NOTES ON DEFORMATION QUANTISATION

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Abstract: Notes on deformation quantisation, formality, etc. Based on lectures and a culled larger set of notes.

Contents

1. Feynman diagrams and formality

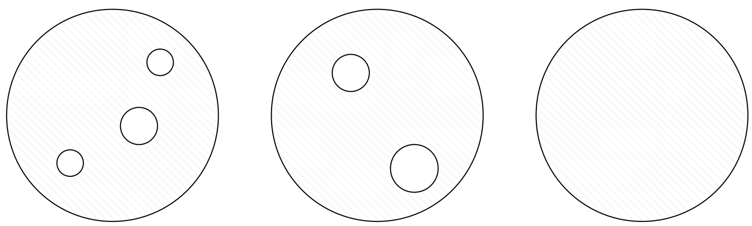
1.1. **Disks.** Let $\mathcal{T}$ be an n-dimensional TQFT valued in symmetric monoidal category $\mathcal{C}$, i.e. a symmetric monoidal functor

$$\mathcal{T} \ : \ \mathrm{Cob}_n \ \to \ \mathcal{C}.$$

Let us now restrict to the symmetric monoidal subcategory $\mathbf{E}_n$ of Cob_n whose objects $[k] = \sqcup S^{n-1}$ are k-tuples of disjoint unit spheres, and writing D^n for the unit disk, the toplogical space of morphisms $[k_1] \to [k_2]$ correspond to inclusions of subdisks

$$\sqcup_{k_1} \mathrm{D}^n \ \hookrightarrow \ \sqcup_{k_2} \mathrm{D}^n.$$

the complement of which is viewed as a cobordism; e.g. a map $[5] \to [3]$ is



Composition in this category is induced by composing cobordisms.

Proposition 1.1.1. *The forgetful functor*

$$F \ : \ \mathbf{E}_n \ \to \ \mathrm{FinSet}_*$$

has the following properties:

- *for every injection $[k_1] \to [k_2]$ there is a preferred (i.e. cocartesian) morphism $(\mathbf{E}_n)_{[k_1]} \to (\mathbf{E}_n)_{[k_2]}$;*
- *the space of maps*

$$\mathrm{Maps}_f(\sqcup_{k_1} \mathrm{D}^n, \sqcup_{k_2} \mathrm{D}^n) \ \simeq \ \prod \mathrm{Maps}_f(\sqcup_{f^{-1}(i)} \mathrm{D}^n, \mathrm{D}^n) \qquad (1)$$

inducing a given map of sets $f : [k_1] \to [k_2]$ is equivalent to the product of the spaces of maps lifting its fibres $f : f^{-1}(i) \to \{i\}$.

Proof. The preferred morphisms are the identity maps on the relevant components, the factorisation isomorphism (1) follows from the definition. □

A category with the above properties is called an *operad*.[1] The elements of $(\mathbf{E}_n)_1$ are called the *colours* of the operad. We can make new operads using categories of cobordisms or disks endowed with extra structures. We write

$$\mathbf{E}_n(k) \ = \ \mathrm{Hom}_{\mathbf{E}_n}([k], [1]).$$

[1] See [Lu] for the precise definition.

Proposition 1.1.2. *The functor*

$$\mathcal{T} \;:\; \mathbf{E}_n \;\to\; \mathcal{C}$$

satisfies the following:

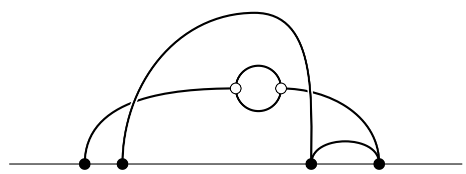
Corollary 1.1.3. *Any TQFT $\mathcal{T}$ induces a unique $\mathbf{E}_n$-algebra structure on $\mathcal{T}(S^{n-1})$.*

1.2. **Feynman diagram operad.**

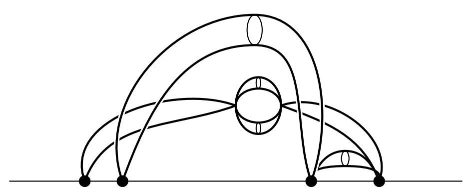
1.2.1. We will now introduce the operad Graphs$(\mathbf{R}^n)$ of Feynman diagrams in $\mathbf{R}^n$. It is valued in $\mathbf{Z}$-graded vector spaces.

A good reference is [LV]. *Warning*: in the literature these are called *admissable* rather than Feynman graphs.

1.2.2. A *preFeynman diagram* is an oriented graph on coloured vertices $\circ, \bullet$ with a total ordering on the $\circ$'s and edges.



