde{P}) = \int_{\mathbb{R}} \tilde{P}_i \frac{d}{dt} \tilde{Q}^i dt.$$

Expectation values may be computed by the formula

$$\langle \mathcal{O}(Q, P) \rangle_0(q, p) = \left\langle \mathcal{O}(q + \tilde{Q}, p + \tilde{P}) \right\rangle_0^{\sim} := \frac{\int e^{\frac{i}{\hbar} S} \mathcal{O}(q + \tilde{Q}, p + \tilde{P}) D\tilde{P}D\tilde{Q}}{\int e^{\frac{i}{\hbar} S} D\tilde{P}D\tilde{Q}}.$$

11.3.1. The propagator. — The nondegenerate quadratic form is of the type considered in section 11.2.2. The Green function of the skew-symmetric operator $\frac{d}{dt}$ is one-half of the sign function:

$$\left(\frac{d}{dt} \right)^{-1} (u, v) = \theta(u - v) = \frac{1}{2} \text{sig}(u - v) = \begin{cases} \frac{1}{2} & \text{if } u > v, \\ -\frac{1}{2} & \text{if } u < v. \end{cases}$$

As a consequence, by (11.2.5), we get

$$\left\langle \tilde{P}_i(u) \tilde{Q}^j(v) \right\rangle_0^{\sim} = i\hbar \theta(v - u) \delta_i^j$$

and, more generally,

$$(11.3.4) \quad \left\langle \tilde{P}_{i_1}(u_1) \cdots \tilde{P}_{i_s}(u_s) \tilde{Q}^{j_1}(v_1) \cdots \tilde{Q}^{j_s}(v_s) \right\rangle_0^{\sim} = \\ = (i\hbar)^s \sum_{\sigma \in \mathfrak{S}} \theta(v_{\sigma(1)} - u_1) \cdots \theta(v_{\sigma(s)} - u_s) \delta_{i_1}^{j_{\sigma(1)}} \cdots \delta_{i_s}^{j_{\sigma(s)}}.$$

Normal ordering. — The normal ordering prescription may now be implemented by setting $\theta(0) = 0$, also in agreement with the skew-symmetry of $\frac{d}{dt}$.

11.3.2. Expectation values. — An example of an observable on \mathcal{M} is given by the evaluation of a smooth function f on $T^*\mathbb{R}^n$ at some point in the path; viz., we set

$$\mathcal{O}_{f;u}(Q, P) := f(Q(u), P(u)), \quad f \in C^\infty(T^*\mathbb{R}^n), \quad u \in S^1 \setminus \{\infty\}.$$

To compute its expectation value, we have first to introduce some notations. Given a multiindex $I = (i_1, \dots, i_r)$, we set $|I| = r$, $p_I = p_{i_1} \cdots p_{i_r}$, $q^I = q^{i_1} \cdots q^{i_r}$ (and similarly for \tilde{P}_I and \tilde{Q}^I),

$$\partial_I = \frac{\partial}{\partial q^{i_1}} \cdots \frac{\partial}{\partial q^{i_r}}, \quad \partial^I = \frac{\partial}{\partial p_{i_1}} \cdots \frac{\partial}{\partial p_{i_r}}.$$

We extend to the case $|I| = 0$ by setting $p_I = q^I = 1$ and $\partial_I = \partial^I = \text{identity operator}$. Performing the change of variables described above and using the Taylor expansion of f , we get

$$\begin{aligned} f(Q(u), P(u)) &= f(q + \tilde{Q}(u), p + \tilde{P}(u)) = \\ &= \sum_{r,s=0}^{\infty} \frac{1}{r!s!} \sum_{|I|=r, |J|=s} \tilde{P}_I(u) \tilde{Q}^J(u) \partial^I \partial_J f(q, p). \end{aligned}$$

Thus,

$$\langle \mathcal{O}_{f;u} \rangle_0(q, p) = \left\langle \mathcal{O}_{f,u}(q + \tilde{Q}, p + \tilde{P}) \right\rangle_0^\sim = f(q, p),$$

for the expectation values of \tilde{Q} s and \tilde{P} s at coinciding points vanish by the normal ordering prescription, see section 11.3.1. If f is integrable, we may also define

$$\langle \mathcal{O}_{f,u} \rangle'_0 = \int_{T^*\mathbb{R}^n} \mu(q, p) f(q, p).$$

Observe that these expectation values do not depend on the point u . We will give in section 11.3.4 an interpretation of this fact.

The next observable we want to consider is more interesting; viz.,

$$\begin{aligned} \mathcal{O}_{f,g;u,v} &= f(Q(u), P(u)) g(Q(v), P(v)), \\ f, g &\in C^\infty(T^*\mathbb{R}^n), \quad u, v \in S^1 \setminus \{\infty\} \cong \mathbb{R}, \quad u < v. \end{aligned}$$

By repeating the analysis above, we may compute its expectation value as follows:

$$\begin{aligned} (11.3.5) \quad \langle \mathcal{O}_{f,g;u,v} \rangle_0(q, p) &= \left\langle f(q + \tilde{Q}(u), p + \tilde{P}(u)) g(q + \tilde{Q}(v), p + \tilde{P}(v)) \right\rangle_0^\sim = \\ &= \sum_{r_1, s_1, r_2, s_2=0}^{\infty} \frac{1}{r_1!s_1!r_2!s_2!} \sum_{|I_1|=r_1, |J_1|=s_1} \sum_{|I_2|=r_2, |J_2|=s_2} \\ &\quad \left\langle \tilde{P}_{I_1}(u) \tilde{Q}^{J_1}(u) \tilde{P}_{I_2}(v) \tilde{Q}^{J_2}(v) \right\rangle_0^\sim \partial^{I_1} \partial_{J_1} f(q, p) \partial^{I_2} \partial_{J_2} g(q, p) = \\ &= \sum_{r,s=0}^{\infty} \frac{1}{r!s!} \left(\frac{i\hbar}{2} \right)^{r+s} (-1)^s \sum_{|I|=r, |J|=s} \partial^I \partial_J f(q, p) \partial^J \partial_I g(q, p) = f \star g(q, p), \end{aligned}$$

where we have used (11.3.4) with the normal ordering prescription and \star denotes the Moyal star product for the Poisson bivector field $\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}$. Observe that again the expectation value is independent of the points u and v . If $f \star g$ is integrable, we may also compute

$$\langle \mathcal{O}_{f,g;u,v} \rangle'_0 = \int_{T^*\mathbb{R}^n} \mu f \star g.$$

If μ is the Liouville measure $\omega^n/n!$, this defines the trace

$$\text{tr}(f \star g) = \int_{T^*\mathbb{R}^n} \frac{\omega^n}{n!} f \star g = \int_{T^*\mathbb{R}^n} \frac{\omega^n}{n!} fg,$$

where the second equality simply follows by using integration by parts in the correction terms to the commutative product. We may now continue along this line and define, e.g.,

$$\begin{aligned} \mathcal{O}_{f_1, \dots, f_k; u_1, \dots, u_k} &= f_1(Q(u_1), P(u_1)) \cdots f_k(Q(u_k), P(u_k)), \\ f_1, \dots, f_k &\in C^\infty(T^*\mathbb{R}^n), \quad u_1, \dots, u_k \in S^1 \setminus \{\infty\} \cong \mathbb{R}, \quad u_1 < \cdots < u_k. \end{aligned}$$

We leave it as an exercise (see also section 11.3.5) to prove that

$$\langle \mathcal{O}_{f_1, \dots, f_k; u_1, \dots, u_k} \rangle_0(q, p) = f_1 \star \cdots \star f_k(q, p).$$

11.3.3. Divergence of local vector fields. — An important consequence of the normal ordering prescription of section 11.3.1 is that local vector fields are divergence free. Recall that a vector at $(\tilde{Q}, \tilde{P}) \in \mathcal{M}(q, p)$ may be identified with a smooth map $\mathbb{R} \rightarrow T^*\mathbb{R}^n$ that vanishes at infinity. A vector field X on $\mathcal{M}(q, p)$ is then the assignment of a map $X(\tilde{Q}, \tilde{P})$ to each path (\tilde{Q}, \tilde{P}) . We say that the vector field is local if $X(\tilde{Q}, \tilde{P})(t)$ is a function of $\tilde{Q}(t)$ and $\tilde{P}(t)$ for all $t \in \mathbb{R}$.

The definition of the divergence of a vector field in a path integral mimics the one in ordinary integrals, restricted to the cases where it makes sense. Formally we would like to have

$$\int_{\mathcal{M}(q,p)} X \left(e^{\frac{i}{\hbar} S} \mathcal{O} \right) = \int_{\mathcal{M}(q,p)} \text{div } X e^{\frac{i}{\hbar} S} \mathcal{O},$$

for all observables \mathcal{O} . The correct requirement can then be expressed in terms of expectation values.

Definition 11.3.1. — If there exists an observable $\text{div } X$ such that for any observable \mathcal{O} we have

$$\langle X(S) \mathcal{O} \rangle_0(q, p) = i\hbar \langle X(\mathcal{O}) \rangle_0(q, p) - i\hbar \langle \text{div } X \mathcal{O} \rangle_0(q, p),$$

then we say that $\text{div } X$ is the divergence of X .

Now we have the

Lemma 11.3.2. — *If X is a local vector field, then $\text{div } X = 0$.*

Proof. — We write $X(\tilde{Q}, \tilde{P})(t) = (X_q(\tilde{Q}(t), \tilde{P}(t)), X_p(\tilde{Q}(t), \tilde{P}(t)))$. Then

$$X(S) = \int \left(X_p \frac{d}{dt} \tilde{Q} - X_q \frac{d}{dt} \tilde{P} \right) dt.$$

By the normal ordering prescription and the locality of X , in the computation of $\langle X(S) \mathcal{O} \rangle_0(q, p)$ we have only to contract the \tilde{P} s (\tilde{Q} s) in \mathcal{O} with the \tilde{Q} s (\tilde{P} s) in $X(S)$ and replace each pair by a propagator. Whenever a \tilde{P} (\tilde{Q}) in \mathcal{O} is contracted with the $\frac{d}{dt} \tilde{Q}$ ($\frac{d}{dt} \tilde{P}$) in $X(S)$, we get the identity operator times $i\hbar$ ($-i\hbar$). Summing up the various terms, we see that this is the same as taking the expectation value of $X(\mathcal{O})$ multiplied by $i\hbar$, which completes the proof. \checkmark

We may also consider the divergence of vector fields on \mathcal{M} . In particular we observe that a local vector field X on \mathcal{M} may be written uniquely as the sum of a vector field X_∞ on $T^*\mathbb{R}^n$ and a section \tilde{X} of local vector fields (i.e., the assignment of a local vector field $\tilde{X}(q, p)$ on $\mathcal{M}(q, p)$ to each $(q, p) \in T^*\mathbb{R}^n$). The above Lemma implies

$$\langle X(S) \mathcal{O} \rangle'_0 = i\hbar \langle X(\mathcal{O}) \rangle'_0 - i\hbar \langle \operatorname{div}_\mu X_\infty \mathcal{O} \rangle'_0,$$

where $\operatorname{div}_\mu X_\infty$ is the ordinary divergence of the vector field X_∞ w.r.t. the measure μ . We may then say that $\operatorname{div}_\mu X_\infty$ is the divergence of the local vector field X on \mathcal{M} .

By using the local vector field $X(\tilde{P}, \tilde{Q})(t) := (\tilde{P}(t), 0)$, it is an exercise to prove that

$$\hbar \frac{d}{d\hbar} (f \star g) = (p_i \partial^i f) \star g + f \star (p_i \partial^i g) - p_i \partial^i (f \star g).$$

11.3.4. Independence from the evaluation points. — In the above computations we have considered observables whose definition relies on the choice of points on $S^1 \setminus \{\infty\}$, but it has always turned out that expectation values are independent of this choice. In other words expectation values are invariant under (pointed) diffeomorphisms of the source manifold S^1 on which the field theory is defined. A quantum field theory with this property is usually called a **topological quantum field theory** TQFT as it depends only on the topological data of the source manifold. In our case we see the topological nature of the theory at the classical level as the action function (11.3.1) is invariant under diffeomorphisms. The fact that this invariance survives at the quantum level indicates that the fictitious measure in the path integral is also diffeomorphism invariant.

We may convince ourselves of the independence of the expectation values from the points where we evaluate the functions if we just recall that the propagator θ is a locally constant function. There exists however another way to obtain this result which will help our intuition in more complicated situations, e.g., in those considered

in Chapter 14. Given $f \in C^\infty(T^*\mathbb{R}^n)$ and two points a and b in \mathbb{R} , we observe that

$$\begin{aligned} f(Q(b), P(b)) - f(Q(a), P(a)) &= \\ &= \int_a^b \left(\partial_i f(Q(t), P(t)) \frac{d}{dt} Q^i(t) + \partial^i f(Q(t), P(t)) \frac{d}{dt} P_i(t) \right) dt = \\ \lim_{r \rightarrow \infty} \int_{-\infty}^{+\infty} \left(\partial_i f(Q(t), P(t)) \frac{d}{dt} Q^i(t) + \partial^i f(Q(t), P(t)) \frac{d}{dt} P_i(t) \right) \lambda_r(t) dt &= \\ &= \lim_{r \rightarrow \infty} \tilde{X}_{f,r}(S) \end{aligned}$$

where $\{\lambda_r\}$ is a sequence of smooth, compactly supported functions that converges almost everywhere to the characteristic function of the interval $[a, b]$ and $\tilde{X}_{f,r}$ is the local vector field⁽²⁾

$$\tilde{X}_{f,r}(t) = (-\partial^i f(Q(t), P(t)), \partial_i f(Q(t), P(t))) \lambda_r(t).$$

We may now restrict $\tilde{X}_{f,r}$ to $\mathcal{M}(q, p)$ and observe that, by Lemma 11.3.2, it is divergence free. Thus,

$$\begin{aligned} \langle (f(Q(b), P(b)) - f(Q(a), P(a))) \mathcal{O} \rangle_0(q, p) &= \\ &= \lim_{r \rightarrow \infty} \langle \tilde{X}_{f,r}(S) \mathcal{O} \rangle_0(q, p) = i\hbar \lim_{r \rightarrow \infty} \langle \tilde{X}_{f,r}(\mathcal{O}) \rangle_0(q, p), \end{aligned}$$

which vanishes if \mathcal{O} does not depend on $(\tilde{Q}(t), \tilde{P}(t))$ for $t \in [a, b]$. This means that we are free to move the evaluation point at least as long as we do not meet another evaluation point.

11.3.5. Associativity. — The associativity of the Moyal star product is not difficult to prove using its explicit formula. It may also be proved from its path-integral representation considering the expectation value of $\mathcal{O}_{f,g,h;u,v,w}$. In its computation, there appear three kinds of propagators, corresponding to the three possible pairings of the points u, v, w . However, the function θ does not see the difference as $\theta(w - u) = \theta(w - v) = \theta(v - u) = 1/2$. We may now decide to group the propagators by considering first only those between u and v and only thereafter the other ones. It is not difficult to see that this implies that $\langle \mathcal{O}_{f,g,h;u,v,w} \rangle_0 = (f \star g) \star h$. On the other hand, we may also group first the propagators between v and w and then the others, thus obtaining $\langle \mathcal{O}_{f,g,h;u,v,w} \rangle_0 = f \star (g \star h)$. This proves associativity.

