CaCS 2022

Coherence, polytopes, and Koszul duality

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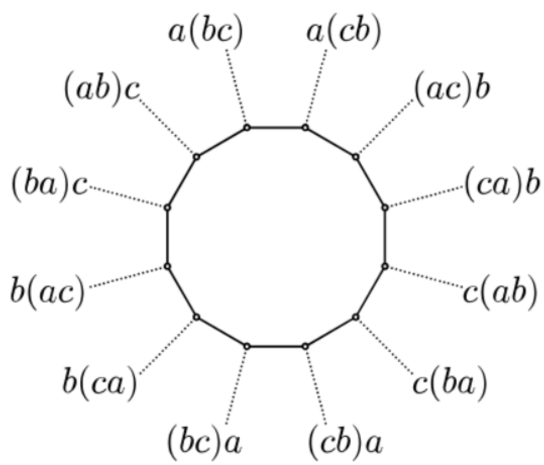
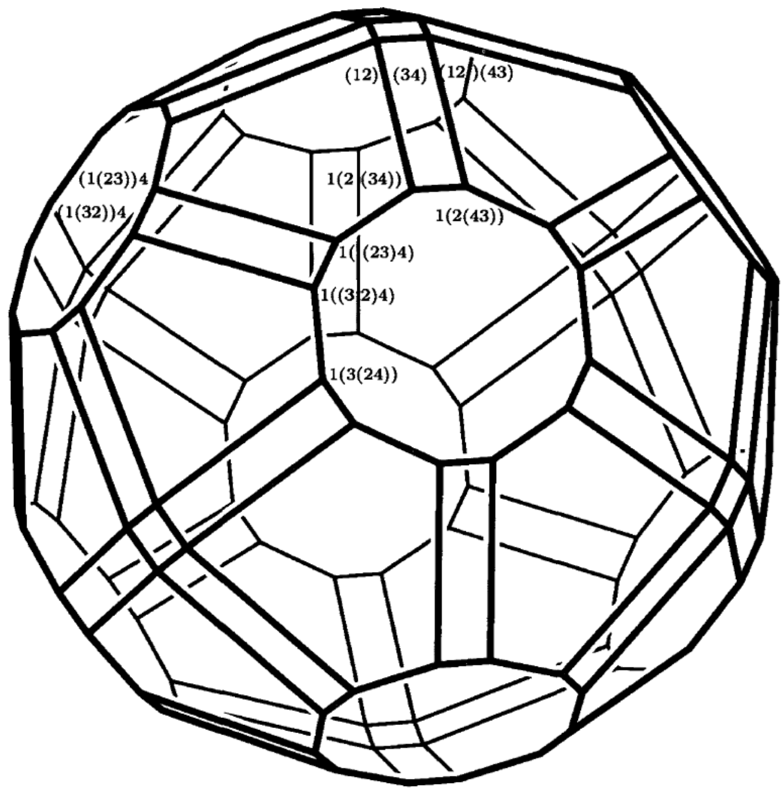
$\boxed{0}$ Introduction

It is natural, therefore, to look for a 'polytope' KP_n whose vertices would correspond to all bracketed and permuted products of n letters so that any n objects in any symmetric (or braided) monoidal category give rise to a diagram of the shape KP_n. We shall construct KP_n, as a CW-complex, in Section 2 and show that it is an $(n-1)$-ball. This gives an instant one-step proof of Mac Lane's theorem in full generality. However, it remains unclear whether KP_n can be realized as a convex polytope, like K_n and P_n.