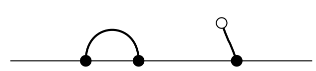
We grade the above, where each $\circ$ contributes degree n and each edge contributes degree $n-1$; in what follows geometrically this will be induced by $\circ$'s moving around in $\mathbf{R}^n$ and spheres around each edge, with the $\bullet$'s fixed:



This forms a $\mathbf{Z}$-graded vector space $\hat{\mathcal{D}}$ which has a differential given by signed[2] sums of contractions of edges

$$d\Gamma \;=\; \sum_e \pm\Gamma/e$$

where we sum over all non-loop edges e touching a $\bullet$ and which are not dead ends



The diagrams $\mathcal{D}(n)$ with n many $\bullet$'s is a *differential graded algebra* structure by taking ordered union of edges and $\circ$'s. Finally, this has a *cooperad* structure where the composition

$$\mathcal{D}(n) \;\to\; \bigoplus_{r,n_i}(\mathcal{D}(n_1)\otimes\cdots\otimes\mathcal{D}(n_r))\otimes\mathcal{D}(r)$$

[2]If n is even, the parity is the ordering on e. Otherwise, it is the maximal ordering of the source and target of e if the source is less than the target in the ordering (otherwise apply -1).

sends Γ to the sum of $\Gamma_1 \otimes \cdots \otimes \Gamma_r \otimes \overline{\Gamma}$, where $\Gamma_i \subseteq \Gamma$ are appropriate subgraphs and $\overline{\Gamma}$ is given by shrinking each Γ_i to a $\bullet$; see [LV, §7.1] for a definition. In particular, the dual $\hat{\mathcal{D}}^\vee$ has an operad structure.

1.2.3. The differential graded complex of *Feynman diagrams* is the quotient

$$\mathcal{D} \;=\; \hat{\mathcal{D}}/I$$

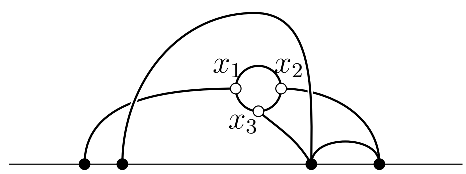
by the subcomplex I generated by:

- *Signs and orderings*. Multiplying a graph by $(-1)^n$ if you reverse an edge or transpose the ordering on the $\circ$'s, and by $(-1)^{n+1}$ if you transpose the ordering on the edges.
- *Disallowed graphs*. Graphs with loops, double edges, $\circ$'s with valence $\leqslant 2$ or not in a connected component with a $\bullet$.

One can then check

Proposition 1.2.4. [LV] $\mathcal{D}$ *forms a cooperad in* $\mathbf{Z}$*-graded differential graded vector spaces.*

Loosely speaking, a Feynman graph



is given by starting with $\bullet$'s and drawing a graph without loops or double edges. The $\circ$'s are where the topology of the graph changes. Indeed, choosing an orientation on the edges and ordering on the edges and $\circ$'s defines an element of $\mathcal{D}$.

1.3. **Formality.**

1.3.1. In next section, we show

Theorem 1.3.2. *Let* $n \geqslant 2$*. Then the spaces*

$$\mathbf{E}_n(k) \;=\; \mathrm{Disk}_k(\mathrm{D}^n)$$

with its element $\mathrm{id} \in \mathbf{E}_n(1)$ *and the structures*

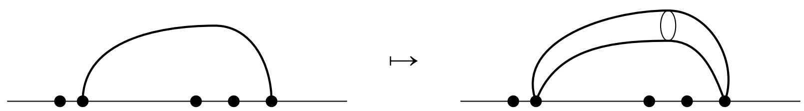
$$m \;:\; (\mathbf{E}_n(k_1) \times \cdots \times \mathbf{E}_n(k_r)) \times \mathbf{E}_n(r) \;\to\; \mathbf{E}_n(k_1 + \cdots + k_r)$$

is formal.

Theorem 1.3.3. *If* $n \geqslant 2$*, there is a unique quasiisomorphism of differential graded algebras with cooperad structure*

$$\overline{\Phi} \;:\; \mathcal{D}(k) \;\xrightarrow{\sim}\; (\mathrm{H}^\bullet(\mathbf{E}_n(k)), 0)$$

killing any graph Γ with a $\circ$ vertex, and sending the ijth chord graph to the pullback of the volume class along the map $\mathbf{E}_n(k) \to S^{n-1}$ given by comparing the two centres of the ijth disks



1.3.4. *What does "formal" mean?* An element A in a dg category $\mathcal{C}$ with t-structure is called *formal* if there is an isomorphism

$$\mathrm{For} \ : \ A \ \tilde{\to} \ \mathrm{H}^\bullet(A)$$

to its cohomology $\bigoplus \tau^{\leqslant n}\tau^{\geqslant n}A$. An element A with extra structure, e.g. an associative product, is called *formal* if there is an isomorphism as above preserving this structure.

