Moduli space of curves and string vertices

Sam Hindson, 04 May 2026

During this note we will become familiar with three Lie algebras:

$$\text{protagonists} := \left\{\mathfrak{g}^+,\ \hat{\mathfrak{g}},\ \hat{\mathfrak{g}}^{\text{comb}}\right\}.$$

All of these are associated with moduli space of curves. The Maurer–Cartan element of of $\mathfrak{g}^+$, known as the *string vertex*, was shown to exist and be unique up to homotopy equivalence by Costello. We will learn what these DGLAs are and that they are quasi-isomorphic. In the coming talks, we will see why the analogue of the string vertex that lives in $\hat{\mathfrak{g}}^{\text{comb}}$ is so important.

In what follows, $C_*(-)$ will denote the functor of normalised singular chains with coefficients in $\mathbb{K}$ from topological spaces to dg-vector spaces.

1. Things to do in moduli space

Let's refresh some details about the moduli space of Riemann surfaces with framed marked points.

Definition 1.1. (Moduli space of curves with inputs and outputs.) Let $g \geq 0$ and $k, l \geq 0$ such that $2g - 2 + k + l > 0$.

(a) We denote by $\mathcal{M}^{\text{fr}}_{g,k,l}$ the moduli space of Riemann surfaces of genus g with $(k + l)$ framed marked points, the first k of which are denoted "inputs" and the final l of which are denoted "outputs".

(b) If one forgets the framing, the corresponding moduli space is denoted $\mathcal{M}_{g,k,l}$.

The framing of the marked points here involves specifying specifying biholomorphic maps

$$\{\varphi_i : \mathbb{D}^2 \to U_{\epsilon_i}(p_i) \mid 1 \leq i \leq k\},$$
$$\{\psi_i : \mathbb{D}^2 \to U_{\delta_j}(q_j) \mid 1 \leq j \leq l\}$$

for each input point p_i and output point q_j for some collection of positive real numbers $\epsilon_1, ..., \epsilon_k, \delta_1, ..., \delta_l$. Here the sets $U_\epsilon(x)$ denote the radius ϵ disk on the curve centred at x with respect to some metric, e.g. the hyperbolic one on $\Sigma \setminus \{p_1, ..., p_k, q_1, ...q_l\}$. We require these maps to extend to some open neighbourhood of $\mathbb{D}^2$, as well as that the U_ϵ of two distinct marked points have disjoint closure. Note that there is an $(\mathbb{S}^1)^{k+l}$ action on $\mathcal{M}^{\text{fr}}_{g,k,l}$ given by precomposing the framing maps with a U(1) rotation.

Since we care about TFT, we care about being able to sew these Riemann surfaces together in various ways. To sew two different Riemann surfaces together at marked points, we need to come up with maps $\mathcal{M}^{\text{fr}}_{g',k',l'} \times \mathcal{M}^{\text{fr}}_{g'',k'',l''} \to \mathcal{M}^{\text{fr}}_{g,k,l}$. We must also think about self-sewing maps, which look like $\mathcal{M}^{\text{fr}}_{g,k,l} \to \mathcal{M}^{\text{fr}}_{g',k',l'}$. We can still think about these sewing maps on the level of individual curves, but we really care about this action under the functor $C_*(-)$ and pre-composed with the Alexander–Whitney map.

Definition 1.2. (The sewing maps.)

(a) For positive integer r satisfying $1 \le r \le l$ we define the *input-output sew map*

$$G_r^{\mathrm{IO}} : C_*(\mathcal{M}^{\mathrm{fr}}_{g',k',l'}) \otimes C_*(\mathcal{M}^{\mathrm{fr}}_{g'',k'',l''}) \to C_*(\mathcal{M}^{\mathrm{fr}}_{g,k,l}),$$

(in which $g = g' + g'' + r - 1$, $k = k' + k'' - r$, and $l = l' + l'' - r$) by gluing the first r input open disks of $\Sigma' \in \mathcal{M}^{\mathrm{fr}}_{g',k',l'}$ to the first r output open disks of $\Sigma'' \in \mathcal{M}^{\mathrm{fr}}_{g'',k'',l''}$.

(b) For $1 \le j' \le l'$ and $1 \le j'' \le l''$ we define the *output-output sew map*

$$G^{\mathrm{OO}}_{j'j''} : C_*(\mathcal{M}^{\mathrm{fr}}_{g',k',l'}) \otimes C_*(\mathcal{M}^{\mathrm{fr}}_{g'',k'',l''}) \to C_*(\mathcal{M}^{\mathrm{fr}}_{g'+g'',k'+k'',l'+l''-2})$$

as the map wrought from the operation which sews the j'^{th} output point on $\Sigma' \in \mathcal{M}^{\mathrm{fr}}_{g',k',l'}$ to the j''^{th} output point on $\Sigma'' \in \mathcal{M}^{\mathrm{fr}}_{g'',k'',l''}$.

(c) For $1 \le j_1 < j_2 \le l$ we define the *self output-output sew map*

$$G^{\mathrm{OOS}}_{j_1 j_2} : C_*(\mathcal{M}^{\mathrm{fr}}_{g,k,l}) \to C_*(\mathcal{M}^{\mathrm{fr}}_{g+1,k,l-2})$$

as the map wrought from the operation which sews the ${j_1}^{\mathrm{th}}$ output point to the ${j_2}^{\mathrm{th}}$ output point on $\Sigma \in \mathcal{M}^{\mathrm{fr}}_{g,k,l}$.

