Practice class 5 – Shadows of polytopes

Aspects of linear algebra in Guillaume's research, at the intersection of discrete geometry, homotopy theory and higher category theory.

Q1. In this question, we will consider d linear transformations with domains and codomains as indicated:

$$\mathbb{R}^d \to \mathbb{R}^{d-1} \to \mathbb{R}^{d-2} \to \cdots \to \mathbb{R} \to \{0\},$$

where the jth arrow is given by the linear transformation

$$P^{(j)} : \mathbb{R}^j \longrightarrow \mathbb{R}^{j-1}$$

for $1 \leq j \leq d$ defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{j-1} \\ x_j \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{j-1} \end{bmatrix}$$

For each of the following maps, compute its associated matrix, its mull-space, its column space, and verify the rank-nullity theorem.

(i) the three maps $P^{(4)}$, $P^{(3)}$, and $P^{(2)}$

(ii) the map $P^{(3)} \circ P^{(4)}$

(iii) the map $P^{(2)} \circ P^{(3)} \circ P^{(4)}$.

(iv) the maps $P^{(d)}$, for any natural number d.

(v) the maps $P_j := P^{(j+1)} \circ \cdots \circ P^{(d-1)} \circ P^{(d)}$, for any $1 \leq j \leq d-1$.

Q2. Given n points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in $\mathbb{R}^d$, their *convex hull* is the set

$$\mathrm{Conv}(\mathbf{x}_1, \ldots, \mathbf{x}_n) := \{\alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n \mid \forall i\ , \alpha_i \geq 0 \text{ and } \alpha_1 + \cdots + \alpha_n = 1\}.$$

A *polytope* is the convex hull of a finite set of points in $\mathbb{R}^d$ for some d. *Polygons* (a triangle, a square, a pentagon,...) and *polyhedra* (a simplex, a cube, an octahedron,...) are examples of polytopes.

(a) Draw the *standard* 3*-simplex*

$$\Delta^3 := \mathrm{Conv}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \subset \mathbb{R}^3.$$

(b) Compute the projections $P^{(3)}(\Delta^3) \subset \mathbb{R}^2$ and $(P^{(2)} \circ P^{(3)})(\Delta^3) \subset \mathbb{R}$, and draw the resulting polytopes.

The d-dimensional *cyclic polytope* on n vertices is the convex hull $C(n,d)$ of the n points

$$\mathbf{x}_j := \begin{bmatrix} (j-1) \\ (j-1)^2 \\ (j-1)^3 \\ \vdots \\ (j-1)^d \end{bmatrix} \in \mathbb{R}^d$$

for $1 \leq j \leq n$.

(c) Draw the cyclic polytopes $C(3,2)$ and $C(4,2)$. *Hint: stretch the x axis.*

(d) Compute the projections $P^{(3)}(C(3,3))$ and $P^{(3]}(C(4,3))$, and draw the resulting polytopes. *Hint: projecting respects linear combinations.*

(e) What is going on here? Make a conjecture.

(f) Prove that conjecture.

(g) [Challenge] Draw the cyclic polytopes $C(3,3), C(4,3)$.

Q3. A *hyperplane* H in $\mathbb{R}^d$ is the set of solutions to a single linear equation.

(a) For $d = 1,2,3$, what are hyperplanes geometrically?

Let $P := \mathrm{Conv}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \subset \mathbb{R}^d$ be a d-dimensional polytope, and consider the intersection of P with an hyperplane H. Either we have

(1) $H \cap P = \varnothing$,

(2) H cuts P into 2 parts,

(3) $H \cap P \neq \varnothing$, and all the points on $P \setminus H$ lie on the same side of H.

The intersection $H \cap P$ of P with an hyperplane H is called a *facet* of P if

- $H \cap P = \mathrm{Conv}(A)$ is the convex hull of a subset $A \subset \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$,
- $H \cap P$ satisfies condition (3) above, and
- the dimension of $H \cap P$ is one less than that of P.

Consider again the standard 3-simplex Δ^3 from Question 1. Its facets are all triangles.

(b) How many facets does Δ^3 have?

(c) Is the following convex hull a facet of Δ^3? Explain your answer.

$$\mathrm{Conv}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \subset \mathbb{R}^3$$

In our class, we have not formally defined dimension yet. However, in the examples of the standard simplex and of cyclic polytopes $C(n,d)$, the facets are exactly the convex hulls of d-element subsets satisfying (3) above.

