2 alternative proofs of formality for $\mathbb{E}_2$

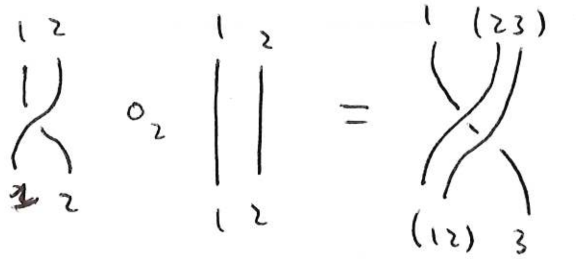
Tamarkin

Def: The operad of parenthesized braids is the operad in groupoids PaB where

$PaB(n)$ has <u>objects</u> fully parenthesized words in $\{1, \dots, n\}$ = planar binary trees with n leaves labelled by $\{1, \dots, n\}$.

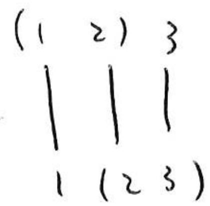
<u>morphisms</u> are braids

Operadic composition is given by insertion



Thm [Bar-Natan '98, Fresse '17]

PaB is generated by $R^{1,2}$ = 

and $\Phi^{1,2,3}$ = 

remark: $PaB(n)$ is the fundamental groupoid of $\mathbb{E}_2(n)$

$$C^*(\mathbb{E}_2(n)) \sim C^*(|N(PaB(n))|)$$

PaB has a Lie algebraic analogue / associated graded

Def: The operad of parenthesized chord diagrams is the operad in groupoids $\mathrm{PaCD} := \exp(\mathfrak{t})$ ← take universal enveloping algebra and restrict to group-like elements

where $\mathfrak{t} = \{\mathfrak{t}_n\}_{n\geq 1}$ is an operad in Lie-algebras defined as follows

$$\mathfrak{t}_n := \mathrm{Lie}\langle t_{ij} \mid 1\leq i<j\leq n\rangle \Big/ \begin{array}{ll} t_{ij} = t_{ji} & \\ {[t_{ij},\, t_{ik}+t_{jk}]} = 0 & i,j,k,l \\ {[t_{ij},\, t_{kl}]} = 0 & \text{distinct} \end{array}$$

For $1\leq \alpha \leq n$ operadic composition $\circ_\alpha : \mathfrak{t}_n \oplus \mathfrak{t}_m \longrightarrow \mathfrak{t}_{n+m-1}$ ← a Lie-alg map!

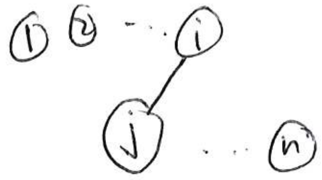
is given by $t_{ij}\circ_\alpha t_{kl} := t_{ij}\circ_\alpha 0 + 0\circ_\alpha t_{kl}$

where

$$t_{ij}\circ_\alpha 0 := \begin{cases} t_{(i+n-1)(j+n-1)} & \alpha < i \\ \sum_{\beta=1}^{n} t_{(i+\beta-1)(j+n-1)} & \alpha = i \\ t_{i(j+n-1)} & i<\alpha<j \\ \sum_{\beta=1}^{n} t_{i(j+\beta-1)} & \alpha = j \\ t_{ij} & \alpha > j \end{cases}$$

$$0\circ_\alpha t_{kl} = t_{(k+\alpha-1)(l+\alpha-1)}$$

In pictures

$t_{ij} =$ 

and operadic composition is given by inserting at a vertex and summing over all possible ways of reconnecting the edges.

Def: A <u>Drinfeld associator</u> is an isomorphism of (pro-unipotent/Malcev completed) operads in groupoids

$$\boxed{PaB \longrightarrow PaCD}$$

which is the identity on objects.

Such an isomorphism is completely determined by the image of the generators

$$R^{1,2} \longmapsto e^{t_{12}/2}$$

$$\Phi^{1,2,3} \longmapsto \phi(t_{12}, t_{23})$$ explicit power series in 2 variables

which are required to satisfy some relations (pentagon, hexagon and unit).

<u>Thm</u> [Drinfeld] later constructions by Alekseev-Torossian and others, too.

There is one such associator.

Tamarkin's strategy

(1) Take PaB as a model for $\mathbb{E}_2$

$$C^*(\mathbb{E}_2) \sim C^*(|N(PaB)|)$$

(2) Use the existence of a Drinfeld associator

$$PaB \xrightarrow[\cong]{\Phi} PaCD$$

which gives a quasi-iso of operads

$$C^*(|N(PaB)|) \sim C^*(|N(PaCD)|)$$

(3) Prove formality for PaCD

$$\mathrm{Gerst} = H^*(\mathbb{E}_2) \longrightarrow PaCD$$

$$\{\,,\,\} \longmapsto t_{12}$$

generated in arity 2

Claim: this is a quasi-iso of operads.

proof: – Serre spectral sequence
– Comparing Betti numbers

same ingredients as for the q.i. $\mathbb{R} \to H^*(FM_d)$!

The idea is to adapt a criterion from Sullivan's rational homotopy theory (theory of "minimal models")

Thm [Sullivan, '77]

Let A be a nilpotent cdga. If a grading automorphism of $H^*(A)$ lifts to a grading automorphism of A, then A is formal.

The idea is to look at the action of the Grothendieck-Teichmüller group

$$GT := \mathrm{Aut}(\widehat{PaB})$$

on $C^*(\mathbb{E}_2)$.

$$\boxed{\begin{array}{l}\text{Thm [Fresse '17]}\\ GT = \mathrm{Aut}^h(\mathbb{E}_2)\end{array}}$$

Thm: $\mathbb{E}_2$ is formal.

proof: the action $GT \curvearrowright C^*(\mathbb{E}_2)$ descends to $\mathbb{Q}^\times \curvearrowright H^*(\mathbb{E}_2)$. $\square$

Note that we need that $GT \longrightarrow \mathbb{Q}^\times$ is surjective, which is deduced by Drinfeld from the existence of an associator.

$\hookrightarrow$ so, in this second strategy, a Drinfeld associator is needed, too.