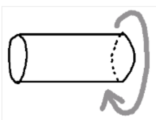
Căldăraru and Tu point out the fact that there is an asymmetry in which sewing maps we care about with respect to inputs and outputs (there are more self-sewing operations one could do, like sewing two inputs together) but that this will be for good reason later on.

Exercise 1.3. What is this good reason?

These gluing maps are quite easy to visualise.

For our purposes, it will be convenient to include two more chains. These correspond to Riemann surfaces which are not stable in the unframed sense since they $2g - 2 + k + l = 0$. However, they are stable when we imbue their marked points with framing. These are as follows:

$M =$  $\in C_0\left(\mathcal{M}^{\mathrm{fr}}_{0,2,0}\right)$

$S =$  $\in C_1\left(\mathcal{M}^{\mathrm{fr}}_{0,1,1}\right)$

We draw S in this way to denote that we are not simply looking at an annulus, but a family of annuli obtained by performing all twists on one end.

It is clear that sewing one input of M to another marked surface produces a surface with one fewer output points and one more input point. If one sews both inputs to outputs of the same surface, one obtains the output-output gluing map described above. The usefulness of S will become apparent in the following sections. See the figure below.

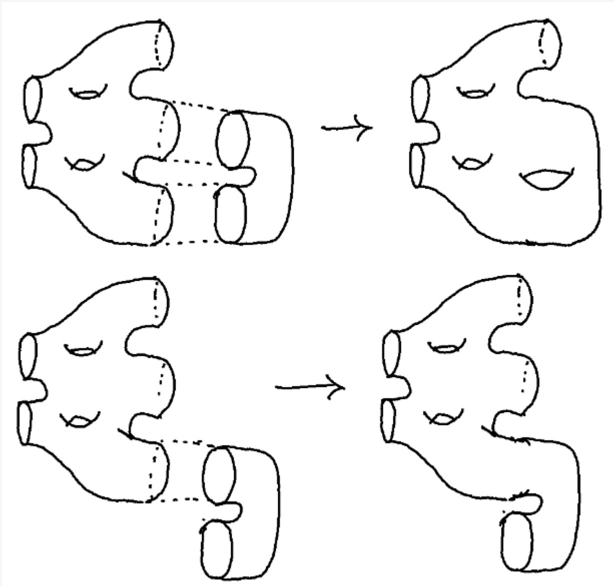


Figure 1.1: The self-gluing and input-to-output operations, respectively.

2. The construction of $\mathfrak{g}$

One can build a DGLA out of these ingredients. For now, let us consider the moduli sapce of Riemann surfaces of genus $g \geq 0$ with $n \geq 0$ output points but no input points; we will write $\mathcal{M}^{\mathrm{fr}}_{g,0,n} = \mathcal{M}^{\mathrm{fr}}_{g,n}$. There is a homological circle action on singular chains on $\mathcal{M}^{\mathrm{fr}}_{g,n}$, denoted by B_i, which sends some chain to the chain corresponding with all possible $\mathbb{S}^1$ rotations of the ith marked point. Acting by B_i is equivalent to sewing S at the corresponding point.

We start by taking coinvariants under our circle actions on the singular chain complex:

$$C_*(\mathcal{M}^{\mathrm{fr}}_{g,n})_{(\mathbb{S}^1)^n} = \left(C_*(\mathcal{M}^{\mathrm{fr}}_{g,n})[u_1^{-1}, ..., u_n^{-1}], \eth := \partial + \sum_{i=1}^{n} u_i B_i \right).$$

The symmetric group S_n acts on $\mathcal{M}^{\mathrm{fr}}_{g,n}$ by permuting marked points. It also acts on $C_*(\mathcal{M}^{\mathrm{fr}}_{g,n})_{(\mathbb{S}^1)^n}$ by permutation of indices of the corresponding u variables. Define

$$C_*(\mathcal{M}^{\mathrm{fr}}_{g,n})_{\mathrm{hS}} = C_*(\mathcal{M}^{\mathrm{fr}}_{g,n})_{(\mathbb{S}^1)^n \rtimes S_n}.$$

> **Exercise 2.1.** Why are we taking a homotopy quotient here? Is this symmetric group action not free?

Let $\hbar, \lambda$ be formal variables of even degree. We then define the super vector space

$$\mathfrak{g} := \left(\bigoplus_{g,n} C_*(\mathcal{M}^{\mathrm{fr}}_{g,n})_{\mathrm{hS}} \right) [\![\hbar, \lambda]\!][1]$$

and use our sewing maps to define the following maps on equivariant chains.

