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École Doctorale Sciences, Technologies, Santé Galilée

Approximations cellulaires d'applications diagonales de polytopes opéradiques

THÈSE DE DOCTORAT
présentée par

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pour l'obtention du grade de
DOCTEUR EN MATHÉMATIQUES

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L'homme peut aujourd'hui accéder à des événements faits de lumière réelle comme jamais auparavant avec, pour l'instant, des lasers, des flashs électroniques, des projecteurs et l'informatique [...] Du coup, on comprend qu'un art nouveau de la lumière qui ne soit ni peinture, ni fresque, ni théâtre, ni ballet, ni opéra, est là sur le pas de notre porte. Un art par définition hors de l'homme, même si comme dans le cas des Polytopes de Persépolis ou de Mycènes, des enfants ou des chèvres porteurs de torches électriques dessinent dans les champs ou sur la montagne des tracés lumineux qui se confondent la nuit avec les constellations célestes. Un art comme la musique, en soi, sans référence anthropomorphique ou réaliste. C'est cela le sens des aventures polytopiennes (des Polytopes de Montréal (1967), de Persépolis (1971), de Cluny (1972), de Mycènes (1978), du Diatope du Centre Georges Pompidou (1978)).

Iannis Xenakis
Les Polytopes

Exercice 4.2.3. Let $\mathcal{O}$ be an operad in a symmetric monoidal category C and $F : C \to D$ a symmetric monoidal functor. Show that [...] $F\mathcal{O}$ [is an operad] in the symmetric monoidal category D. This fact is of an extreme importance in applications -starting with a "geometric" operad in the category, say, of topological spaces, and applying the chain or homology functor one arrives to an operad in the category of vector spaces. This particular property of operads is another manifestation of the amazing unity of mathematics.

Sergei Merkulov
Grothendieck–Teichmüller group, operads and graph complexes : a survey

Résumé

Le but premier de cette thèse, qui se situe à la confluence du calcul opéradique et de la géométrie discrète, est de développer une théorie générale des approximations cellulaires de la diagonale d'une famille de polytopes. L'application de cette théorie aux opéraèdres, encodant la notion d'opérade à homotopie près, et aux multiplièdres, encodant la notion de morphisme infini entre algèbres associatives à homotopie près, permet d'obtenir des modèles topologiques pour ces deux notions, de même que des formules universelles explicites pour leurs produits tensoriels. Les calculs effectués s'avèrent valides pour l'ensemble des permutoèdres généralisés, une classe plus vaste de polytopes comprenant plusieurs autres familles opéradiques.

Mots-clés

Polytopes, opérades, approximation de la diagonale, arrangements d'hyperplans, polytopes de fibre, associaèdres, permutoèdres, graphes-associaèdres, permutoèdres généralisés, multiplièdres, catégories A-infini.

Abstract

The main goal of this thesis, which is situated at the confluence of operadic calculus and discrete geometry, is to develop a general theory of cellular approximations of the diagonal for any family of polytopes. The application of this theory to the operahedra, encoding the notion of homotopy operad, and to the multiplihedra, encoding the notion of infinity-morphism between homotopy associative algebras, provides topological models for these two notions, as well as explicit universal formulas for their tensor products. The calculations are in fact valid for all generalized permutahedra, a larger class of polytopes including several other operadic families.

Keywords

Polytopes, operads, approximation of the diagonal, hyperplane arrangements, fiber polytopes, associahedra, permutahedra, graph-associahedra, generalized permutahedra, multiplihedra, A-infinity categories.

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Cette thèse se situe au carrefour de la topologie algébrique et de la géométrie discrète, et je dois beaucoup à Arnau Padrol et Vincent Pilaud qui m'ont grandement instruit dans ce dernier domaine. J'aimerais aussi remercier Thibaut Mazuir, co-auteur du quatrième chapitre de cette thèse, pour nos échanges aussi passionnants que réguliers.