More precisely, a *facet* of a cyclic polytope $C(n,d)$ is the convex hull $\mathrm{Conv}(\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_d})$ for a subset $I := \{i_1, \ldots, i_d\} \subset \{1, \ldots, n\}$ such that all points $\mathbf{x}_j$ for $j \notin I$ lie on the same side of the hyperplane

$$H(I) := \{\alpha_1 \mathbf{x}_{i_1} + \ldots \alpha_d \mathbf{x}_{i_d} \mid \forall j\ , \alpha_j \in \mathbb{R} \text{ and } \alpha_1 + \cdots \alpha_d = 1\}.$$

A facet of a d-dimensional polytope $C(n,d)$ always has dimension $d-1$.

(d) Draw $C(4,3)$, $C(4,2)$ and $C(4,1)$.

(e) Compute the hyperplanes $H(I)$ for the facets I of each of these three cyclic polytopes. You should obtain 4 equations of the form $ax+by+cz=d$ for $C(4,3)$, 4 equations of the form $ax+by=c$ for $C(4,2)$, and 2 equations of the form $ax=b$ for $C(4,1)$.

A facet I of $C(n,d)$ is an *lower facet* (resp. *upper facet*) if all the points $\mathbf{x}_j$ for $j \notin I$ are above (resp. below) the hyperplane $H(I)$, with respect to the last (dth) coordinate. Let us denote by $\mathcal{U}(C(n,d))$ (resp. $\mathcal{L}(C(n,d))$) the set of upper (resp. lower) facets of $C(n,d)$.

(f) What are the upper and lower facets of $C(4,3)$, $C(4,2)$ and $C(4,1)$? Draw them in different colors.

(g) What do you observe?

(h) Draw $P^{(3)}(\mathcal{U}(C(4,3)))$ and $P^{(3)}(\mathcal{L}(C(4,3)))$, as well as $P^{(2)}(\mathcal{U}(C(3,2)))$ and $P^{(2)}(\mathcal{L}(C(3,2)))$.

(i) What do you observe?

Q4. [Challenge] We will now be interested in the *cyclic simplex* $C(d+1,d)$. Any subset $I \subset \{1,\ldots,d+1\}$ of cardinality $|I| = k+1$ defines, via the convex hull, a k-dimensional face $\mathrm{Conv}(\{x_i\}_{i\in I})$ of $C(d+1,d)$. We define the *k-source* (resp. *k-target*) of $C(d+1,d)$ to be the set of k-faces F of $C(d+1,d)$ such that $P_{k+1}(F)$ belongs to $\mathcal{L}(P_{k+1}(C(d+1,d)))$ (resp. $\mathcal{U}(P_{k+1}(C(d+1,d)))$).

(i) Draw the k-source and k-target of $C(4,3)$, for all $k \geq 0$.

We say that a sequence $F_1,\ldots,F_\ell$ of faces of $C(d+1,d)$ is *k-admissible* if the k-target of F_i and the k-source of F_{i+1} have a k-face in common, for all $1 \leq i \leq \ell-1$. We say that $C(d+1,d)$ is *loop-free* if it does not admit any infinite k-admissible sequence, for any $k \geq 0$.

(ii) Show that $C(4,3)$ is loop-free. *Hint:* do a proof *by exhaustion*.

Q5. [Challenge] A *tiling* of $\mathbb{R}^2$ a family $\mathcal{F}$ of polygons such that

(a) $\bigcup_{F\in\mathcal{F}} F = \mathbb{R}^2$, and

(b) whenever two polygons $F, G \in \mathcal{F}$ intersect, their intersection $F \cap G$ is an edge of both.

Giving such a tiling, suppose moreover that for all polygons $F \in \mathcal{F}$, no edge of F is perpendicular to $\begin{bmatrix}1\\0\end{bmatrix}$. For each polygon $F \in \mathcal{F}$, the k-source and k-target of F for $k = 0,1$ are defined in the same fashion as we did above for $C(n,2)$. Show that $\mathcal{F}$ is loop-free.

If you want to know more about the connections with contemporary research topics, you can look at the following research articles/books (there are many pictures!):

- R.Street, *The algebra of oriented simplices.*
- M. Kapranov & V. Voevodsky, *Pasting schemes and higher Bruhat orders.*
- T. Dyckerhoff & M. Kapranov, *Higher Segal Spaces.*