$\boxed{1}$ MacLane's coherence theorem

Let A be a set. We define a syntax

- object terms $\quad T ::= a \mid T \otimes T \quad , a \in A$
- morphism terms $\quad M ::= \alpha \mid \alpha^{-1} \mid M \circ M \mid id \mid M \otimes M$
- typing rules

$$\frac{}{\alpha : (T_1 \otimes T_2) \otimes T_3 \longrightarrow T_1 \otimes (T_2 \otimes T_3)}$$

$$\frac{}{\alpha^{-1} : [\ldots]} \qquad \frac{}{id : T \to T}$$

$$\frac{M_1 : T_1 \to T_2 \quad M_2 : T_2 \to T_3}{M_2 \circ M_1 : T_1 \to T_3} \qquad \frac{M_1 : T_1 \to T_1' \quad M_2 : T_2 \to T_2'}{M_1 \otimes M_2 : T_1 \otimes T_2 \to T_1' \otimes T_2'}$$

Define $\mathrm{Free}(A) :=$ syntax /
- laws of categories and bifunctors
- MacLane's coherence equation

$$\begin{tikzcd} A \otimes (B \otimes (C \otimes D)) \arrow[r, "a_1"] \arrow[d, "1 \otimes a_3"] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "a_2"] & ((A \otimes B) \otimes C) \otimes D \\ A \otimes ((B \otimes C) \otimes D) \arrow[rr, "a_4"] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "a_5 \otimes 1"'] \end{tikzcd} \tag{3.5}$$

<u>Prop.</u> $\mathrm{Free}(A)$ is the free monoidal category $/ A$,
i.e. $\forall$ monoidal category $\mathcal{C}$, $\forall \rho : A \to \mathrm{Ob}(\mathcal{C})$,
$\exists!$ strict monoidal functor
$[[-]] : \mathrm{Free}(A) \to \mathcal{C}$ extending ρ.

<u>Thm</u> (MacLane, 1963)

> For any two parallel morphisms $M, M' : T \to T'$
> $\forall$ monoidal cat. $\mathcal{C}$, $\forall f : A \to Ob(\mathcal{C})$, we have
>
> $$[\![M]\!] = [\![M']\!]$$

We define a syntax of basic contexts

$$C ::= [-] \mid id \otimes C \mid C \otimes id$$

$$\begin{array}{ccc}
(T_0 \otimes ((T_1 \otimes T_2) \otimes T_3)) \otimes T_4 & \xrightarrow{id \otimes \alpha \otimes id} & (T_0 \otimes (T_1 \otimes (T_2 \otimes T_3))) \otimes T_4 \\
& \big\Downarrow & \\
C[(T_1 \otimes T_2) \otimes T_3] & \xrightarrow[C[\alpha]]{} & C[T_1 \otimes (T_2 \otimes T_3)]
\end{array}$$

<u>Lemma</u>: Every morphism term in Free(A) is equal to a formal composite of canonical isos in context.

__Proof__: By functoriality, every morphism reads

$$T = T_0 \xrightarrow{c_1[K_1]} T_1 \to \cdots \to T_{n-1} \xrightarrow{c_n[K_n]} T_n = T' \ ,$$

where $K_i \in \{\alpha, \alpha^{-1}\}$. $\square$

We are reduced to compare parallel paths $T \to T'$ of canonical isos in context.

__Def__: $M, M' : T \to T'$ are

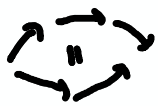
- elementary homotopic if they differ only on
$$\begin{tikzcd} & \cdot \arrow[dr] & \\ \cdot \arrow[ur] \arrow[dr] & & \cdot \\ & \cdot \arrow[ur] & \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} & \cdot \arrow[r] & \cdot \arrow[dr] & \\ \cdot \arrow[ur] \arrow[dr] & & & \cdot \\ & \cdot \arrow[rru] & & \end{tikzcd}$$

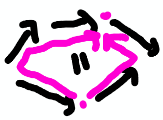
- homotopic if they are related by a finite number of elementary homotopies.

$$\boxed{\text{__Thm__: Any two parallel paths in } \mathrm{Free}(A) \text{ are homotopic.}}$$

Corollary: Maclane's coherence thm

proof: we need two lemmas

1) if  in all empty contexts, so is it in any context.

2) if , any pair of paths commute on it.