There is also a formal argument to prove it—whose generalization to more complex cases (see, e.g., Chapter 14) will help our intuition—that relies on the independence

⁽²⁾Recall that \mathcal{M} is a space of maps. So a vector field on the target $T^*\mathbb{R}^n$ generates a local vector field on \mathcal{M} . Let \tilde{X}_f be the local vector field corresponding to the Hamiltonian vector field X_f of f . Then $\tilde{X}_{f,r}$ is obtained by multiplying \tilde{X}_f by λ_r .

of the evaluation points. This in fact implies that

$$\lim_{v \rightarrow u^+} \langle \mathcal{O}_{f,g,h;u,v,w} \rangle_0(q,p) = \lim_{v \rightarrow w^-} \langle \mathcal{O}_{f,g,h;u,v,w} \rangle_0(q,p).$$

The l.h.s. corresponds then to evaluating first the expectation value of $\mathcal{O}_{f,g;u,v}$, placing the result at u and finally computing the expectation value of $\mathcal{O}_{\langle \mathcal{O}_{f,g;u,v} \rangle_0, h; w}$. The result is then $(f \star g) \star h$. Repeating the computation on the r.h.s. and using the above identity, we get the associativity of the star product.

11.3.6. The evolution operator. — As an application we want now to consider the formal path integral for the full action function (11.3.2). For simplicity we restrict ourselves to considering a time-independent Hamiltonian h and let it act from the instant a to the instant b , $a < b$. Thus we consider S_H with

$$H(q,p,t) = h(q,p) \chi_{[a,b]}(t),$$

where χ denotes the characteristic function of the interval. We want to compute the *evolution operator*

$$U(q,p,T) := \frac{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S_H}}{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S}}$$

with $T = b - a$. First we observe that

$$U(q,p,T) = \left\langle e^{\frac{i}{\hbar} \int H(Q(t), P(t), t) dt} \right\rangle_0(q,p) = \left\langle e^{\frac{i}{\hbar} \int_a^b h(Q(t), P(t)) dt} \right\rangle_0(q,p).$$

Next we regard the integral as a limit of Riemann sums:

$$\int_a^b h(Q(t), P(t)) dt = \lim_{N \rightarrow \infty} \frac{T}{N} \sum_{r=1}^N h(Q(a + rT/N), P(a + rT/N)).$$

Thus we have

$$\begin{aligned} U(q,p,T) &= \lim_{N \rightarrow \infty} \left\langle \prod_{r=1}^N e^{\frac{i}{\hbar} \frac{T}{N} h(Q(a+rT/N), P(a+rT/N))} \right\rangle_0(q,p) = \\ &= \lim_{N \rightarrow \infty} \left\langle \mathcal{O}_{e^{\frac{i}{\hbar} \frac{T}{N} h}, \dots, e^{\frac{i}{\hbar} \frac{T}{N} h}; a+\frac{T}{N}, a+2\frac{T}{N}, \dots, a+T} \right\rangle_0(q,p) = \\ &= \lim_{N \rightarrow \infty} \left(e^{\frac{i}{\hbar} \frac{T}{N} h} \right)^{\star N}(q,p) = \exp_{\star} \left(\frac{i}{\hbar} h \right)(q,p). \end{aligned}$$

Observe that in the above result also negative powers of \hbar appear. The final answer is however well-defined if we observe that each term in the power series expansion of the star exponential is actually a Laurent series in \hbar .

11.4. Perturbative evaluation of integrals

Let \mathcal{M} be an n -dimensional manifold and S a smooth function on \mathcal{M} . We want to compute the integrals appearing in eq. (11.1.1). We consider first the case when S has a unique critical point $x_0 \in \mathcal{M}$ which is moreover nondegenerate. We write

$$S(x_0 + \sqrt{\hbar}x) = S(x_0) + \frac{\hbar}{2}d_{x_0}^2 S(x) + R_{x_0}(\sqrt{\hbar}x),$$

where R_{x_0} is computed from the Taylor expansion of S and is then a formal power series, starting with the cubic term, in $\sqrt{\hbar}x$ with $x \in \mathbb{T}_{x_0} \mathcal{M}$. We denote by \mathbf{A} the Hessian of S at x_0 w.r.t. to the Euclidean metric; i.e., we write $d_{x_0}^2 S(x) = (x, \mathbf{A}x)$. Then the saddle-point approximation to the integral $Z := \int_{\mathcal{M}} e^{\frac{i}{\hbar}S} d^n x$ is given by the formula

$$\begin{aligned} Z &= \hbar^{\frac{n}{2}} e^{\frac{i}{\hbar}S(x_0)} \int_{\mathcal{M}} e^{\frac{i}{2}(x, \mathbf{A}x)} \sum_{r=0}^{\infty} \frac{1}{r!} R_{x_0}^r(\sqrt{\hbar}x) d^n x = \\ &= (2\pi\hbar)^{\frac{n}{2}} e^{\frac{i}{\hbar}S(x_0)} e^{\frac{i\pi \operatorname{sign} \mathbf{A}}{4}} \frac{1}{\sqrt{|\det \mathbf{A}|}} \sum_{r=0}^{\infty} \frac{1}{r!} \left\langle R_{x_0}^r(\sqrt{\hbar}x) \right\rangle_0, \end{aligned}$$

where $\langle \cdot \rangle_0$ denotes the Gaussian expectation value w.r.t. the nondegenerate symmetric matrix \mathbf{A} . This formula yields an asymptotic expansion of Z as a function of \hbar , but for our purposes it will be simply considered as a formal power series⁽³⁾ in \hbar times the oscillatory factor $e^{\frac{i}{\hbar}S(x_0)}$ that we will shortly drop. In this setting R_{x_0} is regarded as a perturbation to the Gaussian theory defined by $d_{x_0}^2 S$ and the above formula is referred to as the perturbative expansion of the integral. Expectation values, denoted by $\langle \cdot \rangle$ as in (11.1.1), may also be computed perturbatively by

$$(11.4.1) \quad \langle \mathcal{O} \rangle = \frac{\sum_{r=0}^{\infty} \frac{1}{r!} \left\langle \mathcal{O}(x_0 + \sqrt{\hbar}x) R_{x_0}^r(\sqrt{\hbar}x) \right\rangle_0}{\sum_{r=0}^{\infty} \frac{1}{r!} \left\langle R_{x_0}^r(\sqrt{\hbar}x) \right\rangle_0}.$$

As the denominator is of the form $1 + O(\hbar)$, the ratio can be computed as a formal power series in \hbar . The computation may be arranged graphically associating a vertex of valence k to the term of degree k in R_{x_0} and a vertex of degree l to the term of degree l in \mathcal{O} . By the Wick theorem (11.2.3), we should then connect in pairs in all possible ways the half-edges emanating from each vertex. As a result we get a collection of graphs—the Feynman diagrams—with a weight associated to each of them, i.e., a number obtained by multiplying the matrix elements of \mathbf{A}^{-1} in the way prescribed by the graph times appropriate combinatorial factors and powers of \hbar . The expectation value is then a sum over all graphs of their weights. A Feynman diagram with vertices coming only from R_{x_0} —i.e., a Feynman diagram appearing in

⁽³⁾Recall that only monomials of even degrees have a nonvanishing Gaussian expectation value. Thus, in the final formula $\sqrt{\hbar}$ is always raised to an even power.

the denominator of (11.4.1)—is called a vacuum diagram. An easy combinatorial effect of the ratio in (11.4.1) is that an expectation value is given by the sum over graphs which do not have a connected component which is a vacuum diagram. We will not enter details here. The interested reader is referred to one of the many books or lecture notes on the subject (see, e.g., [104]).

11.4.1. Infinite dimensions. — The infinite-dimensional generalization is again obtained by taking (11.4.1) verbatim when it is possible to make sense of the r.h.s.

Definition 11.4.1 (Local functions). — A function on a space of fields on some manifold M is called local if it is the integral on M of a function that depends at each point on finite jets of the fields at that point.

If the action is a local function, A will be a differential operator. In the computations we will need its Green function—the propagator—as explained in section 11.2.1, and in the Gaussian expectation values of $\mathcal{O}R^r$ and of R^r we will use (11.2.4). Instead of summing over indices we will have to integrate over Cartesian products of the source manifold. The normal ordering will exclude all graphs with an edge whose endpoints are equal (the so-called tadpoles) and will restrict integration to the configuration space of the source manifold (viz., the Cartesian product minus all diagonals). In general, however, this will not be enough to make all integrals converge as the Green functions are usually very singular when their two arguments approach each other. The general technique to deal with this problem goes under the name of **renormalization**. We will not discuss this issue here because *a nice feature of topological field theories*—of which the Poisson sigma model discussed in the next Sections is an example—is that *configuration-space integrals associated to Feynman diagrams without tadpoles converge*.

In case the action function S has more critical points and all of them are nondegenerate, the asymptotic expansion is obtained by computing the saddle-point approximation around each critical point and then summing all contributions. The formula for the expectation value is then no longer as simple as in (11.4.1). It may happen however that a critical value dominates the others so that one can safely forget them. In physical theories this is usually done by regarding $e^{\frac{i}{\hbar}S}$ as the analytic continuation of the exponential of minus a positive definite function, the “Euclidean action,” so that the dominating critical point is the absolute minimum. There are however examples, as the ones we are going to consider, where there is no way to do it (as all critical points are saddle points). Another option, which is suitable also in these instances, consists in selecting one particular critical point (the “sector”) and expanding only around it. In this case, formula (11.4.1) is still the one to be taken to define the infinite-dimensional generalization.

It often happens however that the critical point is degenerate. A simple case is when the critical points are parametrized by a finite-dimensional manifold $\mathcal{M}_{\text{crit}}$.

Then—by Fubini’s theorem in the finite-dimensional case and by decree in the infinite-dimensional one—one writes, like we did in (11.3.3),

$$\int_{\mathcal{M}} \cdots := \int_{x_0 \in \mathcal{M}_{\text{crit}}} \mu(x_0) \int_{\mathcal{M}(x_0)} \cdots,$$

where $\int_{\mathcal{M}(x_0)}$ denotes the asymptotic expansion of the integral in the complement to $T_{x_0}\mathcal{M}_{\text{crit}}$ of a formal neighborhood of x_0 , while μ is a measure on $\mathcal{M}_{\text{crit}}$ (which is determined in the finite-dimensional case and has to be chosen in the infinite-dimensional case as part of the definition). If moreover the Hessian is constant on $\mathcal{M}_{\text{crit}}$, we may easily write expectation values as

$$\langle \mathcal{O} \rangle = \frac{\int_{x_0 \in \mathcal{M}_{\text{crit}}} \mu(x_0) \sum_{r=0}^{\infty} \frac{1}{r!} \langle \mathcal{O}(x_0 + \sqrt{\hbar}x) R_{x_0}^r(\sqrt{\hbar}x) \rangle_0}{\int_{x_0 \in \mathcal{M}_{\text{crit}}} \mu(x_0) \sum_{r=0}^{\infty} \frac{1}{r!} \langle R_{x_0}^r(\sqrt{\hbar}x) \rangle_0},$$

where $\langle \cdot \rangle_0(x_0)$ denotes the Gaussian expectation value computed expanding around x_0 orthogonally to $T_{x_0}\mathcal{M}_{\text{crit}}$. One possible choice for μ is the delta measure peaked at some point x_0 . In this case, consistently with section 11.3, the expectation value will be denoted by $\langle \cdot \rangle(x_0)$.

11.4.2. A simple generalization. — In the perturbative expansion considered above the only expansion parameter was \hbar . It may happen that there is another small expansion parameter, or that some coefficient appearing in S is much smaller than \hbar (or better, in our setting, is an element of $\hbar^2\mathbb{R}[[\hbar]]$). In this case, the right prescription is to define the Gaussian part using the quadratic, \hbar -independent term of S/\hbar and consider all other terms as the perturbation R . It may thus happen that the perturbation R contains quadratic and linear terms as well. A particular case is when we are in the setting of 11.2.2 on page 124 and our action function has the form

$$(11.4.2) \quad S(y, z) = (y, \mathbf{B}z) + f(y, z),$$

where \mathbf{B} is a nondegenerate matrix and f is a function quadratic in z . If we work around the critical point $y = z = 0$, we may rescale z by \hbar ,

$$S(y, \hbar z) = \hbar(y, \mathbf{B}z) + \hbar^2 f(y, z)$$

and consider f as the perturbation to the Gaussian theory defined by \mathbf{B} and discussed in section 11.2.2. Infinite-dimensional generalizations of this are discussed below and in Chapter 14.

Quantum mechanics. — As we have seen in section 11.3, the topological action (11.3.1) defines the associative Moyal product in terms of the expectation value

(11.3.5) which is independent of the evaluation points. If we now introduce a Hamiltonian as in section 11.3.6, the expectation value

$$f_{\star_{a,b;u,v}}^{\widehat{\star}}(q,p) := \frac{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S_H} \mathcal{O}_{f,g;u,v}}{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S_H}} = \frac{\left\langle e^{\frac{i}{\hbar} \int_a^b h dt} \mathcal{O}_{f,g;u,v} \right\rangle_0(q,p)}{\left\langle e^{\frac{i}{\hbar} \int_a^b h dt} \right\rangle_0(q,p)}$$

will be no longer independent from u and v ($a < u < v < b$) nor will it define an associative product. By the computations in section 11.3.6, we know that the answer is

$$f_{\star_{a,b;u,v}}^{\widehat{\star}} = \frac{\exp_{\star} \left(\frac{i}{\hbar} (u-a)h \right) \star f \star \exp_{\star} \left(\frac{i}{\hbar} (v-u)h \right) \star g \star \exp_{\star} \left(\frac{i}{\hbar} (b-v)h \right)}{\exp_{\star} \left(\frac{i}{\hbar} (b-a)h \right)}.$$

As an application, we want now to perform this computation perturbatively for a Hamiltonian of the form

$$h(q,p) = \frac{1}{2} G^{ij}(q) p_i p_j,$$

at $p = 0$ and for f and g depending only on q . By setting $Q(t) = q + \widetilde{Q}(t)$ and $P(t) = \hbar \widetilde{P}(t)$, we get

$$S_H = \hbar \int_{-\infty}^{\infty} \widetilde{P}(t) \frac{d}{dt} \widetilde{Q}(t) dt + \frac{\hbar^2}{2} \int_a^b G^{ij}(q + \widetilde{Q}(t)) \widetilde{P}_i(t) \widetilde{P}_j(t) dt.$$

Thus, $f_{\star_{a,b;u,v}}^{\widehat{\star}}(q,0)$ is the ratio of the expectation value of $e^{\frac{i\hbar}{2} \int_a^b G^{ij}(q+\widetilde{Q}) \widetilde{P}_i \widetilde{P}_j dt} \mathcal{O}_{f,g;u,v}$ and the expectation value of $e^{\frac{i\hbar}{2} \int_a^b G^{ij}(q+\widetilde{Q}) \widetilde{P}_i \widetilde{P}_j dt}$. As propagators pair \widetilde{P} s to \widetilde{Q} s, in a graphical description it is better to use oriented edges (say, by an arrow going from \widetilde{P} to \widetilde{Q}). Then Feynman diagrams will be oriented graphs with

1. a vertex at u with no outgoing arrows,
2. a vertex at v with no outgoing arrows
3. vertices between a and b with exactly two outgoing arrows.