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Chapitre 1

Introduction

1.1 Contexte

Le but de la topologie algébrique, née au début du XXième siècle grâce aux travaux de Henri Poincaré (1854-1912), est l'étude des espaces topologiques à l'aide de l'algèbre. À chaque espace, on associe une famille de groupes dont la structure reflète la "forme" de l'espace considéré. Ici, la notion de "forme" est à entendre en un sens très général (ou très particulier, selon le point de vue) : pour un topologue, la déformation continue d'un espace n'altère pas sa forme. Ainsi, une tasse et un beignet sont considérés par lui comme équivalents[1], car si on les imagine faits en pâte à modeler, il est possible de déformer l'un en l'autre sans couper ou recoller les morceaux. Le rêve de Poincaré était de décrire fidèlement un espace à travers ses groupes, au point de pouvoir le "reconstruire" à partir de ceux-ci. L'aboutissement de ce projet, s'incarnant dans les récents résultats de Michael A. Mandell [Man01, Man06], a nécessité une transformation profonde (ou plutôt, une généralisation de grande envergure) de la notion de groupe au sens où l'entendait Poincaré.

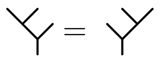
En effet, le dernier siècle a vu l'introduction en algèbre de la notion de déformation continue présente au niveau des espaces (la notion d'homotopie), ce qui a donné naissance à l'*algèbre homotopique*. Cette "algèbre supérieure", comme elle est souvent appelée ces jours-ci, a précisément pour but d'étudier les structures algébriques venant de la topologie. Il s'agit de structures où les relations comme l'associativité et la commutativité sont remplacées par des notions plus souples d'associativité et de commutativité "à homotopie près". Le formalisme des opérades et la théorie de la dualité de Koszul[2] forment un cadre conceptuel permettant de définir, de manipuler et d'étudier ces structures supérieures.

Un exemple fondamental est la notion d'algèbre associative à homotopie près, qui est encodée par une opérade notée A_∞. Rappelons qu'une algèbre associative

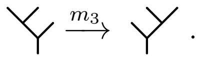
[1] Un mathématicien les dira "homotopes".

[2] Avec ses cygnes.

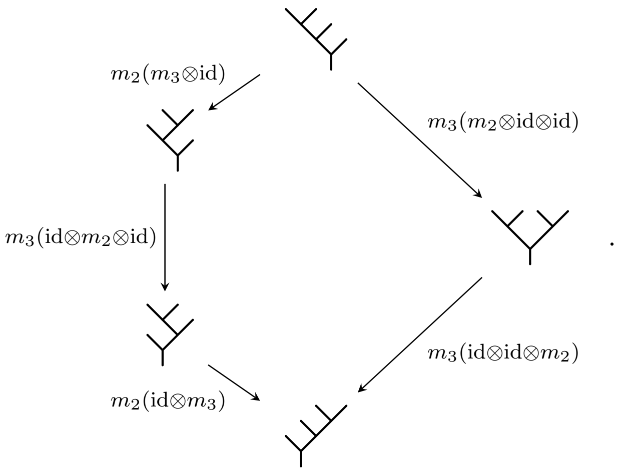
classique est un complexe de chaînes (A, m_1) muni d'une multiplication $m_2 : A \otimes A \to A$ qui vérifie la relation d'associativité, c'est-à-dire que pour tous $a, b, c \in A$, on a $m_2(m_2(a, b), c) = m_2(a, m_2(b, c))$. Une manière commode de représenter cette relation est d'utiliser des arbres, en écrivant



Dans une algèbre associative à homotopie près, ou algèbre A_∞, cette relation est remplacée par une homotopie au sens des complexes de chaînes, c'est-à-dire une application $m_3 : A^{\otimes 3} \to A$ telle que $\partial(m_3) = m_2(m_2 \otimes \mathrm{id}) - m_2(\mathrm{id} \otimes m_2)$. On peut aussi représenter cette homotopie à l'aide des arbres, via le diagramme



Si l'on regarde à présent un mot de quatre lettres, on constate que ses cinq parenthésages possibles, plutôt que d'être égaux, sont reliés par cinq homotopies :



Dans une algèbre A_∞, on déclare ces différentes homotopies équivalentes entre elles à l'aide d'une homotopie supérieure $m_4 : A^{\otimes 4} \to A$, et on poursuit ainsi de manière récursive sur les mots de cinq lettres, puis six lettres, et ainsi de suite. On obtient une tour infinie d'homotopies résolvant l'associativité "stricte" de manière cohérente.