From Thm we get a sequence

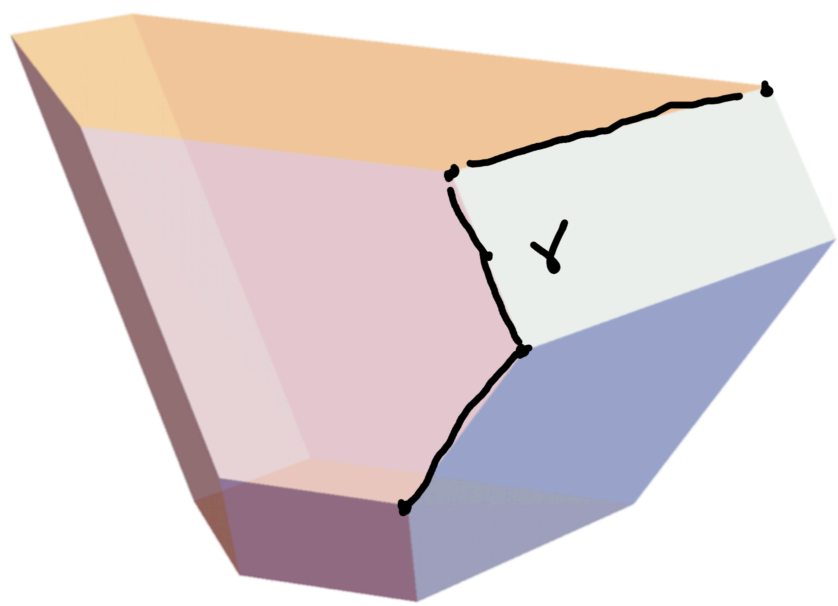
$$M = M_0 \sim M_1 \sim \cdots \sim M_n = M'$$

and using the lemmas we get $[\![M_i]\!] = [\![M_{i+1}]\!]$ $\square$

But how do we prove the Thm ?

$\boxed{2}$ A polytopal coherence theorem

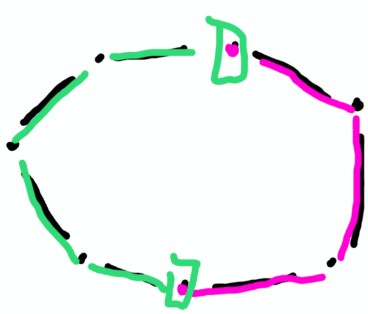
Let $P \subset \mathbb{R}^n$ be a polytope.



<u>Def</u>: A path on P is the cellular image of a continuous, cellular, injective map

$\gamma : [0,1] \longrightarrow P$.

On a 2-dim polytope, $\exists!$ $\gamma^{op} \neq \gamma$ parallel to γ.

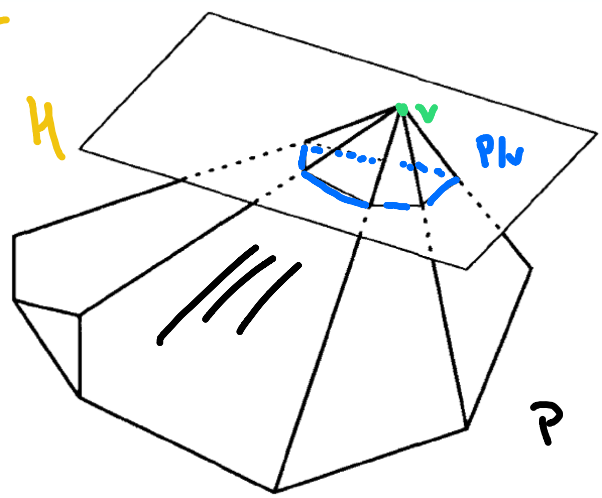


Def: Two paths on P are

- elementary homotopic if they differ only on a 2-face of P
- homotopic if related by elementary homotopies.

