Odense
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Proof of formality of $\mathbb{E}_d$

after Kontsevich ; reference = Lambrechts-Volić 2012

admissible diagrams ③ semi-alg forms compactification of conf. space

Main diagram :

$$\begin{tikzcd} \mathcal{D}(n) \arrow[r, "I", "\sim"'] \arrow[d, "\wr", "②"'] & \Omega_{PA}(FM_d(n)) \arrow[d, "\wr"', "①"] \\ H^*(\mathbb{E}_d) & C^*(\mathbb{E}_d) \end{tikzcd}$$

cooperads

① Definition of FM_d operad and quasi-iso with $C^*(\mathbb{E}_d)$

A a finite set.
$$C(A) := \mathrm{Inj}(A, \mathbb{R}^d) \big/ \mathbb{R}^d \rtimes \mathbb{R}^+_{>0}$$

conf. space of $|A|$ points in $\mathbb{R}^d$ translations positive dilations

Given $a,b,c \in A$ distinct

$$\theta_{a,b} : C(A) \longrightarrow S^{d-1}, \qquad x \longmapsto \frac{x(b)-x(a)}{\|x(b)-x(a)\|}$$

direction between 2 pts

$$\delta_{a,b,c} : C(A) \longrightarrow [0,+\infty], \qquad x \longmapsto \frac{\|x(a)-x(b)\|}{\|x(a)-x(c)\|}$$

rel. distance of 3 pts

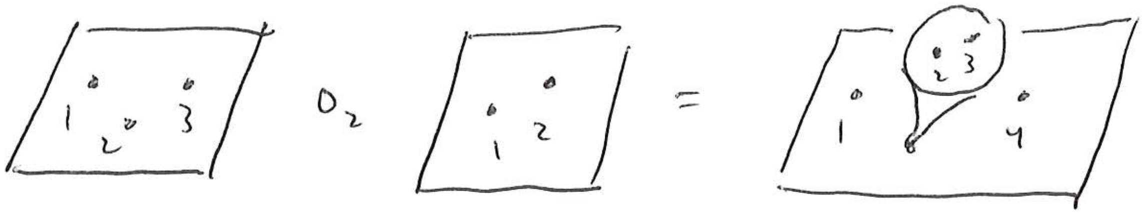
The map
$$i : C(A) \longrightarrow (S^{d-1})^{\binom{A}{2}} \times [0,+\infty]^{\binom{A}{3}}$$
$$x \longmapsto \left(\prod_{a\neq b} \theta_{a,b}(x) , \prod_{a\neq b\neq c} \delta_{a,b,c}(x) \right)$$

is a homeomorphism onto its image.

<u>Def</u>: The Fulton-MacPherson compactification is
$$FM_d(A) := \overline{i(C(A))}$$
closure

2//   Morally: configurations where points can be infinitesimally close while direction & rel. distance are well-defined.

The collection $FM_d := \{FM_d(n)\}_{n\geq 1}$ forms a topological operad

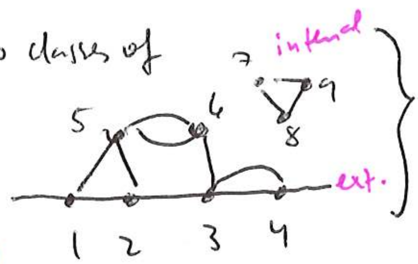


<u>Theorem 1</u>: $\Omega_{PA}(FM_d) \xrightarrow{\sim} C^*(\mathbb{E}_d)$

<u>proof</u>: FM_d & $\mathbb{E}_d$ are homotopy equivalent ; one shows that FM_d is equivalent to the W-construction of $\mathbb{E}_d$.

② Definition of $\mathcal{D}$ cooperad and quasi-iso with $H^*(\mathbb{E}_d)$

$\hat{\mathcal{D}}(A) := \mathbb{K} \{$ iso classes of



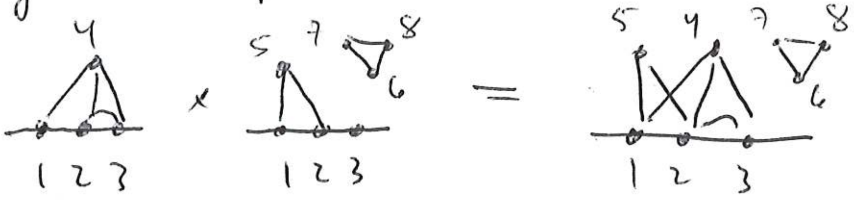
$\}$ / $\sim$ edge inversion $\to \cdot (-1)^d$
internal transposition $\to \cdot (-1)^d$
edge transposition $\to \cdot (-1)^{d+1}$