> **Definition 2.2.** Let $\alpha \in C_*(\mathcal{M}^{\mathrm{fr}}_{g,n})_{(\mathbb{S}^1)^n \rtimes S_n}$ be an equivariant chain.
>
> (a) For some marked point index $1 \leq i \leq n$ define the map π_i

$$\pi_i(\alpha) = \alpha|_{u_i^{-1}=0}.$$

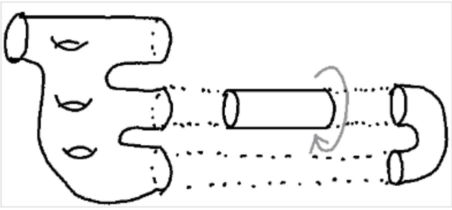
(b) Likewise for some marked point indices $1 \leq i < j \leq n$ define the map π_{ij}

$$\pi_{ij}(\alpha) = \alpha|_{u_i^{-1}=u_j^{-1}=0}.$$

(c) Define the operator

$$\begin{aligned} \Delta : \mathfrak{g}[1] &\longrightarrow \mathfrak{g}[1] \\ \alpha &\longmapsto \Delta(\alpha) := \sum_{1\leq i<j<n} G_{ij}^{\mathrm{OOS}}(\pi_{ij}(B_i\alpha)). \end{aligned}$$

The Δ twisted self-sew can be visualised as follows, using our special annuli S and M:



Note that the this is a little deceptive, since there is more going on here: the projection π_{ij} ensures that the gluing only happens if the appropriate circle parameters u_i and u_j are both zero.

Definition 2.3. For equivariant chains $\alpha \in C_*(\mathcal{M}^{\mathrm{fr}}_{g',n'})_{\mathrm{hS}}, \beta \in C_*(\mathcal{M}^{\mathrm{fr}}_{g'',n''})_{\mathrm{hS}}$ define the symmetric degree one map

$$\begin{aligned} \{-,-\} : \mathfrak{g}[1] \otimes \mathfrak{g}[1] &\longrightarrow \mathfrak{g}[1] \\ (\alpha, \beta) &\longmapsto \{\alpha, \beta\} := \sum_{\substack{1\leq i\leq n' \\ 1\leq j\leq n''}} G_{ij}^{\mathrm{OO}}(\pi_i(B_i\alpha), \pi_j(\beta)). \end{aligned}$$

Theorem 2.4. ([CT24], Theorem 5.1.) The triple $(\mathfrak{g}, \eth + \hbar\Delta, \{-,-\})$ forms a DGLA.

3. String vertices with no inputs

We now have a DGLA. Are there any interesting elements inside of it? Maybe. Let's start by looking for Maurer–Cartan elements satisfying some additional conditions. They even have a fancy name in this context:

Definition 3.1. (String vertices.) A degree zero element $\mathcal{V} \in \mathfrak{g}[1]$ of the form

$$\mathcal{V} = \sum_{g,n} \mathcal{V}_{g,n} \hbar^g \lambda^{2g-2+n} \quad \text{with} \quad \mathcal{V}_{g,n} \in C_{6g-6+2n}(M^{\mathrm{fr}}_{g,0,n})_{\mathrm{hS}}$$

is called a *string vertex* if

(a) $\mathcal{V}$ is a Maurer–Cartan element in $\mathfrak{g}$,

(b) $\mathcal{V}_{0,3} = \frac{1}{6}$ pt, for pt $\in C_0(M^{\mathrm{fr}}_{0,0,3})_{\mathrm{hS}}$ the point class.

Costello showed that these exist and are unique up to gauge equivalence.

Theorem 3.2. ([Cos09]) A string element exists and is unique up to gauge equivalence.

Remark 3.3. Explicitly, condition (a) of Definition 3.1 can be written in terms of the components of $\mathcal{V}$ as

$$\eth\mathcal{V}_{g,n} + \Delta\mathcal{V}_{g-1,n+2} + \frac{1}{2}\sum_{\substack{g_1+g_2=g,\\ n_1+n_2=n+2}} \{\mathcal{V}_{g_1,n_1}, \mathcal{V}_{g_2,n_2}\} = 0$$

by writing out what $(\eth + \hbar\Delta)\mathcal{V} + \frac{1}{2}\{\mathcal{V},\mathcal{V}\} = 0$ means degree-by-degree. This is actually known as a quantum master equation.

Remark 3.4. In the not-too-distant future (*viz.* Proposition 4.8) we will deal instead with the object

$$\mathfrak{g}^+ := \left(\bigoplus_{g,n\geq 1} C_*(\mathcal{M}^{\mathrm{fr}}_{g,n})_{hS}\right)[\![\hbar,\lambda]\!][1],$$

i.e. $\mathfrak{g}$ but without any of (chains of) surfaces without marked points.

Exercise 3.5. Explain why we should also be able to find a string vertex inside $\mathfrak{g}^+$.

4. The construction of $\hat{\mathfrak{g}}$

What if we want input points too? We still have our circle actions, but we can now also permute the inputs. For reasons still slightly mysterious to me, we consider our inputs to be antisymmetrised, as opposed to our outputs which were symmetrised. Concretely:

Definition 4.1. We denote by $\underline{\mathrm{sgn}}$ the rank one local system over $\mathcal{M}^{\mathrm{fr}}_{g,k,l}$ whose fiber is the sign representation of S_k on the set of input points shifted by $[-k]$. Its natural basis vector is $p_1 \wedge \ldots \wedge p_k$.