Moreover no graphs containing tadpoles or vacuum subgraphs will be allowed. See Figs. 1, 2 and 3 for examples (tails of arrows are understood to move on the whole interval $[a, b]$). To a vertex of the first type with r incoming arrows we associate an

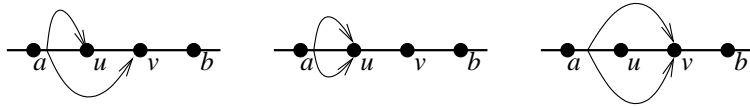


FIGURE 1. Feynman diagrams of order one.

r th derivative of f , to a vertex of the second type with r incoming arrows we associate an r th derivative of g , and to a vertex of the third type with r incoming arrows we associate an r th derivative of G . The order in \hbar is given by the number of vertices of

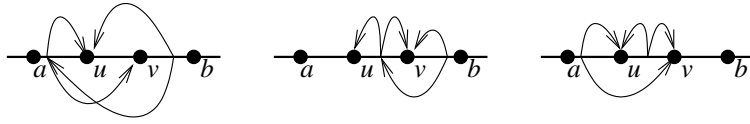


FIGURE 2. Some Feynman diagrams of order two.

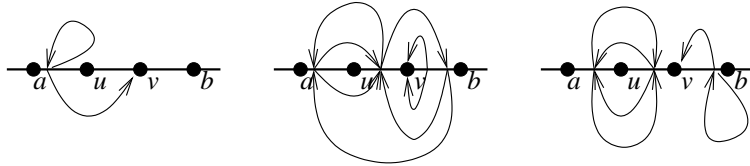


FIGURE 3. Some non allowed Feynman diagrams: tadpoles, vacuum sub-diagrams, or both.

the third type. So, at order 0, we just have f placed at u and g placed at v . This yields the pointwise product of f and g . At order 1, we have the graphs of figure 1. Thus, we get

$$\begin{aligned}
 f \widehat{\star}_{a,b;u,v} g(q, 0) &= f(q)g(q) - \frac{i\hbar}{4}(b - a + 2u - 2v)G^{ij}(q)\partial_i f(q)\partial_j g(q) + \\
 &\quad - \frac{i\hbar}{4}(b - a)G^{ij}(q)[\partial_i \partial_j f(q)g(q) + f(q)\partial_i \partial_j g(q)] + O(\hbar^2)
 \end{aligned}$$

We leave to the reader the computation of some further orders.

CHAPTER 12

SYMMETRIES AND THE BRST FORMALISM

In Chapter 11 we have learnt how to compute integrals of the form given in (11.1.1) under the assumption that the critical point under consideration of the action function S is nondegenerate. This situation does certainly not occur when the action function is invariant under the free action of a Lie group. As this happens, e.g., in the Poisson sigma model for affine Poisson structures, we need to understand how to deal with it as well. In this Section we will explain a method that allows extending to this case the techniques of the perturbative expansion around a nondegenerate critical point.

12.1. The main example

Let \mathcal{M} be a finite-dimensional manifold endowed with a measure μ . Let \mathcal{G} be a compact Lie group endowed with an invariant measure and with a measure-preserving free action on \mathcal{M} . Let us also assume that \mathcal{M}/\mathcal{G} is also a manifold. Then an invariant function f is the pullback of a function \underline{f} on \mathcal{M}/\mathcal{G} and

$$I := \frac{\int_{\mathcal{M}} f \mu}{\text{vol}(\mathcal{G})} = \int_{\mathcal{M}/\mathcal{G}} \underline{f} \underline{\mu},$$

where $\underline{\mu}$ is the measure induced by μ on the quotient.

If there is a section of $\pi: \mathcal{M} \rightarrow \mathcal{M}/\mathcal{G}$, we may also rewrite I as an integral on the image of this section. Let us assume that this image is given locally by the zero set of a function $F: \mathcal{M}' \rightarrow \Gamma$, with $\mathcal{M}' \subset \mathcal{M}$, $\pi(\mathcal{M}') = \mathcal{M}/\mathcal{G}$, and Γ the Lie algebra of \mathcal{G} (in the physics literature the condition $F = 0$ is called **gauge fixing** and the function F is called the **gauge-fixing function**). For each $x \in \mathcal{M}'$, let $A(x)$ be $dF(x)$ restricted to the vertical tangent space at x . As this may also be identified with Γ , we regard $A(x)$ as an endomorphism of Γ and denote by J its determinant (in the physics literature J is usually called the **Faddeev–Popov determinant**). Then we get

$$I = \int_{\mathcal{M}'} f \delta_0(F) J \mu,$$

where δ_0 is the delta-function at $0 \in \Gamma$. We want now to rewrite I in a way that is suitable for saddle-point approximation. To do so we write $\delta_0(F)$ and J in exponential form. To do so, we use the Fourier transform of the delta-function and the Grassmann integration explained in section 11.2.3. Namely, denoting by $\langle \cdot, \cdot \rangle$ the canonical pairing between Γ^* and Γ , we get

$$I = C \int f(x) \mu(x) e^{\frac{i}{\hbar} \langle \lambda, F(x) \rangle} \omega(\lambda) e^{-\frac{i}{\hbar} \langle \bar{c}, A(x)c \rangle},$$

where the integral ranges over $x \in \mathcal{M}'$, $\lambda \in \Gamma^*$, $c \in \text{III}\Gamma$, $\bar{c} \in \text{III}\Gamma^*$, and where C is a constant depending on \hbar and on the choice of the top form $\omega \in \Lambda^{\text{top}}\Gamma^*$. The particular choice of the prefactors $\frac{i}{\hbar}$ is done for later convenience. In the physics literature c is called the **ghost** and \bar{c} the **antighost**; there are various denominations for λ , among which our favorite one is **Lagrange multiplier**. From this point of view a ghost is a “point” in $\text{III}\Gamma$. However, for all computations what is relevant is the algebra of functions $\Lambda\Gamma^*$ which is generated by linear functions, i.e., elements of Γ^* . If we choose a basis $\{c^i\}$ of Γ^* , then $\Lambda\Gamma^*$ may be identified with the algebra generated by the c^i 's with relations $c^i c^j = -c^j c^i$, $\forall i, j$. The generators $c^i \in \Gamma^*$ are often referred to as **ghost variables**. Similarly, one introduces **antighost variables** $\bar{c}_i \in \Gamma$.

Remark 12.1.1. — The determinant of A might also be obtained in terms of an ordinary Gaussian integral over $\Gamma \times \Gamma^*$ at the price however of putting A^{-1} in the exponential. In the finite-dimensional case, this would not make much of a difference. In the infinite-dimensional generalization, however, we may use the Feynman-diagrams techniques only when dealing with local functions, see section 11.4.1. If the action of the group is local, A will be a differential operator, so $\langle \bar{c}, Ac \rangle$ will be a local function (unlike the quadratic function defined in terms of the Green function A^{-1}).

We may now write the expectation value of an invariant function g w.r.t. to a given invariant action function S as

$$(12.1.1) \quad \langle g \rangle = \frac{\int_{\mathcal{M}} e^{\frac{i}{\hbar} S} g \mu}{\int_{\mathcal{M}} e^{\frac{i}{\hbar} S} \mu} = \frac{\int_{\widetilde{\mathcal{M}}} e^{\frac{i}{\hbar} S_F} g \widetilde{\mu}}{\int_{\widetilde{\mathcal{M}}} e^{\frac{i}{\hbar} S} \widetilde{\mu}} =: \langle g \rangle_F,$$

with

$$\begin{aligned} \widetilde{\mathcal{M}} &= \mathcal{M}' \times \text{III}\Gamma \times \Gamma^* \times \text{III}\Gamma^*, \\ S_F &= S + \langle \lambda, F \rangle - \langle \bar{c}, Ac \rangle, \end{aligned}$$

and $\widetilde{\mu} = \mu \omega$. The function S_F is usually called the **gauge-fixed action function**.

12.2. The BRST method

By construction it is clear that the r.h.s. of (12.1.1) is independent of F . We recall that our assumptions so far were rather restrictive: namely, we need a measure-preserving action on \mathcal{M} of a compact Lie group \mathcal{G} , and we have to assume that the

principal bundle $\mathcal{M} \rightarrow \mathcal{M}/\mathcal{G}$ is trivial. On the other hand, in order to define

$$(12.2.1) \quad \langle g \rangle_F = \frac{\int_{\widetilde{\mathcal{M}}} e^{\frac{i}{\hbar} S_F} g \widetilde{\mu}}{\int_{\widetilde{\mathcal{M}}} e^{\frac{i}{\hbar} S} \widetilde{\mu}},$$

we just need the infinitesimal action

$$\begin{aligned} X: \Gamma &\rightarrow \mathfrak{X}(\mathcal{M}) \\ \gamma &\mapsto X_\gamma \end{aligned}$$

of a Lie algebra Γ on \mathcal{M} . In this case \mathbf{A} is simply given by

$$(12.2.2) \quad \mathbf{A}\gamma = \mathbf{L}_{X_\gamma} F, \quad \gamma \in \Gamma.$$

We also want to relax the condition that $F^{-1}(0)$ defines a section and simply require that $\mathbf{A}(x)$ should be nondegenerate $\forall x \in \mathcal{M}'$. Moreover, we obviously require that the integrals are well-defined and that the denominator of (12.2.1) does not vanish. We denote by $\mathcal{F} \subset C^\infty(\mathcal{M}', \Gamma)$ the space of allowed gauge-fixing functions.

As $\langle g \rangle_F$ is well-defined⁽¹⁾ in this case (and susceptible to an infinite-dimensional generalization along the lines discussed in Chapter 11), it makes sense to take it into consideration in more general instances. However, as the gauge-fixing function is chosen arbitrarily, we want conditions for $\langle g \rangle_F$ to be, at least locally, independent of F .

Definition 12.2.1. — A locally constant function on \mathcal{F} is called gauge-fixing independent.

We have the

Theorem 12.2.2. — Let $X: \Gamma \rightarrow \mathfrak{X}(\mathcal{M})$ be an infinitesimal action of the Lie algebra Γ on the manifold \mathcal{M} . If S and g are invariant functions and

$$(12.2.3) \quad \operatorname{div} X_\gamma + \operatorname{tr} \operatorname{ad}_\gamma = 0, \quad \forall \gamma \in \Gamma,$$

then $\langle g \rangle_F$ is gauge-fixing independent.

Here ad denotes the adjoint representation of Γ and $\operatorname{tr} \operatorname{ad}_\gamma$ is regarded as a constant function on \mathcal{M} . The condition in particular says that the divergence of X is a constant function. In the particular case when the Lie algebra is unimodular (i.e., $\operatorname{tr} \operatorname{ad}_\gamma = 0 \forall \gamma$), the condition says that the infinitesimal action must be measure preserving. The case discussed in section 12.1 is covered by the Theorem as the Lie algebra of a compact Lie group is unimodular.

⁽¹⁾Observe that if the group is not compact, an invariant function is not a test function. It is then understood that it is replaced by a test function that in a neighborhood of the zeros of F coincides with the given function. To avoid cumbersome notation, we will never explicitly change the function, so we will write, e.g., $\int_{\mathbb{R}} \delta_0(x) dx = 1$, without mentioning that the constant function 1 is replaced by a test function which is one in a neighborhood of zero.

12.2.1. The BRST operator and a proof of Theorem 12.2.2. — The infinitesimal action of Γ on \mathcal{M} makes $C^\infty(\mathcal{M})$ into a Γ -module. As a consequence, we may consider the Lie algebra complex $\Lambda\Gamma^* \otimes C^\infty(\mathcal{M})$. The Lie-algebra differential δ is in particular a derivation on $\Lambda\Gamma^*$ acting on $C^\infty(\mathcal{M})$ by vector fields. As such it is a derivation on $\Lambda\Gamma^* \otimes C^\infty(\mathcal{M})$ and defines a vector field⁽²⁾ on $\mathcal{M} \times \text{III}\Gamma$. If we choose a basis $\{e_i\}$ of Γ and denote by $\{f_{ij}^k\}$ the corresponding structure constants, then the algebra of functions on $\text{III}\Gamma$ may be identified with the graded commutative algebra with odd generators c^i (the ghost variables) and

$$\delta c^i = -\frac{1}{2} f_{jk}^i c^j c^k.$$

On functions $f \in C^\infty(M)$, δ acts by

$$\delta f = c^i L_{X_{e_i}} f.$$

We may extend the vector field δ to the whole $\widetilde{\mathcal{M}}$ by adding to it a vector field on $\Gamma^* \times \text{III}\Gamma^*$. Using the dual basis $\{e^i\}$ of Γ^* , the algebra of functions may be identified with the graded commutative algebra with odd generators \bar{c}_i and even generators λ_i . On them we define⁽³⁾

$$(12.2.4) \quad \delta \bar{c}_i = \lambda_i, \quad \delta \lambda_i = 0.$$

As a consequence of this definition, we immediately get the

Lemma 12.2.3. — δ is a differential (i.e., an odd derivation that squares to zero) on $\Lambda\Gamma^* \otimes C^\infty(\mathcal{M}) \otimes \Lambda\Gamma \otimes S\Gamma$. It has degree one with respect to the grading $\text{deg}(\alpha \otimes f \otimes \beta \otimes \gamma) := \text{deg}(\alpha) - \text{deg}(\beta)$.

A cohomological vector field is by definition a differential on the algebra of functions. We may restate the Lemma by saying that δ is a cohomological vector field (of degree one) on $\widetilde{\mathcal{M}}$. In the physics literature, it is often referred to as the BRST

⁽²⁾According to section 11.2.3, a vector field on the superspace $\text{III}\Gamma$ is an element of $\Lambda\Gamma^* \otimes \Gamma$. If Γ is a Lie algebra, the commutator may be regarded as an element of $\Lambda^2\Gamma^* \otimes \Gamma$ and as such it is a vector field on $\text{III}\Gamma$. Vector fields on $\mathcal{M} \times \text{III}\Gamma$ may be identified with elements of $\Lambda\Gamma^* \otimes \Gamma \otimes C^\infty(M) \oplus \Lambda\Gamma^* \otimes \mathfrak{X}(M)$. The commutator tensor the constant function 1 is an element of $\Lambda^2\Gamma^* \otimes \Gamma \otimes C^\infty(M)$ while the infinitesimal action of Γ on \mathcal{M} is an element of $\Gamma^* \otimes \mathfrak{X}(M)$. As such they define vector fields on $\mathcal{M} \times \text{III}\Gamma$ and δ is their sum.

⁽³⁾More abstractly, observe that a polynomial vector field on $\Gamma^* \times \text{III}\Gamma^*$ is an element of $\Lambda\Gamma \otimes S\Gamma \otimes (\Gamma^* \oplus \Gamma^*)$. The identity operator may be regarded as an element of $\Gamma \otimes \Gamma^*$. Using the inclusion map

$$\begin{aligned} \iota: \Gamma \otimes \Gamma^* &\rightarrow \Lambda\Gamma \otimes S\Gamma \otimes (\Gamma^* \oplus \Gamma^*) \\ a \otimes b &\mapsto 1 \otimes a \otimes (b \oplus 0) \end{aligned}$$

we may regard it as a vector field on $\Gamma^* \times \text{III}\Gamma^*$. Observe that this vector field corresponds to the de Rham differential on $\text{III}\Gamma^*$.

operator. A function on \mathcal{M} may also be considered as a function on $\widetilde{\mathcal{M}}$. By definition of δ , a function f is invariant iff $\delta f = 0$. By using (11.2.6) one can show easily that

$$\operatorname{div} \delta = \operatorname{div} X_c + \operatorname{tr} \operatorname{ad}_c.$$

Now observe that a gauge-fixing function $F: \mathcal{M} \rightarrow \Gamma$ may be regarded as an element of $C^\infty(\mathcal{M}) \otimes \Gamma$. By the inclusions

$$C^\infty(\mathcal{M}) \otimes \Gamma \hookrightarrow C^\infty(\mathcal{M}) \otimes \Lambda\Gamma \hookrightarrow \Lambda\Gamma^* \otimes C^\infty(\mathcal{M}) \otimes \Lambda\Gamma \otimes S\Gamma,$$

we may then associate to F a function Ψ_F on $\widetilde{\mathcal{M}}$. With the same notations as above, we have $\Psi_F = \bar{c}_i F^i$. The odd function Ψ_F is called the *gauge-fixing fermion*. A very simple computation shows that

$$S_F = S + \delta\Psi_F.$$

We now assume that the action function S is invariant; i.e., $\delta S = 0$.