Construction: the vertex figure

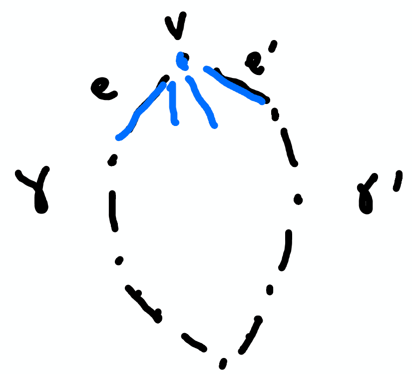
$P/_v$ is the intersection of P with an hyperplane, close to a vertex v



Prop: $P/_v$ is a $(\dim P - 1)$-dim polytope, whose $(k-1)$-faces are in bijection with the k-faces of P which contain v.

Thm: Any two parallel paths on a polytope are homotopic.

proof:



$m :=$ max length of path parallel to γ in P

$n :=$ max length of path between e and e' in $P/_v$.

We proceed by lexicographic induction on (m, n)

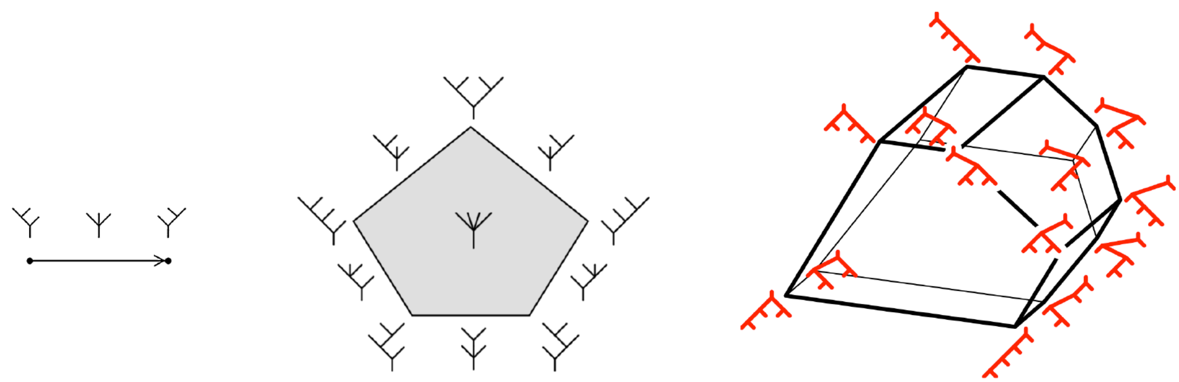
- If $e = e'$, apply induction to $\gamma - e$, $\gamma' - e'$
- If $e \neq e'$, consider F family of 2-faces of P associated to path of minimal length in $P/_v$.

  Write $\gamma = \gamma_1 \gamma_2$, where $e \subset \gamma_1 \subset F$

  Apply induction to $\gamma_1^{op} \gamma_2$ $\square$

Now, MacLane's thm follows from

- syntax $\longleftrightarrow$ 1-skeleton of associahedra



- 2-faces of associahedra are either squares or pentagons.

Remarks :

1) Symmetric monoidal cat. $\longrightarrow$ permutoassociahedron

2) unital " ? $\longrightarrow$ no polytope!

3) work in progress : generalize argument to CW balls

Applications

- Monoidal functors
- Categorified (cyclic) operads
- Permutads, Dioperads, Properads...

[3] MacLane's original proof

Def: A rewriting system is a set A with a binary relation $\rightarrow$.