A space X is called *formal* if its dg algebra of chains $(\mathrm{C}^\bullet(X), \cup)$ is formal. For instance, if X is a smooth projective complex manifold, then Hodge theory says this dg algebra is isomorphic to the de Rham complex, which Hodge theory says is formal, as each cohomology class is represented by a unique harmonic differential form:

$$(\Omega_X^{\bullet,h}, \wedge) \ \to \ (\Omega_X^\bullet, \wedge) \ \to \ (\mathrm{C}^\bullet(X), \cup)$$

Note that a chain complex of vector spaces is always formal. Chain complexes in general abelian categories are not.

2. A proof of formality of $\mathbf{E}_d$

The main diagram is

$$\begin{array}{ccc} \mathcal{D}(n) & \xrightarrow[\sim]{I} & \Omega_{\mathrm{PA}}(\mathrm{FM}_d(n)) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{H}^\bullet(\mathbf{E}_d(n)) & \overset{\sim}{\dashrightarrow} & \mathrm{C}^\bullet(\mathbf{E}_d(n)) \end{array}$$

where all maps are isomorphisms of operads, and the vertical isomorphisms are more obvious; we will do them first. Formality is then induced by the bottom composition.

2.1. **FM and $\mathbf{C}^\bullet(\mathbf{E}_d)$.**

2.1.1. If A is a finite set, we define

$$\overline{\mathrm{FM}}_d(A) \;=\; \mathrm{Inj}(A, \mathbf{R}^d)/\mathbf{R}^d \rtimes \mathbf{R}_{>0}.$$

For each pair of elements $a, b, c \in A$ we have a map

$$\pi_{a,b} \;:\; \overline{\mathrm{FM}}_d(A) \;\to\; S^{d-1}, \qquad\qquad \eta_{a,b,c} \;:\; \overline{\mathrm{FM}}(A) \;\to\; [0,\infty]$$

defined by taking the unit direction vector of $a-b$ and the $|a-b|/|a-c|$, respectively. We define $\mathrm{FM}_d(A)$ as the closure of

$$i \;:\; \overline{\mathrm{FM}}_d(A) \;\to\; (S^{d-1})^{\binom{|A|}{2}} \times [0,\infty]^{\binom{|A|}{3}}.$$

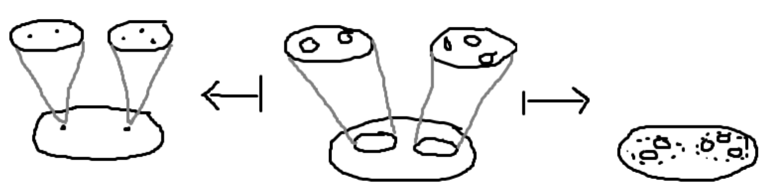
2.1.2. The collection $\mathrm{FM}_d(n)$ forms a topological operad.

2.1.3. *Remark.* Compare this with the Deligne-Mumford operad, where you take the quotient by holomorphic automorphisms of the disk; this is related to associahedra.

Theorem 2.1.4. *There is a quasiisomorphism of cooperads $\Omega_{\mathrm{PA}}(\mathrm{FM}_d(n)) \simeq \mathrm{C}^\bullet(\mathbf{E}_d(n))$ given by a one-step zig-zag.*

Proof. It is given by the "Boardmann-Vogt W-construction" of operads, and there are maps of operads which are homotopy equivalences:

$$\begin{array}{ccc} & W\mathbf{E}_d & \\ \overset{\sim}{\swarrow} & & \overset{\sim}{\searrow} \\ \mathrm{FM}_d & & \mathbf{E}_d \end{array}$$



Points in $W\mathbf{E}_d$ are trees decorated by points of $\mathbf{E}_d$, and the map to FM_d is given by shrinking each disk to a point, and the tree structure is given by the infinitesimal structure on the points, and the map to $\mathbf{E}_d$ is given by forgetting the tree structure. □

2.2. $\mathcal{D}$ and $\mathbf{H}^\bullet(\mathbf{E}_d)$.

2.2.1. *Remark.* The cooperad structure on $\hat{\mathcal{D}}(n)$ does *not* commute with the differential; this is why we need to take the quotient by the ideal I, since the cooperad structure on $\mathcal{D}(n)$ does commute with the differential.