Lemma 12.2.4. — *Let g be a function on $\widetilde{\mathcal{M}}$. If $\delta g = 0$ and δ is divergence-free, then*

$$I_F := \int_{\widetilde{\mathcal{M}}} e^{\frac{i}{\hbar} S_F} g \tilde{\mu},$$

is gauge-fixing independent.

The Lemma together with the result of Exercise 12.2.1 proves Theorem 12.2.2 immediately.

Proof. — Let F_t be a curve of gauge-fixing functions. Then

$$\begin{aligned} \frac{d}{dt} I_{F_t} &= \frac{i}{\hbar} \int_{\widetilde{\mathcal{M}}} \delta \left(\frac{d}{dt} \Psi_{F_t} \right) e^{\frac{i}{\hbar} S_F} g \tilde{\mu} = \frac{i}{\hbar} \int_{\widetilde{\mathcal{M}}} \delta \left(\frac{d}{dt} \Psi_{F_t} e^{\frac{i}{\hbar} S_F} g \right) \tilde{\mu} = \\ &= \frac{i}{\hbar} \int_{\widetilde{\mathcal{M}}} \operatorname{div} \delta e^{\frac{i}{\hbar} S_F} g \tilde{\mu} = 0. \end{aligned}$$

✓

A particular case of a δ -closed function is a δ -exact function. These functions are irrelevant as for computing expectation values. In fact,

$$\int_{\widetilde{\mathcal{M}}} e^{\frac{i}{\hbar} S_F} \delta h \tilde{\mu} = \int_{\widetilde{\mathcal{M}}} \delta \left(e^{\frac{i}{\hbar} S_F} h \right) \tilde{\mu} = 0$$

if S is invariant and δ is divergence free. We may then extend Theorem 12.2.2 to the

Theorem 12.2.5. — *Let $X: \Gamma \rightarrow \mathcal{M}$ be an infinitesimal action of the Lie algebra Γ on the manifold \mathcal{M} . If the action function S is invariant and the BRST operator δ is divergence-free (i.e., if condition (12.2.3) is satisfied), then*

1. $\langle g \rangle_F$ is gauge-fixing independent for $\forall g \in \ker \delta$;
2. $\langle g \rangle_F = 0 \forall g \in \operatorname{im} \delta$.

Thus the expectation value defines a linear function on the δ -cohomology.

Observe that 2. produces identities relating expectation values of different quantities. Such identities are called **Ward identities** and usually have nontrivial content.

12.2.2. Examples. — We discuss two very simple examples.

Translations. — Let $\mathcal{M} = \Gamma = \mathbb{R}$. Let Γ act by infinitesimal translations. Denoting by x a coordinate on \mathcal{M} , we then have $\delta x = c$ and $\delta c = 0$. We assume S and g to be constant. Then

$$\langle g \rangle_F = \frac{e^{\frac{i}{\hbar}S} g \int \delta_0(F(x)) F'(x)}{e^{\frac{i}{\hbar}S} \int \delta_0(F(x)) F'(x)} = g$$

if the denominator does not vanish. Obviously $\langle g \rangle_F$ is gauge-fixing independent.

We treat similarly the case of rotation-invariant functions on $\mathcal{M} = S^1$. A section here is just a point. We take \mathcal{M}' to be a neighborhood of this point. After identifying \mathcal{M}' with \mathbb{R} , we proceed as above with F any function with a single nondegenerate zero corresponding to the image of the section.

Plane rotations. — Assume $\mathcal{M} = \mathbb{R}^2 \setminus \{0\}$ and $\Gamma = \mathfrak{so}(2)$ acting by infinitesimal rotations. Calling x and y the coordinates on \mathbb{R}^2 , we have

$$\delta x = yc, \quad \delta y = -xc, \quad \delta c = 0.$$

Let $\mathcal{M}' = \{x > 0\}$. A possible choice for F is the function $F(x, y) = y$. Then

$$S_F(x, y, c, \lambda, \bar{c}) = S(x, y) + \lambda y + \bar{c}xc,$$

where S is the given rotation-invariant action function. Then

$$\langle g \rangle_F = \frac{\int_0^{+\infty} e^{\frac{i}{\hbar}S(x,0)} g(x,0) x dx}{\int_0^{+\infty} e^{\frac{i}{\hbar}S(x,0)} x dx},$$

which is equal to the expected

$$\frac{\int_{\mathcal{M}} e^{\frac{i}{\hbar}S(x,y)} g(x,y) dx dy}{\int_{\mathcal{M}} e^{\frac{i}{\hbar}S(x,y)} dx dy}$$

for a rotation-invariant function g .

12.3. Infinite dimensions

In the infinite dimensional case, we consider (12.2.1) whenever it makes sense as a perturbative expansion. We assume then to have an infinite-dimensional manifold of fields \mathcal{M} , an infinite-dimensional Lie algebra Γ acting freely on \mathcal{M} , and a local action function S . The space $\widetilde{\mathcal{M}}$ and the BRST operator δ will be defined exactly as above, and δ will still be a cohomological vector field on $\widetilde{\mathcal{M}}$.

A gauge-fixing function F will be allowed if the corresponding A is nondegenerate and the critical point of the action function at a zero of F is also nondegenerate. Then, for suitable functions g , we will be able to define $\langle g \rangle_F$ as a perturbative expansion

in terms of Feynman diagrams. An observable in this new context is a δ -cohomology class g for which $\langle g \rangle_F$ is well-defined.

Theorem 12.2.5 will hold whenever $\text{div } \delta = 0$. The divergence of δ is however not defined a priori and has to be understood in terms of expectation values (as we discussed, e.g., in section 11.3.3). The usual pragmatical way to proceed, however, is to assume Theorem 12.2.5 to hold and to derive from it properties of the expectation values. Once they are properly defined in terms of Feynman diagrams, one checks if the identities hold and calls any deviation an **anomaly**.

Observe that if our aims are of mathematical nature, “Theorem” 12.2.5 then provides a source for a lot of interesting conjectures (which in most cases, fortunately, turn out to be true).

12.3.1. The trivial Poisson sigma model on the plane. — We want to consider a two-dimensional generalization of the example discussed in section 11.3, which will be also at the basis for the study of the Poisson sigma model in the following Sections. Let ξ and η be a zero-form and a one-form on the plane respectively. We assume them to vanish at infinity sufficiently fast (e.g., as Schwarz functions) so that we may define the action function

$$(12.3.1) \quad S := \int_{\Sigma} \eta d\xi,$$

with $\Sigma = \mathbb{R}^2$. Our manifold of fields is then $\mathcal{M} = \Omega_0^0(\mathbb{R}^2) \oplus \Omega_0^1(\mathbb{R}^2)$. On this space we have an action of the abelian Lie algebra $\Gamma = \Omega_0^0(\mathbb{R}^2)$, given by the monomorphism $\iota \circ d$,

$$\Gamma \xrightarrow{d} \Omega_0^1(\mathbb{R}^2) \xrightarrow{\iota} \mathcal{M}$$

with ι the inclusion, and the action function is clearly invariant. The BRST differential is then given on coordinates by

$$\delta\xi = 0, \quad \delta\eta = dc.$$

To define a gauge-fixing function, we choose a Riemannian metric on \mathbb{R}^2 . Denoting by $*$ the induce Hodge-star operator, we have the pairing $(\alpha, \beta) = \int_{\mathbb{R}^2} (*\alpha)\beta$ of forms on \mathbb{R}^2 . Denoting by $d^* = *d*$ the formal adjoint of the de Rham differential, we choose the gauge-fixing function $F(\xi, \eta) = d^*\eta$. Different choices of metrics give gauge-fixing functions connected by paths. It turns out that the corresponding operator A , see (12.2.2), is the Laplace operator on zero-forms which is invertible for the given conditions at infinity. Moreover, there is a unique critical point (i.e., a solution of $d\xi = d\eta = 0$) satisfying the gauge-fixing condition $d^*\eta = 0$: viz., $\xi = \eta = 0$. Thus, the proposed gauge fixing is allowed.

By integration, we identify Γ^* with $\Omega^2(\mathbb{R}^2)$. Then the corresponding gauge-fixing fermion is $\Psi_F = \int_{\mathbb{R}^2} \bar{c} d^* \eta$. This leads to the gauge-fixed action function

$$(12.3.2) \quad S_F = \int_{\mathbb{R}^2} \eta d\xi + \lambda d^* \eta - \bar{c} d^* dc.$$

Using the pairing of forms, we may rewrite it as

$$S_F = \frac{1}{2}(\phi, M\phi) - (*\bar{c}, \Delta c),$$

where $\Delta = d^*d + dd^*$ is the Laplace operator,

$$\phi = \begin{pmatrix} \xi \\ \eta \\ \lambda \end{pmatrix} \in \Omega_0^0(\mathbb{R}^2) \oplus \Omega_0^1(\mathbb{R}^2) \oplus \Omega^2(\mathbb{R}^2),$$

and

$$M = \begin{pmatrix} 0 & *d & 0 \\ *d & 0 & d^* \\ 0 & d^* & 0 \end{pmatrix}.$$

The propagators between ϕ and c or \bar{c} are clearly zero. By (11.2.5) we get

$$\langle *\bar{c}(z) c(w) \rangle_0 = -i\hbar G_0(w, z),$$

where G_0 is the Green function of the Laplace operator acting on functions. By (11.2.3), in order to get the propagator between two fields ϕ , we have to invert the symmetric operator M . We use a little trick as in [8], and compute first its square

$$M^2 = \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix}.$$

Then we observe that $M^{-1} = M M^{-2}$; thus,

$$M^{-1} = \begin{pmatrix} 0 & *d\Delta^{-1} & 0 \\ *d\Delta^{-1} & 0 & d^*\Delta^{-1} \\ 0 & d^*\Delta^{-1} & 0 \end{pmatrix}$$

In particular,

$$\langle \xi(z) \eta(w) \rangle_0 = i\hbar *_z d_z G_1(z, w) = i\hbar *_w d_w G_0(w, z),$$

where G_1 is the Green function of the Laplace operator acting on one-forms. For further convenience we introduce the “superfields”

$$(12.3.3a) \quad \tilde{\xi} = \xi - d^* \bar{c},$$

$$(12.3.3b) \quad \tilde{\eta} = c + \eta,$$

and compute the “superpropagator”

$$(12.3.4) \quad i\hbar\theta(z, w) := \left\langle \tilde{\xi}(z) \tilde{\eta}(w) \right\rangle_0 = \langle \xi(z) \eta(w) \rangle_0 - \langle d^* \bar{c}(z) c(w) \rangle_0 = \\ = i\hbar(*_z d_z + *_w d_w) G_0(w, z) \in \Omega^1(C_2(\mathbb{R}^2)),$$

where $C_2(\mathbb{R}^2)$ denotes the configuration space of two points on \mathbb{R}^2 .

Lemma 12.3.1. — *If we choose the Euclidean metric, then*

$$\theta = \frac{d\phi_E}{2\pi},$$

where d is the differential on $C_2(\mathbb{R}^2)$ and $\phi_E(z, w)$ is the Euclidean angle between a fixed reference line and the line passing through z and w .

Proof. — The Green function for the Euclidean Laplace operator in two dimensions is

$$G_0(z, w) = \frac{1}{2\pi} \log |z - w|,$$

where $|\cdot|$ is the Euclidean norm. It is easier to work with complex coordinates, in which we have

$$G_0(z, w) = \frac{1}{4\pi} \log(z - w)(\bar{z} - \bar{w}).$$

Then

$$d_z G_0(z, w) = \frac{1}{4\pi} \left(\frac{dz}{z - w} + \frac{d\bar{z}}{\bar{z} - \bar{w}} \right).$$

The Euclidean Hodge operator in complex coordinates acts as follows: $*dz = -idz$, $*d\bar{z} = id\bar{z}$. Hence,

$$*_z d_z G_0(z, w) = \frac{1}{4\pi i} \left(\frac{dz}{z - w} - \frac{d\bar{z}}{\bar{z} - \bar{w}} \right).$$

Analogously we get

$$*_w d_w G_0(z, w) = \frac{1}{4\pi i} \left(\frac{dw}{w - z} - \frac{d\bar{w}}{\bar{w} - \bar{z}} \right).$$

Summing up, we obtain

$$\theta = \frac{1}{4\pi i} \left(\frac{dz - dw}{z - w} - \frac{d\bar{z} - d\bar{w}}{\bar{z} - \bar{w}} \right) = \frac{1}{4\pi i} d \log \frac{z - w}{\bar{z} - \bar{w}}.$$

On the other hand, $z - w = |w - z|e^{i\phi}$ yields

$$\phi = \frac{1}{2i} \log \frac{z - w}{\bar{z} - \bar{w}},$$

which completes the proof. √

The cohomology class of θ is then the generator of $H^1(C_2(\mathbb{R}^2); \mathbb{Z})$. It is not difficult to see that other choices of metric yield the same class.⁽⁴⁾

12.3.2. Expectation values. — Any function of ξ is BRST-invariant; e.g., we may consider the evaluation of ξ at some point u . A function of $\int_\gamma \eta$ is also invariant for any closed curve γ . Thus, the expectation value

$$\left\langle \xi(u) \int_\gamma \eta \right\rangle_0 = i\hbar W_\gamma(u), \quad u \notin \text{im } \gamma,$$

is independent of the gauge-fixing. As we also have $i\hbar W_\gamma(u) = \left\langle \tilde{\xi}(u) \int_\gamma \tilde{\eta} \right\rangle_0$, by the previous considerations we see immediately that $W_\gamma(u)$ is the winding number of γ around u . Observe that this number is invariant under deformations of the curve γ or displacement of u , an indication that this theory is also topological. We may formally prove this invariance by using the same techniques as in section 11.3.4. For example, let us deform γ to γ' . Denoting by σ a two-chain whose boundary is $\gamma - \gamma'$, we get

$$W_\gamma(u) - W_{\gamma'}(u) = \left\langle \xi(u) \int_\sigma d\eta \right\rangle_0.$$

If we introduce the sequence of divergence-free vector fields $X_r(\xi, \eta) = \lambda_r \oplus 0$, where $\{\lambda_r\}$ is a sequence of functions that converges almost everywhere to the characteristic function of the image of σ , we get

$$W_\gamma(u) - W_{\gamma'}(u) = \lim_{r \rightarrow \infty} \langle \xi(u) X_r(S) \rangle_0 = i\hbar \lim_{r \rightarrow \infty} \langle X_r(\xi(u)) \rangle_0 = 0,$$

under the assumption that u does not belong to σ .