Consider the super vector space

$$\hat{\mathfrak{g}} := \left(\bigoplus_{\substack{g,k,l\\ k\geq 1}} C_*(\mathcal{M}^{\mathrm{fr}}_{g,n}, \underline{\mathrm{sgn}})_{hS}\right)[\![\hbar,\lambda]\!][2],$$

where the homotopy quotient we've taken here is by $(\mathbb{S}^1)^{k+l} \ltimes S_k \times S_l$, where the S_k action is twisted by $\underline{\mathrm{sgn}}$. The homological degree of an equivariant chain α is written as

$$|\alpha| = \deg(\alpha) - k,$$

where deg denotes the usual singular chain homology degree and k is the number of input points.

Definition 4.2. With $\eth = \partial + \sum_i u_i B_i = \partial + uB$, we have a differential on the complex $C_*(\mathcal{M}^{\mathrm{fr}}_{g,n}, \underline{\mathrm{sgn}})_{\mathrm{hS}}$ which acts as

$$\eth(\alpha, p_1 \wedge \ldots \wedge p_k) := (\eth\alpha, p_1 \wedge \ldots \wedge p_k).$$

Definition 4.3. The twisted self-sewing operation Δ similarly only acts on the equivariant chain α inside the tuple $(\alpha, p_1 \wedge \ldots \wedge p_k) \in C_*(\mathcal{M}^{\mathrm{fr}}, \underline{\mathrm{sgn}})_{\mathrm{hS}}$.

Definition 4.4. Define the operation

$$\iota : C_*(\mathcal{M}^{\mathrm{fr}}_{g,k,l}, \underline{\mathrm{sgn}})_{\mathrm{hS}} \longrightarrow C_*(\mathcal{M}^{\mathrm{fr}}_{g,k+1,l-1}, \underline{\mathrm{sgn}})_{\mathrm{hS}}$$

$$(\alpha, p_1 \wedge \ldots \wedge p_k) \longmapsto (-1)^{\deg(\alpha)} \sum_{j=1}^{l} (\iota_j \alpha, q_j \wedge p_1 \wedge \ldots \wedge p_k)$$

which uses the map $\iota_j : \mathcal{M}^{\mathrm{fr}}_{g,k,l} \to \mathcal{M}^{\mathrm{fr}}_{g,k+1,l-1}$ that is defined to relabel one of the output points as an input point. Our map ι just sums over all possible ways to do this.

As mentioned earlier, this can be seen as an operation which represents all the ways one can affix one end of our horseshoe annulus M to an output point.

It turns out that $(\eth + \iota + \hbar\Delta)^2 = 0$ ([CT24], Lemma 5.4.) We'll now look at defining a Lie bracket. Morally, this operation takes two equivariant chains and produces a weighted sum of chains which resembles this:

$$(\Psi, \Phi) \mapsto \sum_{r \geq 1} \hbar^{r-1}(\text{normalised outcomes of gluing } r \text{ outputs of } \Psi \text{ to } r \text{ inputs of } \Phi \text{ and vice versa})$$

It requires a few more definitions to get there explicitly:

Definition 4.5.

(a) Define the twisted sewing map

$$G^B_r : C_*\left(\mathcal{M}^{\mathrm{fr}}_{g',k',l'}, \underline{\mathrm{sgn}}\right) \otimes C_*\left(\mathcal{M}^{\mathrm{fr}}_{g'',k'',l''}, \underline{\mathrm{sgn}}\right) \longrightarrow C_*\left(\mathcal{M}^{\mathrm{fr}}_{g,k,l}, \underline{\mathrm{sgn}}\right)$$

by setting

$$G^B_r((\alpha, p'_1 \wedge \ldots \wedge p'_{k'}), (\beta, p''_1 \wedge \ldots \wedge p''_{k''}))$$
$$= (-1)^{\deg(\beta)(k'+r)}\left(G_r\left(\alpha, B_{q''_1} \ldots B_{q''_r}\beta, p'_{r+1} \wedge \ldots \wedge p'_{k'} \wedge p''_1 \wedge \ldots \wedge p''_{k''}\right)\right)$$

where $B_{q_f}\beta$ denotes result of the circle action on β around the marked point q_f

(b) Define the twisted sewing map $\tilde{G}^B_r$ on equivariant chains Ψ, Φ by

$$\tilde{G}^B_r(\Psi, \Phi) = G^B_r\left(\Psi_{w_1^{-1}=\ldots=w_r^{-1}=0}, \Phi_{u_1^{-1}=\ldots=u_r^{-1}=0}\right)$$

where w_i are the variables associated with inputs on Ψ and u_i are variables associated with outputs on Φ.

(c) Define the map

$$\circ_r : C_*\left(\mathcal{M}^{\mathrm{fr}}_{g',k',l'}, \underline{\mathrm{sgn}}\right)_{hS} \otimes C_*\left(\mathcal{M}^{\mathrm{fr}}_{g'',k'',l''}, \underline{\mathrm{sgn}}\right)_{hS} \longrightarrow C_*\left(\mathcal{M}^{\mathrm{fr}}_{g,k,l}, \underline{\mathrm{sgn}}\right)_{hS}$$

given by

$$\Psi \circ_r \Phi := \sum_{\substack{\sigma\in \mathrm{Shuffle}(r,k'-r)\\ \tau\in \mathrm{Shuffle}(r,l''-r)}} \widetilde{G}^B_r(\sigma\Psi, \tau\Phi)$$

where the shuffle permutation σ acts on $\{p'_1, ..., p'_{k'}\}$ and τ acts on $\{q''_1, ..., q''_{l''}\}$.