$A = \{1, .., 9\}$

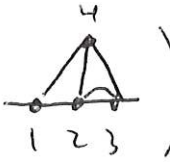
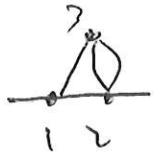
+ total order on edges

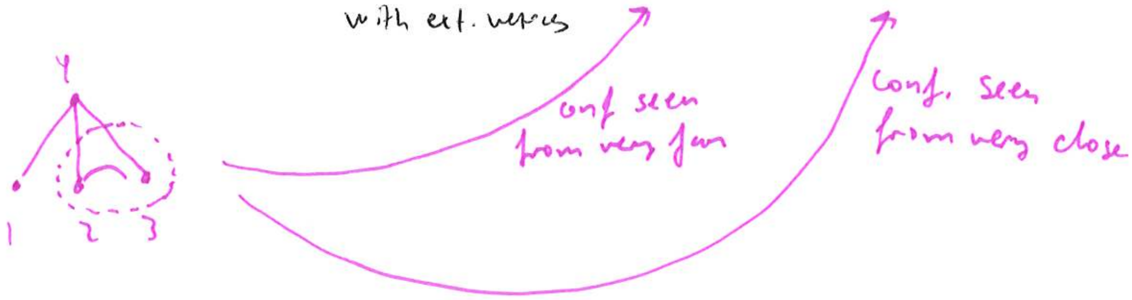
$\mathcal{D}(A) := \hat{\mathcal{D}}(A) / N(A)$ ← non-admissible diagrams
- loops
- double edges
- internal vertices with valence ≤ 2
- internal vertices not connected to an external one

<u>Def</u>: dga and cooperad structures

a) 

3/1

c) $\Delta\Big($  $\Big) = \sum\limits_{\substack{\text{subgraphs}\\ \text{with ext. vertices}}}$  $\otimes$ 

morally 

d) $d\Gamma := \sum\limits_{\substack{\text{contractible}\\ \text{edges}}} \pm \Gamma/e$

ex: $d\Big($  $\Big) =$  $+ \dots$

<u>Proposition</u>: We have that

(i) Points a) and d) give $\hat{\mathcal{D}}(A)$ a dga structure

(ii) $N(A)$ is a dg ideal ($\Rightarrow \mathcal{D}(A)$ is a dga)

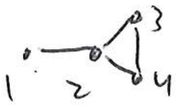
(iii) Point c) gives $\hat{\mathcal{D}}(A)$ a cooperad structure

(iv) Δ commutes with the differential in $\mathcal{D}(A)$, but not in $\hat{\mathcal{D}}(A)$.

<u>proof</u>: (i) compatibility of d with $\sim$ straightforward but tedious keep track of signs and cases

(ii) case by case, not difficult

(iii) this is elaborate (technical, graph combinatorics): bijection between condensations of Γ and Γ/e

(iv) take $\Gamma =$  and commute that $d(\Delta(d\Gamma)) \neq 0$.

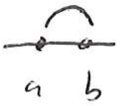
4// **Theorem 2**: $\mathcal{D} \xrightarrow{\sim} H^*(\mathbb{E}_d)$ is a weak-equivalence of cooperads.

proof: It is equivalent to prove $\mathcal{D} \xrightarrow{\sim} H^*(FM_d)$ ← zero differential

first ingredient = Cohen computation of

$$H^*(FM_d) = \frac{\Lambda(\{g_{ab} \mid a \neq b \in A\})}{(g_{ab}g_{bc} + g_{bc}g_{ca} + g_{ca}g_{ab} \,;\, (g_{ab})^2 \,;\, g_{ab} - (-1)^d g_{ba})}$$

Here, $g_{ab} := \theta^*_{a,b}([vol]) \in H^{d-1}(FM_d)$ for $[vol] \in H^{d-1}(S^{d-1})$
orientation class

Then, map  $\longmapsto g_{ab}$ and everybody else to 0.