12.3.3. The trivial Poisson sigma model on the upper half plane. — We now consider the action function in (12.3.1) with Σ the upper half plane $H = \mathbb{R} \times \mathbb{R}^+$. As a boundary condition, we impose that the one-form η vanishes when restricted to the boundary $\partial H = \mathbb{R} \times \{0\}$. The Lie algebra that acts on the space of fields consists now of zero-forms on H vanishing on ∂H (more generally we might consider zero-forms that are constant on ∂H , but then the action would not be free). The BRST

⁽⁴⁾A simple observation is that $*_w \theta(z, w)$ is the Green function of the operator $P = *d\Delta^{-1}*$ which is a parametrix for the de Rham differential on forms that vanish at infinity; viz.,

$$dP + Pd = \text{id}.$$

The convolution relating P to θ may be written as

$$P\alpha = -\pi_{2*}(\theta \pi_1^* \alpha), \quad \alpha \in \Omega_0^*(\mathbb{R}^2),$$

with π_1 and π_2 the two projections to \mathbb{R}^2 . Then

$$dP\alpha + Pd\alpha = -\pi_{2*}(d\theta \pi_1^* \alpha) + \pi_*^\partial(\theta) \alpha,$$

where $\pi_*^\partial(\theta)(w)$ denotes the integral of θ along a limiting small circle around w . Since P is a parametrix and α is arbitrary, we see that in general θ is closed and has integral one along the generator of $H_1(C_2(\mathbb{R}^2); \mathbb{Z})$.

complex may be defined exactly as above, and we may choose the same gauge-fixing function. We define the superpropagator as in (12.3.4), but we denote it by ϑ instead of θ to avoid confusion. Then we have the following generalization of Lemma 12.3.1:

Lemma 12.3.2. — *If we choose the Euclidean metric, then*

$$\vartheta = \frac{d\phi_h}{2\pi},$$

where d is the differential on $C_2(H)$ and $\phi_h(z, w)$ is the angle between the vertical line through w and the geodesic joining w to z in the hyperbolic Poincaré metric.

Proof. — We use the classical method of the images. The Green function G_0^H of the Laplace operator on H is the restriction to H of the Green function of the Laplace operator on \mathbb{R}^2 plus a harmonic function such that the sum satisfies the boundary conditions. In complex coordinates, we need $G_0^H(w, z) = 0$ if w is real. This is achieved by

$$G_0^H(w, z) = G_0(w, z) - G_0(\bar{w}, z).$$

Then

$$\vartheta(z, w) = \theta(z, w) - \theta(z, \bar{w}).$$

Since the hyperbolic angle is given by

$$\phi_h(z, w) = \frac{1}{2i} \log \frac{(z-w)(\bar{z}-w)}{(\bar{z}-\bar{w})(z-\bar{w})},$$

this completes the proof. ✓

Observe that ϑ is the generator of $H^1(C_2(H), H \times \partial H; \mathbb{Z})$.

12.3.4. Generalizations. — The simplest generalization of the trivial Poisson sigma model described in section 12.3.1 and section 12.3.3 consists in allowing more fields. Namely, we take a collection of n zero-forms ξ^i and of n one-forms η_i , $i = 1, \dots, n$, and consider the action function $\int_{\Sigma} \eta_i d\xi^i$. We may also think of ξ and η as forms taking values in \mathbb{R}^n . The Lie algebra Γ of symmetries will now consist of the direct sum of n copies of the previous one; in other words it will be the abelian Lie algebra of \mathbb{R}^n -valued zero-forms. Denoting by c_i , $i = 1, \dots, n$, the generators of the algebra of functions of $\Pi\Gamma$, we may write the BRST operator as $\delta\xi^i = 0$, $\delta\eta_i = c_i$. If we choose the gauge-fixing function to be $F_i(X, \eta) = d^*\eta_i$, the computation is exactly analogous to the one given above. In particular, we may introduce again “superfields” $\tilde{\xi}^i = \xi^i - d^*\bar{c}^i$ and $\tilde{\eta}_i = c_i + \eta_i$ (where the c^i s are the generators of the algebra of functions of $\Pi\Gamma$) and compute the “superpropagator”

$$(12.3.5) \quad \left\langle (\tilde{\xi}^i(z) \tilde{\eta}_i(w)) \right\rangle_0 = \begin{cases} i\hbar\delta_j^i\theta(z, w) & \text{on the plane,} \\ i\hbar\delta_j^i\vartheta(z, w) & \text{on the upper half plane.} \end{cases}$$

The next generalization consists in dropping the assumption that the zero-form field vanishes at infinity. More precisely, we denote by X^i a collection of maps to \mathbb{R}^n with no conditions on the boundary or at infinity and consider the action function

$$(12.3.6) \quad S = \int_{\Sigma} \eta_i dX^i,$$

where Σ is the plane or the upper half plane and the η_i s are one-forms vanishing on the boundary and at infinity. Critical points are pairs of constant maps together with closed one-forms. They are now degenerate also after modding out by the action of the abelian Lie algebra of zero-forms. However, the degeneracy is now of a very simple type as it is parametrized by the finite-dimensional manifold \mathbb{R}^n —the value of the constant map. As discussed in section 11.4.1, it is enough to choose a measure on \mathbb{R}^n and impose Fubini's theorem. We choose a delta-measure picked at a point $x \in \mathbb{R}^n$ and require the X to map infinity to x . If we write $X^i = x^i + \xi^i$, then ξ^i vanishes at infinity and we are reduced to the previous situation.

The last generalization is to replace \mathbb{R}^n by a manifold M . We think of X^i as a local coordinate expression of a map $X: \Sigma \rightarrow M$. For the action function to be covariant, we should assume $\eta_i(u)$, $u \in \Sigma$, to be the local coordinate expression of a one-form on Σ taking values in the cotangent space of M at $X(u)$. Namely, we assume $\eta \in \Gamma(\Sigma, T^*\Sigma \otimes X^*T^*M)$. The manifold of fields \mathcal{M} may then be identified with the manifold of bundle maps $T\Sigma \rightarrow T^*M$ and the action function may be written in an invariant way as

$$S = \int_{\Sigma} \langle \eta, dX \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between the tangent and the cotangent bundle of M , and dX is the differential of the map X regarded as a section of $T^*\Sigma \otimes X^*T^*M$. If we now require X to map infinity to a given point $x \in M$, we may expand around the critical solution by setting $X = x + \xi$, $\xi: \Sigma \rightarrow T_x M$ and by regarding η as a one-form taking values in T_x^*M . Choosing coordinates we may identify $T_x M$ with \mathbb{R}^n ($n = \dim M$) and reduce to the previous case.

Finally, we may allow Σ to be any two-manifold. The above discussion will change drastically if Σ is not simply connected, as the space of solutions modulo symmetries, with $X = x$, will now be parametrized by $H^1(\Sigma, T_x^*M)$ and one has to choose a measure on this vector space as well.

CHAPTER 13

THE POISSON SIGMA MODEL

We want now to discuss deformations of the trivial Poisson sigma model described in sections 12.3.1, 12.3.3 and 12.3.4. This chapter is included only to provide a motivation for the action function that describes the deformation quantization of affine Poisson structures, see Chapter 14, and may be safely skipped.

13.1. The Poisson sigma model

Our starting point is the trivial Poisson sigma model (12.3.6). We want to deform its action function without introducing extra structure on Σ (the plane or the upper half plane). This means that the terms we are allowed to add must be two-forms on Σ built in terms of the fields X^i and η_i ; viz., they must be linear combinations of terms $\alpha^{ij}(X)\eta_i\eta_j$, $\beta_j^i(X)\eta_i dX^j$ and $\gamma_{ij}(X)dX^i dX^j$ (we do not consider a term $\phi^i(X)d\eta_i$ as integration by parts reduces it to a term of the second type). The second and third terms may however be absorbed by a redefinition of η adding to it terms linear in η and dX . Thus, modulo field redefinitions, the most general deformation of the action function has the form

$$S = \int_{\Sigma} \left(\eta_i dX^i + \frac{1}{2} \epsilon \alpha^{ij}(X) \eta_i \eta_j \right) + O(\epsilon^2),$$

where ϵ is the deformation parameter and α^{ij} is assumed to be skew-symmetric. The aim of this Section is to show that it makes sense to consider only those deformations in which the α^{ij} s are the components of a Poisson bivector field and that the BRST method is available only if the Poisson structure is affine.

We want first to deform the symmetries accordingly (without adding extra structure on Σ). We do it already in the BRST framework. We recall that in the trivial

case the BRST operator acted by $\delta X^i = 0$, $\delta \eta_i = dc_i$ and $\delta c_i = 0$, with $c \in \Pi\Gamma$ and $\Gamma = \Omega_0^0(\Sigma, \mathbb{R}^n)$. We want to deform the trivial δ so that $\delta S = O(\epsilon^2)$ for the new S .⁽¹⁾

Lemma 13.1.1. — *Modulo field redefinitions, there is a unique BRST operator deforming the trivial one such that $\delta S = O(\epsilon^2)$ and $\delta^2 = O(\epsilon^2) + R$ with R vanishing at critical points. It acts by*

$$\begin{aligned}\delta X^i &= -\epsilon \alpha^{ij}(X) c_j + O(\epsilon^2), \\ \delta \eta_i &= dc_i + \epsilon \partial_i \alpha^{jk}(X) \eta_j c_k + O(\epsilon^2), \\ \delta c_i &= -\frac{1}{2} \epsilon \partial_i \alpha^{jk}(X) c_j c_k + O(\epsilon^2).\end{aligned}$$

Moreover, R vanishes on the whole $\mathcal{M} \times \Pi\Gamma$ if α is at most linear.

Proof. — Recalling that δ applied to X or η must be linear in the ghosts c , the most general deformation of δ (without adding extra structure on Σ) is of the form

$$\begin{aligned}\delta X^i &= \epsilon v^{ij}(X) c_j + O(\epsilon^2), \\ \delta \eta_i &= dc_i + \epsilon (a_i^{jk}(X) \eta_j c_k + b_i^j(X) dc_j + d_{ij}^k(X) dX^j c_k) + O(\epsilon^2),\end{aligned}$$

for some functions v^{ij} , a_i^{jk} , b_i^j and d_{ij}^k on \mathbb{R}^n . Thus,

$$\begin{aligned}\delta S &= \epsilon \int_{\Sigma} ((a_i^{jk}(X) \eta_j c_k + b_i^j(X) dc_j + d_{ij}^k(X) dX^j c_k) dX^i + \\ &\quad + \eta_i (dX^r \partial_r v^{ij}(X) c_j + v^{ij}(X) dc_j) + \alpha^{ij}(X) dc_i \eta_j) + O(\epsilon^2).\end{aligned}$$

As the identity $\delta S = O(\epsilon^2)$ must hold for any η , we get the following two equations

$$(13.1.1a) \quad a_i^{jk}(X) c_k dX^i + dX^r \partial_r v^{jk}(X) c_k + v^{jk}(X) dc_k + \alpha^{kj}(X) dc_k = 0,$$

$$(13.1.1b) \quad \int_{\Sigma} (b_i^j(X) dc_j + d_{ij}^k(X) dX^j c_k) dX^i = 0.$$

In particular, choosing X to be a constant map (with value x), we deduce from the first equation

$$v^{jk}(x) dc_k + \alpha^{kj}(x) dc_k = 0,$$

and as this has to hold for any c , we finally have

$$\alpha^{jk} = v^{jk}.$$

Inserting this into (13.1.1a) yields

$$a_i^{jk}(X) c_k dX^i - dX^i \partial_i \alpha^{jk}(X) c_k = 0,$$

and as this also has to hold for all c and X , we get

$$a_i^{jk} = \partial_i \alpha^{jk}.$$

⁽¹⁾We consider only the restriction of δ to $\mathcal{M} \times \Pi\Gamma$ as its restriction to $\Gamma^* \times \Pi\Gamma^*$ needs no deformation.

Integrating by parts, (13.1.1b) yields

$$\int_{\Sigma} (-dX^r \partial_r b_i^j(X) c_j + d_{ij}^k(X) dX^j c_k) dX^i = 0,$$

and as this has to hold for all X and c , we finally get

$$d_{ij}^k = \partial_j b_i^k.$$

Thus, we have proved that

$$\begin{aligned} \delta X^i &= -\epsilon \alpha^{ij}(X) c_j + O(\epsilon^2), \\ \delta \eta_i &= dc_i + \epsilon (\partial_i \alpha^{jk}(X) \eta_j c_k + d(b_i^j(X) c_j)) + O(\epsilon^2), \end{aligned}$$

which, after the redefinition $c_i \mapsto c_i - \epsilon b_i^j(X) c_j + O(\epsilon^2)$, yields the the first two equations in the Lemma. As for the last equation, we recall that the BRST operator on c must be quadratic in c , so its general form is

$$\delta c_i = \frac{1}{2} \epsilon f_i^{jk}(X) c_j c_k + O(\epsilon^2).$$

To determine the “structure” functions f_i^{jk} , we compute δ^2 . Observe that $\delta^2 X^i = \delta^2 c_i = 0(\epsilon^2)$ automatically. On the other hand,

$$\begin{aligned} \delta^2 \eta_i &= \epsilon \left(\frac{1}{2} d(f_i^{jk}(X) c_j c_k) + \partial_i \alpha^{jk}(X) dc_j c_k \right) + 0(\epsilon^2) = \\ &= \epsilon \left((f_i^{jk}(X) + \partial_i \alpha^{jk}(X)) dc_j c_k + \frac{1}{2} dX^r \partial_r f_i^{jk}(X) c_j c_k \right) + 0(\epsilon^2). \end{aligned}$$

At a critical point (where $dX^i = O(\epsilon)$) the third summand of the last equation vanishes. Thus, $\delta^2 = O(\epsilon^2)$ at critical points implies

$$f_i^{jk} = -\partial_i \alpha^{jk},$$

which proves the last equation in the Lemma. Observe that

$$\delta^2 \eta_i = -\frac{1}{2} \epsilon dX^r \partial_r \partial_i \alpha^{jk}(X) c_j c_k,$$

which is identically zero (not only at critical points) if α is at most linear. \checkmark

We want now to extend deformations beyond the first order in ϵ . Even without knowing the following terms, we already have the following

Lemma 13.1.2. — $\delta^2 = O(\epsilon^3)$ at critical points only if α is Poisson.

Proof. — We have

$$\delta^2 X^i = -\epsilon^2 \left(\alpha^{rk}(X) c_k \partial_r \alpha^{ij}(X) c_j + \frac{1}{2} \alpha^{ij}(X) \partial_j \alpha^{rk}(X) c_r c_k \right) + O(\epsilon^3).$$

As this has to hold for all c , we get the Jacobi identity for α . \checkmark

It is now possible to prove that, under the assumption that α is Poisson, this deformation is not only infinitesimal. Namely, we have the

Theorem 13.1.3. — *Given a Poisson bivector field α , the odd vector field*

$$\begin{aligned}\delta X^i &= -\epsilon \alpha^{ij}(X) c_j, \\ \delta \eta_i &= dc_i + \epsilon \partial_i \alpha^{jk}(X) \eta_j c_k, \\ \delta c_i &= -\frac{1}{2} \epsilon \partial_i \alpha^{jk}(X) c_j c_k,\end{aligned}$$

is cohomological for α at most linear or at critical points $\forall \epsilon$. Moreover,

$$S := \int_{\Sigma} \left(\eta_i dX^i + \frac{1}{2} \epsilon \alpha^{ij}(X) \eta_i \eta_j \right)$$

is δ -closed $\forall \epsilon$.

For a proof see [53], [86] or [16] (or do the simple computation as an exercise). The geometrical meaning of this Theorem is that there is a distribution of vector fields on \mathcal{M} under which the action function is invariant. In general, this distribution is involutive only on the submanifold of critical points of S . It is involutive on the whole of \mathcal{M} if α is at most linear and in this case it can be regarded as the free, infinitesimal action of a Lie algebra (see Chapter 14 for more details in this case).

