Tamarkin's proof: using formality of $\mathbb{E}_2$ to prove deformation quantization of Poisson manifolds

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GL

I Deformation quantization of Poisson manifolds

Let M be a smooth manifold. A <u>star product</u> on M is an $\mathbb{R}[[\hbar]]$-linear product $*$ on $\mathcal{C}^\infty(M)[[\hbar]]$ such that

(1) $(f*g)*h = f*(g*h)$

(2) $f*g = f\cdot g + \sum_{k\geq 1} \hbar^k B_k(f,g) \quad \forall f,g \in \mathcal{C}^\infty(M)$

where B_k are bidifferential operators.

(3) $f = 1*f = f*1$.

Given a star product $*$, one can define a bracket

$$\{f,g\}_* := B_1(f,g) - B_1(g,f) \quad \text{for } f,g \in \mathcal{C}^\infty(M)$$

which defines a Poisson structure on M.

<u>Def</u>: Let $(M, \{-,-\})$ be a Poisson manifold. A <u>deformation quantization</u> of M is a star product $*$ s.t. $\{-,-\}_* = \{-,-\}$.

<u>Q</u>: Does every Poisson manifold admit a deformation quantization?

Let us define the Lie algebra in which star products naturally live:

$$D^d_{poly}(M) := \left\{ D : \mathcal{C}^\infty(M)^{\otimes d} \longrightarrow \mathcal{C}^\infty(M) \;\middle|\; D \underset{\text{locally}}{=} \sum f \frac{\partial}{\partial x_{I_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{I_d}} \right\}$$

polydifferential operators

2/ $D^{\bullet}_{poly}$ is naturally a dg Lie algebra

↳ differential $D \in D^2_{poly} \rightsquigarrow d(D)(f_1,f_2,f_3) = f_1 D(f_2,f_3) - D(f_1 f_2, f_3)$
$+ D(f_1, f_2 f_3) - D(f_1,f_2) f_3$

↳ bracket $[D, D'] := D \circ D' - D' \circ D$.

$$\boxed{\text{Prop: } * \text{ is associative } \Leftrightarrow d(*) + \frac{1}{2}[*,*] = 0 \Leftrightarrow * \in MC(D_{poly}[[\hbar]])}$$

On the other hand, a Poisson structure on M is encoded by the Poisson bivector $\pi \in \Lambda^2 T_M$

These naturally live in the Lie algebra of polyvector fields

$$T^d_{poly}(M) := \Gamma(M, \Lambda^d T_M)$$

whose dg structure is given by

↳ differential $= 0$

↳ bracket = extension of the Schouten-Nijenhuis bracket on T^1_{poly} via $[X, Y \wedge Z] = [X,Y] \wedge Z + Y \wedge [X,Z]$

$$\boxed{\text{Prop: } \pi \text{ is Poisson } \Leftrightarrow [\pi,\pi] = 0 \Leftrightarrow \pi \in MC(T_{poly})}$$

Theorem [Kontsevich]

There is an L_∞-morphism

$$\phi : T_{poly}(M) \longrightarrow D_{poly}(M)$$

which is a quasi-isomorphism.

proof: Completely explicit construction by integration over configuration spaces; for $M = \mathbb{R}^n$

$$\phi_n = \sum_{\Pi \in Graph(n,0)} \omega_\Pi^{\in \mathbb{R}} \phi_\Pi \quad \phi_\Pi : T_{poly}(\mathbb{R}^d)^{\wedge n} \longrightarrow D^{\circ}_{poly}(\mathbb{R}^d)$$

ω_Π : integral over conf. space $\int_M \bigwedge_{edges} \omega_e$

$\square$

Corollary: There is a bijection

$$MC(T_{poly}[[\hbar]])/\text{gauge eq.} \longleftrightarrow MC(D_{poly}[[\hbar]])/\text{gauge eq.}$$

$$\Pi \longmapsto \sum_{n \geq 1} \frac{1}{n!} \phi_n(\Pi, \dots, \Pi)$$

proof: This is a classical result of deformation theory; see Doubek-Markl-Zima lecture notes, Thm. 7.8.

$\square$

In fact, Kontsevich's ϕ extends a well-known map ϕ_1, which was known to be a quasi-isomorphism by the Hoschild-Konstant-Rosenberg theorem, but which is not compatible with the Lie brackets.

$\boxed{2}$ Tamarkin's proof

Starting from the fact that

$$D^{\bullet}_{poly}(M) \simeq \underset{\substack{\uparrow \\ \text{Hoschild complex}}}{CH^{\bullet}(\mathcal{C}^{\infty}(M))} \quad \text{and} \quad T^{\bullet}_{poly}(M) \simeq \underset{\substack{\uparrow \\ \text{Hoschild homology}}}{HH^{\bullet}(\mathcal{C}^{\infty}(M))}$$

Let us set $A := \mathcal{C}^{\infty}(M)$. The idea is to use

(1) Thm (Deligne conjecture):

$CH^{\bullet}(A)$ is an algebra over $C^{*}(\mathbb{E}_2)$.

(2) Thm (Formality):

$C^{*}(\mathbb{E}_2)$ is quasi-isomorphic as an operad to $H^{*}(\mathbb{E}_2) = Gerst$.

Combining these two facts, we get a homotopy Gerstenhaber algebra structure on $CH^{*}(A)$

$$(*) \quad Gerst_{\infty} \xrightarrow{\sim} Gerst \xrightarrow[(2)]{\sim} C^{*}(\mathbb{E}_2) \xrightarrow[(1)]{\sim} End(CH^{*}(A)).$$

Now, we use a third non-trivial fact

(3) Thm (Intrinsic formality of $HH^{*}(A)$):

$$H^{*}(CH^{*}(A)) \underset{\substack{\uparrow \\ \text{as Gerst alg}}}{\cong} HH^{*}(A) \implies CH^{*}(A) \underset{\substack{\uparrow \\ \text{as } Gerst_{\infty}\text{-alg.}}}{\cong} HH^{*}(A)$$

The last step is to show that

(4) Prop: The Gerst$_\infty$ equivalence $CH^*(A) \xrightarrow{\simeq} HH^*(A)$ restricts to an h_∞ equivalence

This gives the existence of an h_∞-morphism $T_{poly} \to D_{poly}$, proving the desired result.

About the proofs

There are several proofs of (1), using different models of $\mathbb{E}_2$ to construct an action explicitly.

The proof of (2) amounts, as we have seen, to the construction of a Drinfeld associator, which is highly non-trivial.

The proof of (3) is algebraic, and amounts to showing that the first homology $H^1(\mathrm{Def}(HH^*(A)))$ is trivial.

Showing (4) amounts to showing that the h_∞ part of the Gerst$_\infty$ algebra structure obtained in (*) is independent of the choice of a Drinfeld associator. This is not obvious, but follows "for degree reasons".