Definition 4.6. Define a map $\{-,-\}_\hbar$ by

$$\begin{aligned} \{\Psi,\Phi\}_\hbar : \hat{\mathfrak{g}}[1] \otimes \hat{\mathfrak{g}}[1] &\longrightarrow \hat{\mathfrak{g}}[1] \\ \Psi \otimes \Phi &\longmapsto (-1)^{|\Psi|} \sum_{r\geq 1} \left(\Psi \circ_r \Phi - (-1)^{|\Psi||\Phi|} \Phi \circ_r \Psi\right) \times \hbar^{r-1}. \end{aligned}$$

Theorem 4.7. ([CT24], Theorem 5.6.) The triple $(\hat{\mathfrak{g}}, \eth + \iota + \hbar\Delta, \{-,-\}_\hbar)$ forms a $\mathbb{Z}$-graded DGLA.

As promised, here is the callback to last section's protagonist:

Proposition 4.8. There exists a quasi-isomorphism of DGLAs $q : \mathfrak{g}^+ \to \hat{\mathfrak{g}}$.

Exercise 4.9. The proof of the above uses firstly that the underlying moduli spaces at fixed genus having $n = k + l$ are all isomorphic. However, I did not understand the next step involving an exact Koszul complex. What's going on here?

5. Combinatorial string vertices

We now have a Lie algebra built from the totalisation of the PROP of chains on $\mathcal{M}^{\mathrm{fr}}_{g,k,l}$ (together with the two special elements M and S). How does one actually compute the string vertices here? Boundary operators on singular homology as well as circle actions are a bit hard to visualise for arbitrary chains, and so except for some particularly nice examples (see e.g. Section 2.1. of [CT24], which discusses an example originally due to Costello) one might yearn for a combinatorial way to do calculations and solve for string vertex coefficients. Luckily for us, there is a way to do this. We pursue it by constructing another Lie algebra coming from a combinatorial picture of the moduli spaces we are interested in.

5.1. Fat graphs with black and white vertices

Before giving any definitions, here's a picture of the objects we'll be dealing with.

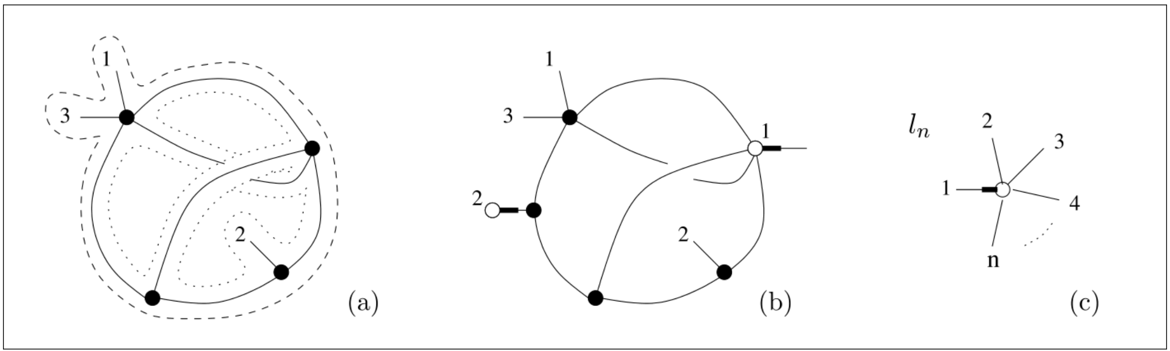


Figure 5.3: From [WW16]

Here's what they are in words:

Definition 5.1.

- A **graph** is a tuple (V, H, s, i) with vertices V, half-edges H, a "source map" $s : H \to V$, and an involution $i : H \to H$.
- Fixed points of the involution are called **leaves**. Pairs $\{h, i(h) \neq h\}$ are called **edges**.
- A **fat graph** is a graph together with a cyclic ordering of the sets $s^{-1}(v)$ for each $v \in V$.
- The **genus** of a fat graph is the minimal genus of a surface one would need on which to draw the graph without there being any over/undercrossings of edges.
- The **orientation** of a graph is a unit vector in $\det(\mathbb{R}(V \sqcup H))$.
- A **black and white (BW) graph** is a fat graph whose set of vertices is split up $V = V_b \sqcup V_w$ in which
  - the black vertices are unlabelled and must be at least trivalent,
  - the white vertices are labelled $1, \ldots, |V_w|$, can have any valence, and have a distinguished half-edge which attaches to it, called its **starting edge**, which is drawn in thick. (So, there is an honest ordering to $s^{-1}(v)$ for any $v \in V_w$.)
- A $\binom{p}{m}$**-graph** is a BW graph with p white vertices and m leaves labelled $\{1, \ldots, m\}$. It is allowed to have additional unlabelled leaves, provided these are the start half-edge of a white vertex.
- A **boundary cycle** of a graph is a sequence of consecutive half-edges corresponding to the boundary components one would get by thickening the graph.

The leftmost graph in Figure 5.3 has two boundary cycles: one demarcated with a dotted line and the other with a dashed line. We will eventually only be considering graphs with one boundary cycle per labelled leaf.