We remark that the action function S may be generalized to the case when one wants to consider a Poisson manifold (M, α) which is not an \mathbb{R}^n . To do so, one regards X as a map $\Sigma \rightarrow M$ and, for a given map X , η is taken to be a section of $T^*\Sigma \otimes X^*T^*M$. Denoting by $\langle \cdot, \cdot \rangle$ the canonical pairing between the tangent and the cotangent bundle of M and by $\alpha^\#$ the bundle map $T^*M \rightarrow TM$ induced by the Poisson bivector field α , we may write

$$S = \int_{\Sigma} \left(\langle \eta, dX \rangle + \frac{1}{2} \epsilon \langle \eta, \alpha^\#(X)\eta \rangle \right).$$

13.2. Observables

In the case when Σ is the upper half plane, c has to vanish on the boundary. This implies that $\delta X(u) = 0$ for $u \in \partial\Sigma$. As a consequence,

$$\begin{aligned}\mathcal{O}_{f_1, \dots, f_k; u_1, \dots, u_k} &:= f_1(X(u_1)) \cdots f_k(X(u_k)), \\ f_1, \dots, f_k &\in C^\infty(\mathbb{R}^n), \quad u_1, \dots, u_k \in \partial\Sigma \cong \mathbb{R}, \quad u_1 < \cdots < u_k,\end{aligned}$$

are observables (i.e., δ -closed functions). It is shown in [16] that, with the gauge-fixing $d^*\eta = 0$ for the Euclidean metric on Σ , one has

$$\langle \mathcal{O}_{f_1, \dots, f_k; u_1, \dots, u_k} \rangle(x) = f_1 \star \cdots \star f_k(x),$$

where $\langle \cdot \rangle(x)$ denotes the expectation value for $X(\infty) = x$ (and expanding only around the trivial critical solution $X = x, \eta = 0$) while \star is Kontsevich's star product

for the given Poisson structure. In Chapter 14 we will derive this result in the case when α is at most linear so that the BRST method is available.

CHAPTER 14

DEFORMATION QUANTIZATION OF AFFINE POISSON STRUCTURES

An affine Poisson structure on a vector space \mathbb{R}^n is the datum of Poisson bivector field α that is at most linear. The linear part of α gives the dual of \mathbb{R}^n a Lie algebra structure while the constant part is a 2-cocycle in the Lie algebra cohomology with trivial coefficients. From now on we will denote by \mathfrak{g} this Lie algebra. The fields of the Poisson sigma model for an affine Poisson structure are then a map $X: \Sigma \rightarrow \mathfrak{g}^*$ and a one-form η on Σ taking values in \mathfrak{g} . As \mathfrak{g} is a Lie algebra, we may regard η as a connection one-form for a trivial principal bundle P over Σ (with structure group any Lie group whose Lie algebra is \mathfrak{g}). For definiteness, we will fix Σ to be the upper half plane and we will require η to vanish at infinity and on the boundary. The action function reads

$$S = \int_{\Sigma} \left(\eta_i dX^i + \frac{1}{2} \alpha^{ij}(X) \eta_i \eta_j \right),$$

where

$$\alpha^{ij}(x) = \chi^{ij} + x^k f_k^{ij}$$

is the given affine Poisson structure on \mathfrak{g}^* . Integrating by parts, we may also rewrite it as

$$S = \int_{\Sigma} \left(\langle X, F_{\eta} \rangle + \frac{1}{2} \chi(\eta, \eta) \right),$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between \mathfrak{g} and \mathfrak{g}^* while

$$(F_{\eta})_i = d\eta_i + \frac{1}{2} f_i^{jk} \eta_j \eta_k$$

is the curvature two-form of the connection one-form η . In accordance with Chapter 13, there is a Lie algebra Γ —which as a vector space consists of functions $\Sigma \rightarrow \mathfrak{g}$ vanishing at infinity and on the boundary—that acts on the manifold of fields \mathcal{M} and leaves the action function invariant. The BRST operator on $\mathcal{M} \times \Pi\Gamma$ has the form displayed in Theorem 13.1.3 (with $\epsilon = 1$). Geometrically then we may regard Γ as

the Lie algebra of infinitesimal gauge transformations of the principal bundle P ; the field η actually transforms as a connection one-form, while X transform as a section of the coadjoint bundle in case $\chi = 0$. For $\chi \neq 0$, we may regard $X \oplus 1$ as a section of the coadjoint bundle for the Lie algebra $\widehat{\mathfrak{g}} \simeq \mathfrak{g} \oplus \mathbb{R}$ obtained by central extension of \mathfrak{g} through χ . The BRST operator on $\Pi^* \times \Gamma^*$ has the usual form displayed in (12.2.4).

14.1. Gauge-fixing and Feynman diagrams

As in sections 12.3.1 and 12.3.3, we choose a metric on Σ and define the gauge-fixing function $F(X, \eta) = d^*\eta$. The gauge-fixing fermion is then $\Psi_F = \int_{\Sigma} \langle \bar{c}, d^*\eta \rangle$ and the gauge-fixed action function reads

$$S_F = \int_{\Sigma} \left(\eta_i dX^i + \frac{1}{2} \alpha^{ij}(X) \eta_i \eta_j + \lambda^i d^* \eta_i - \bar{c}^k d^*(dc_k + \partial_k \alpha^{ij}(X) \eta_i c_j) \right).$$

We now fix the value of X at infinity to x . We write $X = x + \xi$, where the field ξ has to vanish at infinity. We observe that S_F has the form given in (11.4.2) on page 135 (with y the collection of ξ , \bar{c} , λ , and z the collection of η and c). Thus, we write $S_F = S_0 + S_1$, with

$$\begin{aligned} S_0 &= \int_{\Sigma} (\eta_i dX^i + \lambda^i d^* \eta_i - \bar{c}^k d^* dc_k), \\ S_1 &= \int_{\Sigma} \left(\frac{1}{2} \alpha^{ij}(x + \xi) \eta_i \eta_j - \bar{c}^k d^*(\partial_k \alpha^{ij}(x + \xi) \eta_i c_j) \right), \end{aligned}$$

and regard S_1 as a perturbation of S_0 . The unperturbed action function S_0 consists of $\dim \mathfrak{g}$ copies of the action function (12.3.2) which we have studied before. The perturbation S_1 may be rewritten, integrating by parts, as

$$S_1 = \int_{\Sigma} \left(\frac{1}{2} \alpha^{ij}(x + \xi) \eta_i \eta_j - d^* \bar{c}^k \partial_k \alpha^{ij}(x + \xi) \eta_i c_j \right).$$

Introducing “superfields” $\tilde{\xi}$ and $\tilde{\eta}$ as in (12.3.3), we have finally

$$S_1 = \int_{\Sigma} \frac{1}{2} \alpha^{ij}(x + \tilde{\xi}) \tilde{\eta}_i \tilde{\eta}_j,$$

where integration on Σ is understood to select the two-form component. This shows that, as long as also the observables under consideration may be written as functions of the superfields, expectation values are computed in terms of the superpropagators (12.3.5) only. If we denote the superpropagator graphically as an arrow from $\tilde{\eta}$ to $\tilde{\xi}$, the perturbation S_1 is represented by the two vertices in figure 1, with the bivalent vertex corresponding to $\alpha^{ij}(x) = \chi^{ij} + x^k f_k^{ij}$ and the trivalent vertex corresponding to the structure constants.

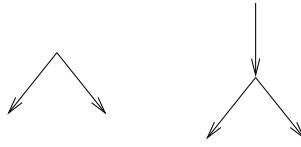


FIGURE 1. The two vertices.

We now consider the observables $\mathcal{O}_{f_1, \dots, f_k; u_1, \dots, u_k}$ introduced in section 13.2. As evaluation at a point, i.e., integration along a zero-cycle, is understood to select the zero-form component of a differential form, we may write

$$\begin{aligned} \mathcal{O}_{f_1, \dots, f_k; u_1, \dots, u_k} &= f_1(x + \tilde{\xi}(u_1)) \cdots f_k(x + \tilde{\xi}(u_k)), \\ f_1, \dots, f_k &\in C^\infty(\mathbb{R}^n), \quad u_1, \dots, u_k \in \partial\Sigma \cong \mathbb{R}, \quad u_1 < \cdots < u_k. \end{aligned}$$

As a consequence, in computing the expectation value $\langle \mathcal{O}_{f_1, \dots, f_k; u_1, \dots, u_k} \rangle(x)$, we only need the superpropagator. The corresponding Feynman diagrams then have three kinds of vertices:

1. bivalent vertices in the upper half plane corresponding to $\alpha^{ij}(x)$;
2. trivalent vertices in the upper half plane corresponding to f_k^{ij} ;
3. l -valent vertices, $l \geq 0$ with only incoming arrows at one of the boundary points u_i corresponding to the l th derivative of f_i .

As observed in section 11.4.1, the normal ordering prescription excludes all graphs containing a tadpole (i.e., an edge whose head and tail are the same vertex). The combinatorics prevents automatically vacuum subgraphs. See figure 2 for examples.

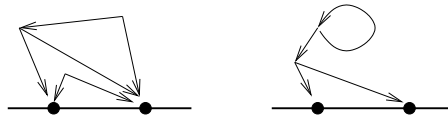


FIGURE 2. An allowed graph and a non-allowed graph (tadpole).

In the case $k = 2$ and considering the gauge-fixing $d^*\eta = 0$ w.r.t. the Euclidean metric on the upper half plane, i.e., with the superpropagator determined by the one-form ϑ in Lemma 12.3.2, we get

$$\langle \mathcal{O}_{f, g; 0, 1} \rangle(x) = f \star g(x),$$

where \star denotes Kontsevich's star product for the given affine Poisson structure. We leave the (now) easy proof of this fact as an exercise.

14.2. Independence from the evaluation points

Reasoning as in section 11.3.4, we give a formal proof of the independence of the expectation values of $\mathcal{O}_{f_1, \dots, f_k; u_1, \dots, u_k}$ from the points u_1, \dots, u_k . The first observation is that

$$f(X(v)) - f(X(u)) = \int_u^v dX^i \partial_i f(X) = \int_u^v (dX^i + d^* \lambda^i) \partial_i f(X) - \delta \Phi,$$

with $\Phi = \int_u^v d^* \bar{c}^i \partial_i f(X)$. Let ω_r be a sequence of one-forms on Σ vanishing on the boundary and at infinity that converges to the measure concentrated on the interval $(u, v) \in \Sigma$.⁽¹⁾ Let $Y_{f,r}$ be the local vector field on $\widetilde{\mathcal{M}}$ corresponding to the infinitesimal displacement of η_i by $\omega_r \partial_i f(X)$. Then

$$f(X(v)) - f(X(u)) = \lim_{r \rightarrow \infty} Y_{f,r}(S_F) - \delta \Phi.$$

If \mathcal{O} is a BRST-observable depending on fields outside the closed interval $[u, v]$, we have

$$\langle (f(X(v)) - f(X(u))) \mathcal{O} \rangle = i\hbar \lim_{r \rightarrow \infty} \langle Y_{f,r}(\mathcal{O}) \rangle - \langle \delta(\Phi \mathcal{O}) \rangle = 0.$$

14.3. Associativity

The independence from the evaluation points shows in particular that

$$\lim_{v \rightarrow u^+} \langle \mathcal{O}_{f,g,h;u,v,w} \rangle_0(x) = \lim_{v \rightarrow w^-} \langle \mathcal{O}_{f,g,h;u,v,w} \rangle_0(x).$$

Intuitively the l.h.s. corresponds to evaluating first the expectation value of $\mathcal{O}_{f,g;u,v}$, placing the result at u and finally computing the expectation value of $\mathcal{O}_{\langle \mathcal{O}_{f,g;u,v} \rangle_0, h; w}$. The result is then $(f \star g) \star h$. Repeating the computation on the r.h.s. we get $f \star (g \star h)$.

This rather formal argument explains why one should expect the star product defined by the Poisson sigma model to be associative.

⁽¹⁾Denoting by (a, b) , $b \geq 0$, the coordinates on the upper half plane Σ , a possible choice for this sequence is

$$\omega_r(a, b) = rb e^{-\frac{rb^2}{2}} \chi_r(a) da,$$

where χ_r is a sequence of smooth, compactly supported functions converging to the characteristic function of the interval (u, v) almost everywhere.

PART IV

APPENDICE A

ESPACES DE CONFIGURATIONS (PAR A. BRUGUIÈRES)

On explique la construction des compactifications des variétés de configurations décrites dans [60] en suivant les notes de l'exposé [81]. Ce sont des variétés à coins. Leur construction s'inspire de celles de [38]. On démontre un lemme fondamental d'annulation sur l'intégration des 3-formes sur une variété de configurations.

A.1. Terminologie

Dans tout l'exposé, et sauf mention contraire, les "variétés" sont des variétés analytiques réelles et les "morphisms", des applications analytiques réelles.

Notations. On note T le tore \mathbb{R}/\mathbb{Z} , et, si X est une variété et I un ensemble fini, on note $X_*^I \subset X^I$ l'ouvert des applications injectives $I \hookrightarrow X$.

Germes. Si $X \subset E$ et $Y \subset E'$ sont des parties localement fermées (pour la topologie ordinaire) de variétés E, E' , une application $f : X \rightarrow Y$ est dite *analytique* si pour tout $x_0 \in X$, il existe un voisinage ouvert U de x_0 dans E et une application analytique $g : U \rightarrow E'$ telle que $\forall x \in U \cap X, f(x) = g(x)$.

On définit une 'catégorie de germes' dont les objets sont les couples (X, Z) , où X est une partie localement fermée d'une variété E , et Z une partie localement fermée de X . L'objet (X, Z) est appelé *germe de X le long de Z* . En particulier si $x \in X$, $(X, \{x\})$ est le *germe de X en x* . Les morphismes d'un germe (X, Z) vers un germe (X', Z') sont les éléments de l'ensemble

$$\varinjlim_U \{f : U \rightarrow X' \mid f \text{ analytique et } f(Z) \subset Z'\}.$$

où U décrit l'ensemble des voisinages ouverts de Z dans X .

Variétés à coins. Une *variété à coins* est une partie localement fermée Z d'une variété E dont le germe en tout point est isomorphe au germe de $\mathbb{R}^{d-h} \times \mathbb{R}_+^h$ à l'origine, pour $d, h \in \mathbb{N}$ dépendant du point.

L'entier h , vu comme fonction sur une variété à coins X , définit une partition :

$$X = \bigsqcup_{h \in \mathbb{N}} \partial^h X.$$

Les composantes connexes des $\partial^h X$ forment une stratification, qu'on appelle la *stratification canonique de X* . Chaque strate est une variété sans coins, dont l'adhérence est une variété à coins.

L'intérieur de X est la variété $\partial^0 X$, et le bord de X , le fermé $X - \partial^0 X \subset X$.

A.2. L'espace de configurations C^I

Soit I un ensemble fini de cardinal n . Le groupe G des permutations de \mathbb{C} de la forme $z \mapsto az + b$, avec $a \in \mathbb{R}_+^*$ et $b \in \mathbb{C}$, opère sur \mathbb{C}_*^I , et son action est libre si $n \geq 2$. Sous cette hypothèse, le quotient \mathbb{C}_*^I/G est une variété connexe de dimension $2n - 3$, orientée, qu'on note C^I .

La construction de C^I est fonctorielle en I en ce sens que toute application injective $J \hookrightarrow I$, avec $\#J \geq 2$, induit par restriction un morphisme $C^I \rightarrow C^J$.

Notons $C^n = C^{\{1,2,\dots,n\}}$. L'application $\phi : C^2 \rightarrow \mathbb{T}$ définie par $\phi(z_1, z_2) = \frac{1}{2\pi} \text{Arg}(z_2 - z_1)$ est un isomorphisme de variétés de C^2 sur \mathbb{T} . Ainsi, C^2 est compact. Pour $n \geq 3$, l'application $\theta : \mathbb{T} \times (\mathbb{C} - \{0, 1\})_*^{n-2} \rightarrow C^n$ définie par

$$(t, (\zeta_1, \dots, \zeta_{n-2})) \mapsto e^{2i\pi t}(0, 1, \zeta_1, \dots, \zeta_{n-2})$$

est un isomorphisme de variétés, et C^n n'est pas compact. En particulier, $C^3 \simeq \mathbb{T} \times (\mathbb{C} - \{0, 1\})$.