Theorem 2.2.2. *There is a quasiisomorphism of cooperads $\mathcal{D}(n) \simeq \mathrm{H}^\bullet(\mathrm{FM}_d(n))$.*

Proof. This basically follows from the classical fact that

$$\mathrm{H}^\bullet(\mathrm{FM}_d(A)) \;=\; \mathrm{Sym}\mathbf{C}\{g_{a,b} \;:\; a,b \in A\}/(g_{ab}g_{bc} + g_{bc}g_{ca} + g_{ca}g_{ab},\; g_{ab}^2,\; g_{ab} - (-1)^d g_{ba}).$$

The map from $\mathcal{D}(n)$ is given by sending the a, bth chord to $g_{a,b}$. □

Note that $d(\Gamma) = g_{ab}g_{bc} + g_{bc}g_{ca} + g_{ca}g_{ab}$, where Γ is the graph on three $\bullet$'s connected to a single $\circ$ each by a single edge.

2.3. The map $I : \mathcal{D}(n) \xrightarrow{\sim} \Omega_{\mathbf{PA}}(\mathbf{FM}_d(n))$.

2.3.1. To begin, note that the we have the following $d-1$-form on $\mathbf{R}^d$:

$$\kappa_d \sum (-1)^i t_i dt_1 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_d$$

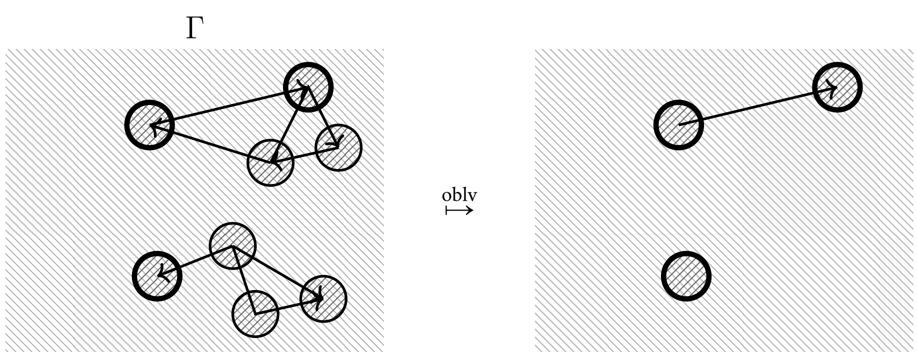
whose exterior derivative is the volume form. Define $\mathrm{vol} \in \Omega^{d-1}(S^{n-1})$ to be its restriction to S^{d-1}.

2.3.2. If $\Gamma \in \hat{\mathcal{D}}(n)$ is a graph with $n_\bullet, n_\circ$ many dots of type $\circ$ and $\bullet$ dots and with E many edges, we consider

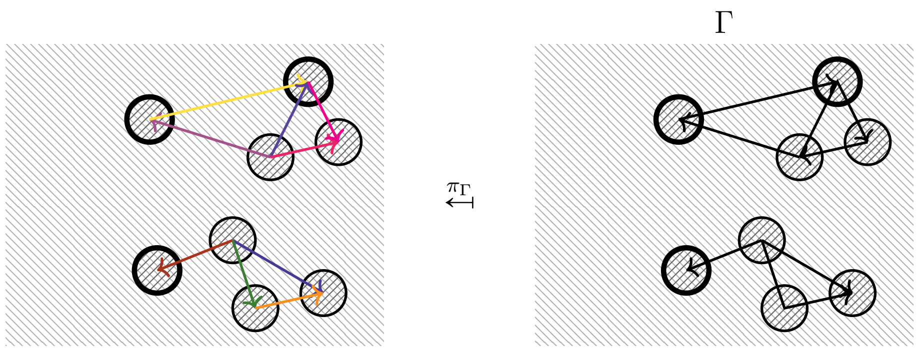
$$\begin{array}{ccc} & \mathrm{FM}_d(n_\bullet + n_\circ) & \\ {}^{\pi_\Gamma}\swarrow & & \searrow^{\mathrm{oblv}} \\ (S^{d-1})^E & & \mathrm{FM}_d(n_\bullet) \end{array}$$

where we write $\pi_\Gamma = \prod_{e:i\to j} \pi_{i,j} : \mathrm{FM}_d(V) \to (S^{d-1})^E$. We define

$$I(\Gamma) \;=\; \mathrm{oblv}_* \pi_\Gamma^*(\textstyle\prod_{e:i\to j[]} \mathrm{vol}_{i,j})$$



where the vertices of the first kind are drawn in bold. Likewise,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