Isomorphisms of BW-graphs are assumed to preserve orientation and leaf labellings.

Definition 5.2. For a BW graph G with a non-cycle edge e which moreover does not join two white vertices, define G/e to be the set of isomorphism classes of BW graphs that can be obtained from G by collapsing e, identifying its endpoint verties, and coloring the resulting vertex white if either of the collapsed vertices were white, and black otherwise. If the new vertex is white and one did not collapse along the start half-edge of a white vertex, then the collapse is well-defined. If you collapsed along this marked edge, you'll land up with several possible collapses.

A **blow-up** of a BW graph G has a blow-up $\tilde{G}$ if there exists an edge $\tilde{e}$ of $\tilde{G}$ such that $G \in \tilde{G}/\tilde{e}$.

Check out Figure 5.4 for some examples of collapses.

Orientation (if applicable) is inherited: if $e = \{h_1, h_2\}$ with $s(h_1) = v_1$ and $s(h_2) = v_2$, then if G was oriented as $v_1 \wedge v_2 \wedge h_1 \wedge h_2 \wedge \ldots$ then the orientation G/e is written as $v \wedge \ldots$.

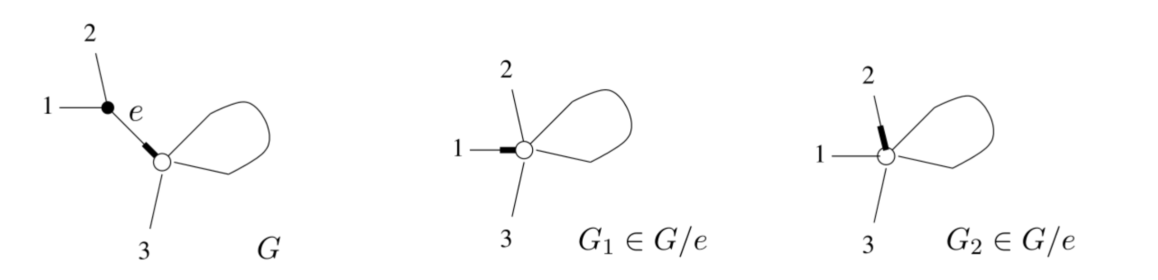


Figure 5.4: Also from [WW16]

5.1.1. The chain complex of BW graphs

Definition 5.3. We let BWG denote the chain complex generated as a $\mathbb{Z}$-module by isomorphism classes of oriented BW graphs modulo that -1 acts by reversing orientation. The **degree** of a BW graph is

$$\deg(G) = \sum_{v \in V_b} (|v| - 3) + \sum_{v \in V_w} (|v| - 1).$$

Degenerate graphs (single leaves with no vertices; circle with no vertices) have degree 0. There is a differential $\hat{d}$ on BWG given by

$$\hat{d}G := \sum_{(\tilde{G}, \tilde{e}) : G \in \tilde{G}/\tilde{e}} \tilde{G}.$$

Note that underlying any BW graph G with p white vertices and m labelled leaves is a $\binom{p}{m}$-graph $\lfloor G \rfloor$. Finding $\lfloor G \rfloor$ for a given G involves checking whether G has unlabelled leaves which are not the start edge of a white vertex. If it does not have these, then $\lfloor G \rfloor = G$. If it does have these:

- if the leaf is attached to a trivalent black vertex, both the leaf and this vertex are forgotten when presenting $\lfloor G \rfloor$
- if the leaf is attached to a black vertex of valence ≥ 4 or a white vertex, set $\lfloor G \rfloor = 0$.

Definition 5.4. We let (p, m)G denote the chain complex generated as a $\mathbb{Z}$-module by isomorphism classes of oriented $\binom{p}{m}$ graphs modulo that -1 acts by reversing orientation. The **degree** of such a graph is the same as before.

The differential of such an underlying $\binom{p}{m}$-graph $\lfloor G \rfloor$ is given by

$$d\lfloor G \rfloor = \left\lfloor \hat{d}G \right\rfloor.$$

Exercise 5.5. Show that $\hat{d}, d$ are differentials.

There is a way to compose graphs. The full details of these operations can be found in [WW16]; the sewing of graphs we are interested in follows almost directly from the sewing of Riemann surfaces. That is, the composition operations we will only ever need require identifing marked leaves with invalent white vertices. It is a nontrivial fact that these graph composition rules (in their full generality) give us chain maps however, and the fact that these correspond with the geometric sewing maps of the Riemann surfaces we know and love is an even more involved result. This is a result from [San15], which shows that the combinatorial sewing maps (graph compositions) we have above agree with the geometric sewing maps of Riemann surfaces. We therefore have another chain complex, which we call $C_*^{\mathrm{comb}}(\mathcal{M}_{g,k,l}^{\mathrm{fr}})_{\mathrm{hS}}$, and therefore another Lie algebra:

$$\hat{\mathfrak{g}}^{\mathrm{comb}} := \left(\bigotimes_{g,k\geq 1,l} C_*^{\mathrm{comb}}(\mathcal{M}_{g,k,l}^{\mathrm{fr}}, \underline{\mathrm{sgn}})\right)[\![\hbar,\lambda]\!]$$

which is quasi-isomorphic to $\hat{\mathfrak{g}}$.