Pour $k, l \in I$, $k \neq l$, on définit une fonction analytique $\phi_{k,l} : C^I \rightarrow \mathbb{T}$ par $\phi_{k,l}((z_i)_{i \in I}) = \phi(z_k, z_l)$. Les fonctions $\phi_{k,l}$ et leurs différentielles sont appelées à jouer un rôle important.

L'un des buts de l'exposé est de construire une compactification \overline{C}^I de C^I qui soit fonctorielle en I ; ce caractère fonctoriel aura pour conséquence avantageuse que les fonctions $\phi_{k,l}$ se prolongeront analytiquement à \overline{C}^I . En revanche, nous devons quitter le cadre des variétés ordinaires : \overline{C}^I sera une variété à coins.

Dans un premier temps, nous allons construire une compactification plus naïve de C^I , que nous noterons \tilde{C}^I . Pour $c \in C^I$, notons

$$\underline{c} = \frac{1}{\#I} \sum_{i \in I} c_i \quad \text{et} \quad \|c\|^2 = \frac{1}{\#I} \sum_{i \in I} |c_i - \underline{c}|^2,$$

de sorte que $\|c\| = 0 \iff c : I \rightarrow \mathbb{C}$ est constante. Notons

$$C_0^I = \{c \in C^I \mid \|c\| \neq 0\} \quad \text{et} \quad \tilde{C}^I = \{c \in C^I \mid \underline{c} = 0 \text{ et } \|c\| = 1\}.$$

L'ensemble \tilde{C}^I est une sous-variété compacte de \mathbb{C}^I , qu'on voit comme l'ensemble des configurations normalisées (de moyenne 0 et d'écart-type 1).⁽¹⁾

Tout $c \in \mathbb{C}_0^I$ s'écrit de manière unique $c = g \cdot \tilde{c}$, avec $g \in G$ et $\tilde{c} \in \tilde{C}^I$, ce qui nous permet d'identifier \tilde{C}^I à \mathbb{C}_0^I/G , et C^I à $\tilde{C}^I \cap \mathbb{C}_*^I$. Ainsi \tilde{C}^I est une compactification de C^I . Toutefois elle n'est pas fonctorielle en I car les fonctions $\phi_{k,l}$ ne s'y prolongent pas continûment.

Afin de construire une compactification fonctorielle, considérons le plongement

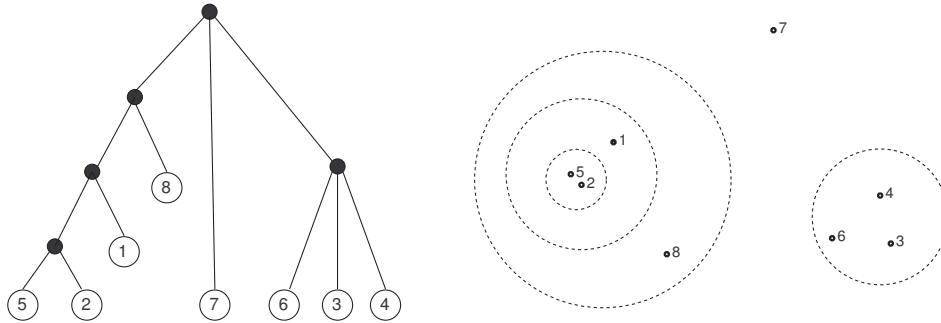
$$C^I \rightarrow \prod_{J \subset I, \#J \geq 2} \tilde{C}^J$$

défini par $c \mapsto (\xi_J)_{J \subset I, \#J \geq 2}$, où $\xi_J = \tilde{c}_J$.

On note \overline{C}^I l'adhérence de l'image de ce morphisme; c'est une compactification de C^I , fonctorielle en I par construction. En particulier, pour $j, k \in I, j \neq k$, la projection $\prod_J \tilde{C}^J \rightarrow \tilde{C}^{\{j,k\}} \simeq \mathbb{C}^2$, composée avec $\phi : \mathbb{C}^2 \xrightarrow{\sim} \mathbb{T}$, est un prolongement analytique de $\phi_{j,k} : C^I \rightarrow \mathbb{T}$.

A.2.1. Structure de \overline{C}^I . — Nous allons démontrer que \overline{C}^I est une variété à coins, et pour cela, nous allons tout d'abord définir une stratification de cet espace, qui sera en définitive sa stratification canonique.

De manière informelle, voici l'idée. Lorsqu'une configuration $c \in C^I$ s'approche du bord, certains points se confondent. À l'échelle macroscopique, on voit certains paquets s'effondrer. Mais, si l'on observe chacun de ces paquets au microscope avec un grossissement adéquat, on verra de nouveau une configuration de points où certains sous-paquets s'effondrent à leur tour. On peut répéter cette opération de grossissement jusqu'à isoler un seul point dans l'oculaire. La structure combinatoire sous-jacente à cette exploration est un arbre ayant pour feuilles les éléments de I , et ces arbres vont précisément indexer les strates.



Graphe et configuration

⁽¹⁾En fait, \tilde{C}^I est une sphère de dimension $2n - 3$.

Définition A.2.1. — On appelle *I-arbre* un arbre dont les feuilles sont indexées par les éléments de I , c'est-à-dire, plus formellement, une partie T de $\mathcal{P}(I)$ vérifiant :

- (i) T contient I et les singletons, mais pas \emptyset ;
- (ii) deux éléments de T sont emboîtés ou disjoints.

Les éléments de T sont les sommets de l'arbre ; I est la racine, et les singletons sont les feuilles. On note $T_0 = \{A \in T \mid \#A \geq 2\}$ l'ensemble des sommets intérieurs, $T_* = T_0 - \{I\}$, et pour $A \in T$, on note $\text{St } A$ l'ensemble des éléments maximaux de $\{A' \in T, A' \subsetneq A\}$, qui s'identifie à l'ensemble des arêtes issues du sommet A (étoile de A).

À tout point $\xi = (\xi_J)$ de \overline{C}^I est naturellement associé un *I-arbre* $T(\xi)$ défini comme suit : pour $J \subset I$, $\#J \geq 2$, et $J' \subset I$, on écrit $J' \prec J$ si J' est une fibre de l'application $\xi_J : J \rightarrow \mathbb{C}$. Une partie $A \subset I$ est un sommet de $T(\xi)$ si et seulement s'il existe une suite finie $A = A_r \prec A_{r-1} \prec \cdots \prec A_0 = I$.

Si T est un *I-arbre*, notons $\partial_T \overline{C}^I$ l'ensemble des $\xi \in \overline{C}^I$ tels que $T(\xi) = T$.

Proposition A.2.2. — Le compactifié \overline{C}^I est une variété à coins d'intérieur C^I , dont les strates canoniques sont les $\partial_T \overline{C}^I$ pour T décrivant l'ensemble des *I-arbres*.

De plus, la strate $\partial_T \overline{C}^I$ est de codimension $\#T_*$ et s'identifie à $\prod_{A \in T_0} C^{\text{St } A}$.

Exemple A.2.3. —

$$\overline{C}^2 = \bigcirc \quad \text{et} \quad \overline{C}^3 = \bigcirc \times \bigcirc \otimes \bigcirc$$

Démonstration. — Soit T un *I-arbre*, $S_T = \prod_{A \in T_0} C^{\text{St } A}$, et \mathcal{U}_T la variété à coins $S_T \times \mathbb{R}_+^{T_*}$.

Nous allons construire un morphisme partiellement défini $\Phi : \mathcal{U}_T \dashrightarrow \overline{C}^I$.

Nous montrerons ensuite que Φ définit un isomorphisme du germe de \mathcal{U}_T le long de $S_T \times \{0\}$ sur le germe de \overline{C}^I le long de $\partial_T \overline{C}^I$, d'où la proposition.

Pour $A, B \in T$, on écrira $B \prec A$ si $B \in \text{St } A$.

Soit $\gamma \in \mathcal{U}_T$, $\gamma = ((c_A)_{A \in T_0}, (s_B)_{B \in T_*})$. Pour $A \in T_0$, on définit un élément $x_A = (x_{A,i})_{i \in A}$ de \mathbb{C}^A par la formule :

$$x_{A,i} = c_{A_0, A_1} + s_{A_1} c_{A_1, A_2} + \cdots + s_{A_1} \cdots s_{A_{r-1}} c_{A_{r-1}, A_r}$$

où $\{i\} = A_r \prec A_{r-1} \prec \cdots \prec A_0 = A$.

Soit $J \subset I$, $\#J \geq 2$, A le plus petit élément de T contenant J , et posons $\xi_J = \widetilde{x_{A|J}}$ si $x_{A|J} \in \mathbb{C}_0^J$. Cette condition est ouverte, et elle est satisfaite si les s_B sont tous nuls.

On définit Φ par : $\Phi(\gamma) = (\xi_J)_{J \subset I, \#J \geq 2}$. C'est un morphisme défini dans un voisinage ouvert de $S_T \times \{0\}$.

On vérifie sans peine les faits suivants :

- a) si $\gamma \in S_T \times \{0\}$, $\Phi(\gamma) \in \partial_T \overline{C}^I$;

b) si $\gamma \in \mathcal{U}_T$ est intérieur et proche du bord, $\Phi(\gamma) \in C^I$, de sorte que Φ est à valeurs dans \overline{C}^I .

Reste à construire $\Psi : \overline{C}^I \dashrightarrow \mathcal{U}_T$ inverse à Φ . Soit $\xi = (\xi_J)_{J \subset I, \#J > 2} \in \overline{C}^I$. Pour $A \in T$, notons $T_A = \{B \in T, B \subset A\}$. On construit un élément z_A de \mathbb{C}^{T_A} récursivement par les formules

$$\begin{aligned} z_{A,B} &= \xi_{A,i} \text{ si } B = \{i\}, i \in A; \\ z_{A,B} &= \frac{1}{\#B} \sum_{C \in \text{St } B} z_{Q,C} \text{ si } B \subset A, \#B > 1. \end{aligned}$$

Pour ξ voisin de $\partial_T \overline{C}^I$, $z_A|_{\text{St } A}$ est un élément de $\mathbb{C}_*^{\text{St } A}$; on note c_A sa classe dans $C^{\text{St } A}$. Pour $B \in T_*$, $B \prec A$, on note

$$s_B = \frac{\|(z_A)|_{\text{St } B}\|}{\|(z_A)|_{\text{St } A}\|}$$

On pose $\Psi(\xi) = ((c_A)_{A \in T_0}, (s_B)_{B \in T_*})$. On vérifie que $\Psi(\partial_T \overline{C}^I) \subset S_T \times \{0\}$ et que les germes de Φ et de Ψ sont inverses l'un de l'autre. \checkmark

A.3. L'espace $C^{I,J}$

Soit \mathcal{H} le demi-plan de Poincaré. Soient I, J deux ensembles finis, $n = \#I, m = \#J$. On note $\text{Conf}^{I,J}$ la variété à bord $\mathcal{H}_*^I \times \mathbb{R}_*^J$.

Le groupe G' des permutations de \mathbb{C} de la forme $z \mapsto az + b$, $a \in \mathbb{R}_*^+, b \in \mathbb{R}$ opère sur $C^{I,J}$, librement si $2n + m \geq 2$. Sous cette hypothèse, on note $C^{I,J} = \text{Conf}^{I,J} / G'$.

C'est une variété à bord de dimension $2n + m - 2$, orientable, à $m!$ composantes connexes qui sont en bijection canonique avec les numérotations de l'ensemble J .

On se propose de compactifier $C^{I,J}$. Pour cela, posons $K = I \sqcup \overline{I} \sqcup J$, réunion disjointe de deux copies de I et d'une copie de J , muni de l'involution $\sigma : k \rightarrow \overline{k}$ (conjugaison) qui échange les deux copies de I et fixe les éléments de J . Les éléments de J, I et \overline{I} indexeront respectivement les points réels, ceux situés dans le demi-plan de Poincaré, et les conjugués de ces derniers.

L'application $\text{Conf}^{I,J} \rightarrow \mathbb{C}_*^K$,

$$(z, x) = ((z_i)_{i \in I}, (x_j)_{j \in J}) \mapsto (z, \overline{z}, x) = ((z_i)_{i \in I}, (\overline{z_i})_{\overline{i} \in \overline{I}}, (x_j)_{j \in J}),$$

induit un plongement fermé de $C^{I,J}$ dans C^K .

On identifie $C^{I,J}$ à son image dans C^K , et on note $\overline{C}^{I,J}$ l'adhérence de $C^{I,J}$ dans \overline{C}^K .

Afin d'élucider la structure de $\overline{C}^{I,J}$, il est naturel d'étudier ses intersections avec les strates de \overline{C}^K , et on constate que la strate $\partial_T \overline{C}^K$ ne peut rencontrer $\overline{C}^{I,J}$ que si le K -arbre T satisfait les conditions suivantes :

- (a) $A \in T \implies \sigma(A) \in T$;
- (b) pour $A \in T$, on a $A \subset I$ ou $A \subset \overline{I}$ ou $\sigma(A) = A$.

Définition A.3.1. — On appelle (I, J) -arbre tout K -arbre T satisfaisant les conditions (a) et (b) ci-dessus.

Si P est un ensemble de parties de K , on notera P^σ (resp. P^I , resp. $P^{\bar{I}}$) l'ensemble des $A \in P$ t. q. $\sigma(A) = A$ (resp. $A \subset I$, resp. $A \subset \bar{I}$).

Avec ces notations, un (I, J) -arbre est un K -arbre T vérifiant $\sigma(T) = T$ et $T = T^I \sqcup T^{\bar{I}} \sqcup T^\sigma$.

Si T est un (I, J) -arbre, on pose $\partial_T \bar{C}^{I,J} = \bar{C}^{I,J} \cap \partial_T \bar{C}^K$.

Proposition A.3.2. — Le compactifié $\bar{C}^{I,J}$ est une variété à coins d'intérieur $C^{I,J}$, dont les strates canoniques sont les composantes connexes des $\partial_T \bar{C}^{I,J}$, pour T décrivant l'ensemble des (I, J) -arbres.

De plus, $\partial_T \bar{C}^{I,J}$ est de codimension $\#(T_*^\sigma \sqcup T_0^I)$, et s'identifie à

$$\prod_{A \in T_0^\sigma} C^{\text{St}^I A, \text{St}^\sigma A} \times \prod_{A \in T_0^I} C^{\text{St} A}.$$

Démonstration. — On adapte la démonstration de la proposition 1. Soit $S_T = \prod_{A \in T_0} C^{\text{St} A}$, $\mathcal{U}_T = S_T \times \mathbb{R}_+^{T_*}$, et $\Phi : \mathcal{U}_T \dashrightarrow \bar{C}^K$ le morphisme construit ci-dessus. Posons $Z_T = \prod_{A \in T_0^\sigma} C^{\text{St}^I A, \text{St}^\sigma A} \times \prod_{A \in T_0^I} C^{\text{St} A}$, et soit \mathcal{V}_T la variété à coins $Z_T \times \mathbb{R}_+^{T_* \sqcup T_0^I}$.