Compute recursively that $d\Gamma \mapsto 0$ because of $g_{ab}g_{bc} + \dots$ relation.
We get a surjection in homology. It is then sufficient to show the isomorphism at the level of $\mathbb{K}$-modules; this is done by comparing Betti numbers via Serre spectral sequence argument.

③ Definition of the morphism $I : \mathcal{D} \longrightarrow \Omega_{PA}(FM_d)$ and quasi-iso compatible with cooperad structures.

$$vol := K_d \sum_{i=1}^{d} (-1)^i \, t_i \, dt_1 \wedge \dots \wedge \widehat{dt_i} \wedge \dots \wedge dt_d \in \Omega^{d-1}(S^{d-1})$$

volume form on S^{d-1} — normalizing constant

finite ordered set $E \longrightarrow vol_E := \prod_{e \in E} vol_e \in \Omega((S^{d-1})^E)$

Let $\Gamma \in \mathcal{D}$. For an edge $e \in E_\Gamma$, we set $\theta_e := \theta_{s(e),t(e)}$, and define

$$\theta_\Gamma := \prod_{e \in E_\Gamma} \theta_e : FM_d(V_\Gamma) \longrightarrow (S^{d-1})^{E_\Gamma}$$

vertices of Γ — edges of Γ

5// We get a minimal form $\theta_{\Gamma}^*(\mathrm{vol}_{E_\Gamma}) \in \Omega(\mathrm{FM}_d(V_\Gamma))$

For $V_\Gamma \supseteq A$, we have a canonical projection

external vertices

$$\pi_\Gamma : \mathrm{FM}_d(V_\Gamma) \longrightarrow \mathrm{FM}_d(A)$$

which is an oriented semi-algebraic bundle. Integrating along the fiber gives $(\pi_\Gamma)_* : \Omega_{min}(\mathrm{FM}_d(V_\Gamma)) \longrightarrow \Omega_{PA}(\mathrm{FM}_d(A))$.

Combining these, we can finally define

$$\boxed{\hat{I}(\Gamma) := (\pi_\Gamma)_*\left(\theta_\Gamma^*(\mathrm{vol}_{E_\Gamma})\right) \in \Omega_{PA}(\mathrm{FM}_d(A))}$$

Proposition: We have

(i) $\hat{I}$ is well-defined

(ii) $\hat{I} = 0$ on non-admissible graphs ($\Rightarrow I : \mathcal{Q}(A) \longrightarrow \Omega_{PA}(\mathrm{FM})$)

(iii) $\hat{I}, I$ respects the dga structure

(iv) $\hat{I}, I$ respect the cooperad structures

this is why we can't use de Rham forms!

proof: (i) compatibility of $\hat{I}$ with $\sim$ is what motivated the defn of $\sim$

(ii) case by case; using properties of SA forms

ex: loop $\Rightarrow$ 1 component of θ_Γ is cst $\Rightarrow$ θ_Γ factors through space of lower dimension $\Rightarrow$ pullback of maximal degree form is 0

(iii) algebra: properties of pullback of forms

differential: explicit non-trivial computation!

need decomposition of fibrewise boundary into faces that are images of operadic composition

+ use fibrewise Stokes formula

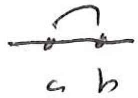
6// (iv) Ω_{PA} is not comonoidal, so $\Omega(FM)$ is not a cooperad, but the following diagram commutes.

$$\begin{tikzcd}
\hat{\mathcal{D}}(A) \arrow[r, "I"] \arrow[ddd, "\Delta"'] \arrow[dddr, phantom, "\circlearrowright"] & \Omega_{PA}(FM(A)) \arrow[d, "\Omega_{PA}(\Delta)"] \\
& \Omega_{PA}\left(\prod_p FM(A_p)\right) \\
& \\
\bigotimes_p \hat{\mathcal{D}}(A_p) \arrow[r, "\bigotimes_p I"] & \bigotimes_p \Omega_{PA}(FM(A_p)) \arrow[uu]
\end{tikzcd}$$

proving this is absolutely tedious!

Theorem 3: $I : \mathcal{D} \longrightarrow \Omega_{PA}(FM_d)$ is a quasi-iso.

proof: It is a morphism of algebras. Sending

 $\longmapsto \theta^*_{ab}(vol)$ gives generator of $H^*(FM_d)$

we get a surjection in cohomology.

Since we have proved $H^*(\mathcal{D}) \cong H^*(FM_d)$, the result follows.