5.2. Summary: things to do with graphs

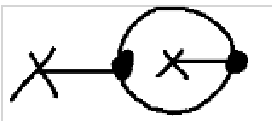
A word of warning: *The graph manipulation rules laid out here are mostly reverse-engineered from superceded arXiv preprints and the final five minutes of a Zoom lecture given by Căldăraru in 2020. Several as-yet unreleased papers with tantalising titles (like Căldăraru's "Efficiently computing with ribbon graphs" and Căldăraru–Cheung's "Explicit formulas for the Kontsevich-Soibelman PROP") might help clear up any confusions once they are available for public consumption. As such, do not base any serious research on the little drawings given below. Thanks!*

Since the whole point of introducing this formalism is to identify certain graphs with certain (equivariant) chains of Riemann surfaces with marked points, only certain graphs will end up playing a role in the calculations which follow. The correspondence is this:

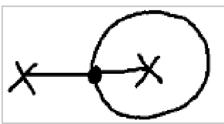
- Marked leaves correspond to *output points*. These are labelled by powers of the corresponding u_i variable. We use the convention that if there is a "×" symbol but no other label, the edge is labelled by u_i^0. Each such marked leaf will be unique in its boundary cycle.
- White vertices correspond to *input points*.
- The degree of a graph is the same as the degree of the chain it represents.

Let us make all of the sewing maps extremely explicit.

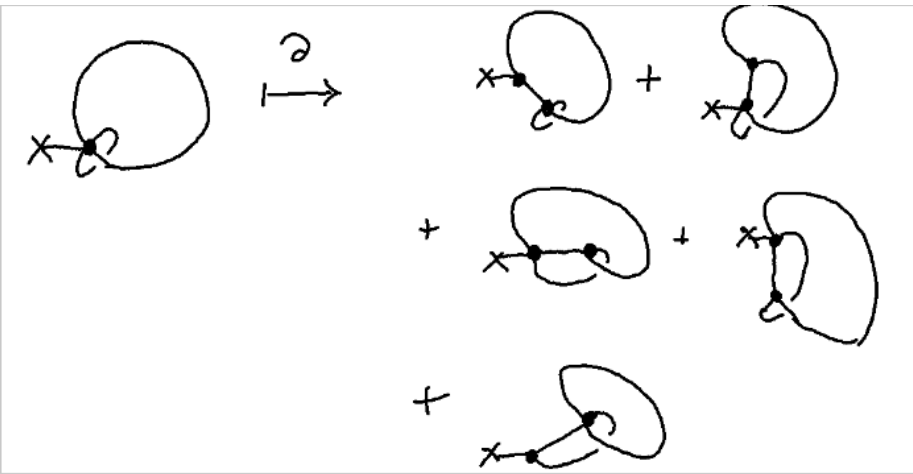
The ι_j map which turns an output into an input is achieved in graph language by affixing to a white vertex the following graph:



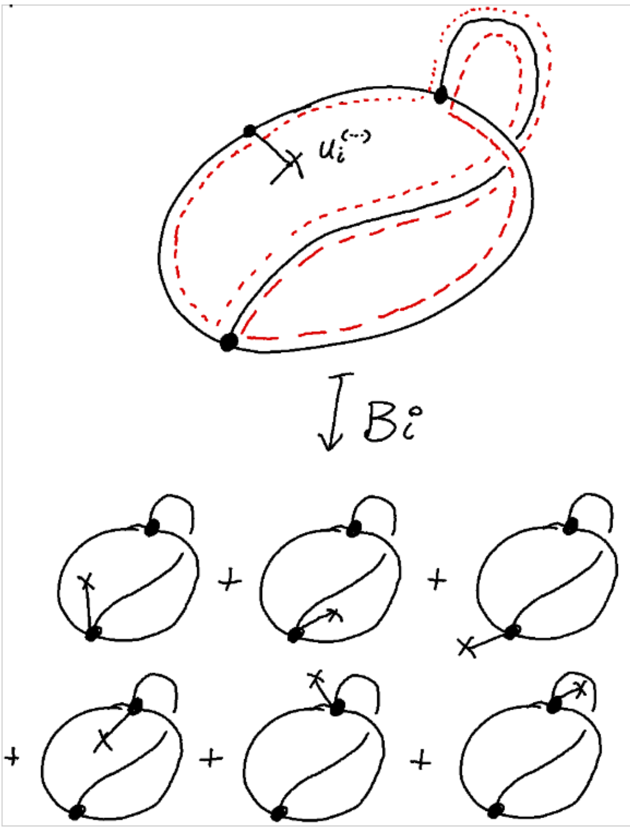
The action of the twisted self-sew map Δ corresponds to affixing to a given white vertex the graph



The (boundary) differential δ involves splitting black vertices in possible ways that leave at least trivalent black vertices:



The circle action B_i corresponds to the sum of all possible graphs one gets by attaching the marked leaf of interest to the other black vertices one encounters in the corresponding boundary cycle. For example:



5.3. Actually using the combinatorial model

Since we now have a roof diagram

$$\mathfrak{g}^{+} \xrightarrow{q} \hat{\mathfrak{g}} \xleftarrow{\tilde{q}} \hat{\mathfrak{g}}^{\mathrm{comb}},$$

we can ask about the MC elements in $\hat{\mathfrak{g}}^{\mathrm{comb}}$, since we know they must exist and are unique up to gauge equivalence.