On identifie \mathcal{V}_T à un fermé de \mathcal{U}_T via les plongements suivants :

- (i) $C^{\text{St}^I A, \text{St}^\sigma A} \hookrightarrow C^{\text{St} A}$ pour $A \in T_0^\sigma$ ($(z, x) \mapsto (z, \bar{z}, x)$);
- (ii) $C^{\text{St} A} \hookrightarrow C^{\text{St} A} \times C^{\text{St} \bar{A}}$ pour $A \in T_0^I$ ($z \mapsto (z, \bar{z})$);
- (iii) $(\mathbb{R}_+)^{T_* \sqcup T_0^I} \hookrightarrow (\mathbb{R}_+)^{T_*}$ ($s \mapsto s'$, où s' prolonge s par $s'_B = s_{\bar{B}}$ pour $B \in T_0^{\bar{I}}$).

On vérifie que $\Phi|_{\mathcal{V}_T}$ est à valeurs dans $\bar{C}^{I,J}$, et induit un isomorphisme de germes de $(\mathcal{V}_T, Z_T \times \{0\})$ sur $(\bar{C}^{I,J}, \partial_T \bar{C}^{I,J})$ d'inverse induit par Ψ . \checkmark

Les strates de codimension 1 dans $\bar{C}^{I,J}$ joueront un rôle important. Elles correspondent à deux types d'arbres :

- 1) cas $T_* = \{A, \bar{A}\}$ avec $A \subset I$, $\#A \geq 2$. Notant $\partial_A \bar{C}^{I,J} = \partial_T \bar{C}^{I,J}$, on a

$$\partial_A \bar{C}^{I,J} \simeq C^{I-A \sqcup \{*\}, J} \times C^A;$$

- 2) cas $T_* = \{A \sqcup \bar{A} \sqcup B\}$, avec $A \subset I$, $B \subset J$ (l'une des deux inclusions étant stricte), et $2\#A + \#B \geq 2$. Notant $\partial_{A,B} \bar{C}^{I,J} = \partial_T \bar{C}^{I,J}$, on a

$$\partial_{A,B} \bar{C}^{I,J} \simeq C^{A,B} \times C^{I-A, J-B \sqcup \{*\}}.$$

Exemple A.3.3. — Pour $n, m \in \mathbb{N}$ (avec $2n + m \geq 2$), posons $C^{n,m} = C^{I,J}$, où $I = \{1, \dots, n\}$ et $J = \{n+1, \dots, n+m\}$. On a :

$$\bar{C}^{1,1} = \text{arc} \quad \text{et} \quad \bar{C}^{2,0} = \text{œil}$$

Les paupières de l'œil sont les strates $\partial_{A,\emptyset}$ pour $A = \{1\}$ et $\{2\}$ respectivement, et le contour de l'iris est la strate ∂_A pour $A = \{1, 2\}$.

A.4. Formes différentielles sur C^I et $C^{I,J}$

Soit I un ensemble fini de cardinal $n \geq 2$. On appelle *formes angulaires* les 1-formes différentielles $d\phi_{j,k}$ sur \overline{C}^I pour $j, k \in I$ distincts.

Proposition A.4.1. — (Lemme d'annulation). *Soit I un ensemble fini de cardinal $n \geq 3$. L'intégrale d'un produit de $2n - 3$ formes angulaires sur C^I est nulle.*

Démonstration. — L'intégrale converge absolument, les formes angulaires étant analytiques sur la variété à coins compacte \overline{C}^I .

Quitte à numérotter les éléments de I , on se ramène à montrer que sur C^n , l'intégrale de la forme $d\phi_{j_0, k_0} \wedge \cdots \wedge d\phi_{j_{2n-4}, k_{2n-4}}$ est nulle; et on peut supposer $j_0 = 1, k_0 = 2$.

Soit X la variété algébrique complexe $(\mathbb{C} - \{0, 1\})_*^{n-2}$, et θ l'isomorphisme analytique (réel) $T \times X \xrightarrow{\sim} C^n, (t, (\zeta_1, \dots, \zeta_{n-2})) \mapsto e^{2i\pi t}(0, 1, \zeta_1, \dots, \zeta_{n-2})$ que nous avons déjà rencontré. On a $\theta^* \phi_{1,2} = t$, et, pour $l > 0$, $\theta^* \phi_{j_l, k_l}$ s'écrit $t + \frac{1}{2\pi} \text{Arg } Z_l$, où Z_l est une fonction rationnelle non identiquement nulle sur X . D'où : $\theta^* d\phi = (\frac{1}{2\pi})^{2n-4} dt \wedge d \text{Arg } Z_1 \wedge \cdots \wedge d \text{Arg } Z_{2n-4}$.

On conclut par le théorème suivant.

Théorème A.4.2. — *Soit X une variété algébrique complexe irréductible de dimension $n > 0$, et Z_1, \dots, Z_{2n} des fonctions rationnelles non nulles sur Z . Soit ω la $2n$ -forme différentielle réelle $d \text{Arg } Z_1 \wedge \cdots \wedge d \text{Arg } Z_{2n}$. Alors $\int_X \omega$ converge absolument, et sa valeur est nulle.*

Démonstration. — Deux remarques pour commencer.

1) On a $\omega = d \text{Log } |Z_1| \wedge \cdots \wedge d \text{Log } |Z_{2n}|$.

En effet, pour $x \in X$, notons $J_x : T_x X \rightarrow T_x X$ la structure complexe. Si Z est une fonction holomorphe non nulle en x , $d_x \text{Log } Z = d_x \text{Log } |Z| + id_x \text{Arg } Z$ et, $\text{Log } Z$ étant holomorphe en x , on en déduit : $d_x \text{Arg } Z \circ J_x = d_x \text{Log } |Z|$. On conclut en remarquant que J_x est de déterminant réel 1.

2) On peut supposer que X est une variété projective lisse, et qu'il existe un ouvert de Zariski dense $U \subset X$ vérifiant :

- (i) $X - U$ est un diviseur à croisements normaux ;
- (ii) les Z_i sont régulières et partout non nulles sur U .

En effet, le problème est birationnel (les fermés de Zariski $F \subsetneq X$ étant négligeables). Soit U un ouvert dense affine lisse de X où les Z_i sont régulières et partout non nulles. Il existe un plongement ouvert de U dans une variété projective \overline{U} , et, par résolution des singularités, on peut faire en sorte que \overline{U} soit lisse et $\overline{U} - U$ soit un diviseur à croisements normaux.

L'idée de la démonstration du théorème est maintenant simple. On a $\omega = d\alpha$, où $\alpha \in A^{2n-1}(U)$ est la forme différentielle

$$\text{Log}|Z_1| d \text{Log}|Z_2| \wedge \cdots \wedge d \text{Log}|Z_{2n}|.$$

Si la forme différentielle α se prolongeait à X , le résultat serait immédiat : $\int_X \omega = \int_X d\alpha = 0$. En général, α ne se prolonge pas à X , mais satisfait une condition plus faible qui permettra de conclure.

Si M est une variété \mathcal{C}^∞ orientée de dimension N , et $d \in \mathbb{N}$, notons $\mathcal{A}^d(M)$ le dual de l'espace $A_c^{N-d}(M)$ des formes différentielles à support compact de degré $N - d$. On définit une 'différentielle formelle' $D : \mathcal{A}^d(M) \rightarrow \mathcal{A}^{d+1}(M)$ par la formule $\langle D\phi, \xi \rangle = (-1)^{d+1} \langle \phi, d\xi \rangle$. Par ailleurs, appelons ' d -forme différentielle L^1 loc' une d -forme différentielle sur M dont les coefficients en coordonnées locales sont des fonctions localement L^1 (plutôt que des fonctions \mathcal{C}^∞). Une telle forme différentielle β peut être vue comme un élément de $\mathcal{A}^d(M)$ en posant $\langle \beta, \xi \rangle = \int_M \beta \wedge \xi$.

Pour conclure, il suffira de montrer que la forme α satisfait la propriété de régularité suivante : α est L^1 loc sur X , presque partout différentiable, sa différentielle $d\alpha$ est L^1 loc, et $D\alpha = d\alpha$ dans $\mathcal{A}^{2n}(X)$. En effet, on aura alors $\int_X d\alpha = \int_X D\alpha \wedge 1 = \pm \int_X \alpha \wedge d1 = 0$.

Or on observe que cette propriété de régularité est de nature $A^*(M)$ -linéaire et se vérifie localement. Au voisinage de tout point $p \in X$, il existe des coordonnées locales complexes (z_1, \dots, z_n) telles que chaque composante de $X - U$ soit l'une des hypersurfaces $z_i = 0$. Sur un tel voisinage, α est combinaison linéaire à coefficients dans $A^*(M)$ de formes de l'un des deux types suivants :

- (a) $\bigwedge_{j \in J} d \text{Log}|z_j|$ ($J \subset \{1, \dots, n\}$);
- (b) $\text{Log}|z_i| \cdot \bigwedge_{j \in J} d \text{Log}|z_j|$ ($\{i\} \cup J \subset \{1, \dots, n\}$);

On se ramène donc par séparation des variables à vérifier la propriété de régularité pour chacune des formes $\text{Log}|z|$, $d \text{Log}|z|$ et $\text{Log}|z| \cdot d \text{Log}|z|$ sur \mathbb{C} , ce qui se calcule facilement à la main. (Indication : soit β l'une de ces formes, et d son degré. On voit facilement que β et $d\beta$ sont L^1 loc, et on se ramène par Stokes à vérifier que pour $\eta \in A_c^{1-d}(\mathbb{C})$, l'intégrale de $\beta \wedge \eta$ sur le cercle de rayon r centré en 0 tend vers 0 lorsque $r \rightarrow 0$.)

Ceci achève la démonstration du théorème et de la proposition. ✓

✓

A.4.1. Formes différentielles sur $C^{I,J}$ et graphes admissibles. — Soient $n, m \in \mathbb{N}$ avec $2n+m \geq 2$, $I = \{1, \dots, n\}$, $J = \{n+1, \dots, n+m\}$, et posons $C^{n,m} = C^{I,J}$. On note $C_0^{n,m}$ la composante connexe de $C^{n,m}$ formée des configurations où les m points réels sont en ordre croissant.

Pour $i \in I, j \in I \sqcup J, i \neq j$, on définit une fonction $\theta_{i,j} : C^{n,m} \rightarrow \mathbb{T}$ par

$$(z_k)_{k \in I \sqcup J} \mapsto \frac{1}{2\pi} \text{Arg} \left(\frac{z_j - z_i}{z_j - \bar{z}_i} \right).$$

On observe que $\theta_{i,j}$ est la restriction à $C^{I,J}$ de $\phi_{i,j} - \phi_{\bar{i},j}$ et à ce titre, se prolonge analytiquement à $\overline{C}^{I,J}$.

On note $\omega_{i,j}$ la 1-forme différentielle $d\theta_{i,j}$ sur $\overline{C}^{I,J}$, en convenant que $\omega_{i,i} = 0$.

Plus généralement, on associe une forme différentielle sur l'espace de configurations à tout 'graphe admissible'. Un graphe admissible est, de manière informelle, un graphe orienté fini dont chaque arête a pour source un élément de I et pour but un élément de $I \sqcup J$, les arêtes de source donnée étant numérotées. Voici une définition plus formelle. DÉFINITION. On appelle (n, m) -*graphe admissible* (ou simplement *graphe admissible*) toute application $\gamma : S_\gamma \rightarrow I \sqcup J$, où $S_\gamma \subset I \times \mathbb{N}$ est de la forme

$$S_\gamma = \{(i, j) \mid i \in I \text{ et } 1 \leq j \leq k_i\},$$

k_1, \dots, k_n étant des entiers naturels.

Les *sommets* de γ sont les éléments de $I \sqcup J$; les éléments de I (resp. J) sont les sommets de type I (resp. II). Les *arêtes* de γ sont les éléments de S_γ , $e = (i, l) \in S_\gamma$ étant vu comme arête de source $s(e) = i$ et de but $t(e) = \gamma(i, l)$. La source d'une arête est donc toujours de type I. On ordonne lexicographiquement l'ensemble des arêtes S_γ , et on appelle *degré* de γ le cardinal de cet ensemble.

Si γ est un graphe admissible de degré d , on pose

$$\omega_\gamma = \bigwedge_{e \in S_\gamma} \omega_{s(e), t(e)}.$$

C'est une forme différentielle fermée de degré d sur $\overline{C}^{n,m}$, qui est nulle si γ comporte une boucle ou une arête multiple.

Enfin, si γ est de degré $2n + m - 2$, on pose :

$$W_\gamma = \int_{C_0^{n,m}} \omega_\gamma.$$

Les scalaires W_γ sont les poids qui interviennent dans la formule de Kontsevich. On observe que toute renumérotation des éléments de I , ou de J , ou de l'ensemble des arêtes issues d'un élément $i \in I$ modifie W_γ par un signe.

Nous allons voir maintenant que les poids W_γ satisfont certaines relations qui résultent de la formule de Stokes.

Soit γ un (n, m) -graphe admissible de degré $2n + m - 3$. Par la formule de Stokes, il vient :

$$\int_{\partial \overline{C}_0^{n,m}} \omega_\gamma = \int_{\overline{C}_0^{n,m}} d\omega_\gamma = 0.$$

Or le premier terme est la somme des intégrales de ω_γ sur les strates de codimension 1 de $\overline{C}_0^{n,m}$, que l'on sait décrire.

1) Strate du type $\partial_A \overline{C}^{n,m} \simeq C^A \times \overline{C}^{I-A \sqcup \{*\}, J}$. Soit $\iota : C^A \times \overline{C}^{I-A \sqcup \{*\}, J} \hookrightarrow \overline{C}^{I, J}$.
On a

$$\iota^* \theta_{i,j} = \begin{cases} p_1^* \theta_{i,j} & \text{si } i, j \in A; \\ p_2^* \theta_{i,j} & \text{si } i, j \notin A; \\ p_2^* \theta_{i,*} & \text{si } i \notin A, j \in A; \\ p_2^* \theta_{*,j} & \text{si } i \in A, j \notin A. \end{cases}$$

Ainsi $\iota^* \omega_\gamma$ est nul sauf dans le cas où le nombre d'arêtes de A à A est égal à $2\#A - 3$. De plus, en vertu du lemme d'annulation, son intégrale sur la strate ne peut être non nulle que si $\#A = 2$.

Dans ce cas, l'intégrale étudiée est égale à $\pm W_{\gamma'}$, où γ' est le $(n-1, m)$ -graphe admissible obtenu en contractant l'unique arête de A à A .

2) Strate du type $\partial_{A,B} \overline{C}^{n,m} \simeq C^{A,B} \times C^{I-A, J-B \sqcup \{*\}}$. Soit $\iota : C^{A,B} \times C^{I-A, J-B \sqcup \{*\}} \hookrightarrow \overline{C}^{I, J}$. On a

$$\iota^* \theta_{i,j} = \begin{cases} p_1^* \theta_{i,j} & \text{si } i \in A \text{ et } j \in A \sqcup B; \\ p_2^* \theta_{i,j} & \text{si } i \notin A \text{ et } j \notin A \sqcup B; \\ p_2^* \theta_{i,*} & \text{si } i \notin A, j \in A \sqcup B; \\ 0 & \text{si } i \in A, j \notin A \sqcup B. \end{cases}$$

Ainsi, posant $a = \#A$, $b = \#B$, $\iota^* \omega_\gamma$ est nulle sauf dans le cas où le nombre d'arêtes de A à $A \sqcup B$ est égal à $2a + b - 2$, et toute arête issue de A va dans $A \sqcup B$.

Dans ce cas, l'intégrale étudiée est égale à $\pm W_\gamma W'_\gamma$, où γ est le (a, b) -graphe admissible obtenu par restriction de γ à $A \sqcup B$, et γ' est le $(n-a, m-b+1)$ -graphe admissible obtenu en contractant les éléments de $A \sqcup B$ en un unique sommet de type II.

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