Denote $\xrightarrow{*}$ the reflexive-transitive closure of $\rightarrow$

A is confluent if $\forall\, a, a_1, a_2,$

$a \xrightarrow{*} a_1,\ a \xrightarrow{*} a_2,\ \exists\, a',\ a_1 \xrightarrow{*} a',\ a_2 \xrightarrow{*} a'.$

Original proof of MacLane:

Our syntax defines a rewriting system

A = object terms

$\rightarrow$ = basic morphism terms, with only α and not α^{-1}

I) This rewriting system is confluent (use Newman's lemma)

$$\begin{tikzcd} & F \arrow[dl] \arrow[dr] & \\ \cdot \arrow[dr, squiggly] \arrow[ddr, bend right] & & \cdot \arrow[dl, squiggly] \arrow[ddl, bend left] \\ & G \arrow[d] & \\ & H & \end{tikzcd}$$

either pentagon or square

II) General coherence follows from directed one.

$$\begin{tikzcd} F \arrow[r] \arrow[dr] & F_1 \arrow[d] & \cdots \arrow[l] \arrow[r] & G \arrow[dll] \\ & H & & \end{tikzcd}$$

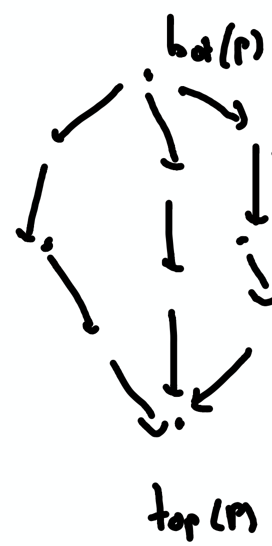
$\square$

Def: A polytope P is oriented by $\vec{v}$ if $\vec{v}$ is not $\perp$ to any edge of P.

$\longrightarrow$ rewriting system on the 1-skeleton of P.

<u>Thm</u>: For any oriented polytope, the induced rewriting system on the 1-skeleton is confluent.

<u>proof</u>:



□

MacLane's coherence follows from

1) There is an orientation on the associahedra whose rewriting system agrees with MacLane

2) Every 2-face of associahedra is either a square or a pentagon.

Remark: Proof of confluence does not use Newman's lemma!

4 Koszul duality

Let $\mathcal{P}(E,R)$ be a quadratic ns operad.

Suppose that there is an ordered basis $\{e_i\}$ of E and a compatible order on planar trees, such that composition in $\mathcal{T}(E)$ is strictly increasing.

Then, relations in R can be written

$$t(e_i, e_j) = \sum_{t'(k,l) < t(i,j)} t'(e_k, e_l)$$

This defines a rewriting system on $\mathcal{T}(E)$.

Thm: If this rewriting system is confluent, then $\mathcal{P}$ is Koszul.

proof:

$$\begin{tikzcd}
\{t(e_i)\} \arrow[dd, mapsto] & T(E) \arrow[d, two heads] \arrow[dr, two heads] & \\
 & T(E)/(R_{\text{lead}}) \arrow[r, two heads] & \mathrm{gr}\, P \cong P \\
\{t(e_i)\} \arrow[rr, mapsto] & & \{\bar{t}(e_i)\}
\end{tikzcd}$$

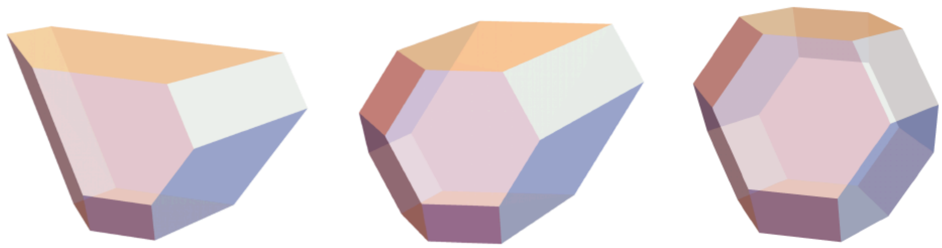
confluence $\Rightarrow$ elements of the spanning set are linearly independent.

$\square$

The rewriting system for the associative operad As is MacLane's! Therefore As is koszul.

Thm: The colored operad encoding ns operads is Koszul.

proof: use the operahedra ...



□

Remark: In this case we can avoid Newman's lemma!

Applications

Operads, Dioperads, Permutads...

Thank you for your attention!