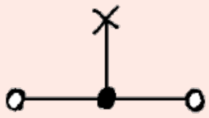
Theorem 5.6. There exists a unique (up to homotopy) degree -1 element $\hat{\mathcal{V}}^{\mathrm{comb}} \in \hat{\mathfrak{g}}^{\mathrm{comb}}$ of the form

$$\hat{\mathcal{V}}^{\mathrm{comb}} = \sum_{g,k\geq 1,l} \hat{\mathcal{V}}^{\mathrm{comb}}_{g,k,l} \hbar^g \lambda^{2g-2+k+l},$$

called the *combinatorial string vertex*, which satisfies

(a) being a Maurer–Cartan element of $\hat{\mathfrak{g}}^{\mathrm{comb}}$

(b) one has

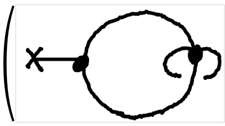
$$\hat{\mathcal{V}}^{\mathrm{comb}}_{0,1,2} = \frac{1}{2}$$ 

Why that graph? It turns out that this is the only genus 0, degree 0 BW fat graph that with 1 marked leaf (corresponding to the input point) and 2 white vertices (corresponding to the two output points). Let's try and compute the next coefficient. The quantum master equation at the next order reads

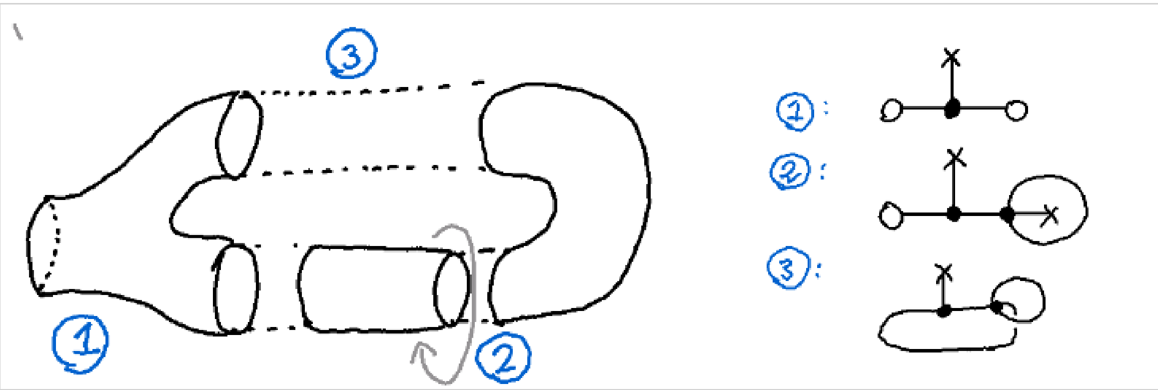
$$\Delta\left(\hat{V}^{\mathrm{comb}}_{0,1,2}\right) = -(\partial + uB)\hat{V}^{\mathrm{comb}}_{1,1,0}.$$

The LHS of the quantum master equation is given by

$$\Delta\left(\hat{\mathcal{V}}^{\mathrm{comb}}_{0,1,2}\right) = \frac{1}{2}\left(\quad \right) \in C_1(\mathcal{M}^{\mathrm{fr}}_{1,1,0})_{hS}, \tag{5.1}$$

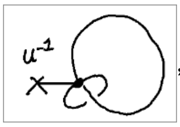


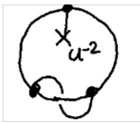
which can be gleaned by the looking at the following three calculation steps:



Step (1) comes from condition (b) in Theorem 5.6. Step (2) is the twisted self-sewing operation we described before. Finally, step (3) connects up this new input point with the other output point. Also remember that the diagram in step (1) comes with a factor of $\frac{1}{2}$.

Let's try the RHS. We know that $V^{\mathrm{comb}}_{1,1,0}$ has to be some rational linear combination of the following two BW graphs:

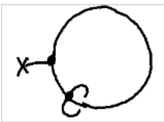
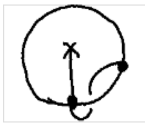
$V^{\text{comb}}_{1,1,0} = aX + bY, \quad X =$ 

$, Y =$ 

,

since these are the only two ribbon graphs of genus 1, degree 2 in our chain complex. (Remember that u is of degree 2!)

There are also only two genus 1, degree 1 graphs:

$Z =$  $, W =$  .

Actually computing the action of our differential on the linear combination $aX + bY$ is something we have already sneakily done in the previous section (check the figures!):

$$\begin{aligned} \partial X &= -2W - Z \qquad & \partial Y &= 0 \\ uB(X) &= 0 \qquad & uB(Y) &= 6Z, \end{aligned} \tag{5.2}$$

which means (owing to linear indepence of the graphs W, Z) one has from (5.1) and (5.2) that

$$\frac{1}{2}W = -a(-2W - Z) + -b(6Z)$$
$$\leftrightarrow a = \frac{1}{4}, b = \frac{1}{24}.$$

Exercise 5.7. Work out the next string vertex coefficient, $V^{\text{comb}}_{0,2,1}$.

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