Définition 1.1. *Une* algèbre A_∞ *est un espace vectoriel gradué A muni d'applications $m_n : A^{\otimes n} \to A, n \geq 1$ de degré $|m_n| = n - 2$, vérifiant les relations*

$$\partial(m_n) = -\sum_{p+q+r=n} (-1)^{p+qr} m_{p+1+r}(\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r}) . \qquad (1.1)$$

Notons que cette notion d'algèbre généralise strictement celle d'algèbre associative : une algèbre associative n'est rien d'autre qu'une algèbre A_∞ où $m_n = 0$ pour tout $n \geq 3$. Le produit tensoriel $A \otimes B$ de deux algèbres associatives peut être muni d'une structure d'algèbre associative en prenant

$$\begin{aligned} m_1^{A\otimes B} &:= m_1^A \otimes \mathrm{id} + \mathrm{id} \otimes m_1^B && (1.2) \\ m_2^{A\otimes B} &:= m_2^A \otimes m_2^B \ . && (1.3) \end{aligned}$$

On peut maintenant se demander si cette structure se généralise au cas des algèbres A_∞.

Problème. *Comment munir le produit tensoriel de deux algèbres* A_∞ *d'une structure d'algèbre* A_∞ *qui généralise le produit tensoriel des algèbres associatives ?*

En d'autres mots, on cherche à construire à partir des opérations m_n^A et m_n^B des nouvelles opérations $m_n^{A\otimes B}$ étendant (1.2) et (1.3) et satisfaisant les relations (1.1), sachant que les m_n^A et m_n^B les satisfont. Plus que cela, on est intéressé par une solution *universelle* à ce problème, c'est-à-dire par une formule qui s'applique à toute paire d'algèbres A_∞. Le langage des opérades[3] permet de reformuler ce problème de manière conceptuelle.

Définition 1.2. *Une* opérade *(non-symétrique) est une famille de complexes de chaînes* $\{\mathcal{P}(n)\}_{n\geq 1}$ *munie d'applications de chaînes*

$$\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m-1+n) \ , \ 1 \leq i \leq m$$

vérifiant les relations

$$\begin{aligned} (\lambda \circ_i \mu) \circ_{i-1+j} \nu &= \lambda \circ_i (\mu \circ_j \nu) \ , 1 \leq i \leq l, 1 \leq j \leq m && (1.4) \\ (\lambda \circ_i \mu) \circ_{k-1+m} \nu &= (-1)^{|\mu||\nu|} (\lambda \circ_k \nu) \circ_i \mu \ , 1 \leq i < k \leq l && (1.5) \end{aligned}$$

pour tous $\lambda \in \mathcal{P}(l), \mu \in \mathcal{P}(m), \nu \in \mathcal{P}(n)$ *de même que d'un élément* $\mathrm{id} \in \mathcal{P}(1)$ *vérifiant* $\mathrm{id} \circ_1 \nu = \nu$ *et* $\mu \circ_i \mathrm{id} = \mu$ *pour tous* μ, ν.

Définition 1.3. L'opérade A_∞ *est l'opérade quasi-libre sur un ensemble de générateurs* $\{\mu_n\}_{n\geq 2}$ *de degrés* $|\mu_n| = n-2$ *dont la différentielle est définie sur les générateurs par*

$$\partial(\mu_n) := - \sum_{p+q+r=n} (-1)^{p+qr} \mu_{p+1+r} (\mathrm{id}^{\otimes p} \otimes \mu_q \otimes \mathrm{id}^{\otimes r}) \ .$$

Proposition 1.4. *La donnée d'une structure universelle d'algèbre* A_∞ *sur le produit tensoriel de deux algèbres* A_∞ *est équivalente à la donnée d'un morphisme d'opérades*

$$\mathrm{A}_\infty \to \mathrm{A}_\infty \otimes \mathrm{A}_\infty \ ,$$

c'est-à-dire à une diagonale pour l'opérade A_∞.

[3]Qui n'a rien à voir avec celui de l'opéra ; le terme "opérade" est en effet un mot-valise formé de la fusion des mots "opérations" et "monade", ce qui suggère plutôt un lien avec la philosophie de Leibniz (voir par exemple https://ncatlab.org/nlab/show/monad+%28disambiguation%29).

L'existence d'une telle diagonale est facile à prouver, en se servant du fait que l'opérade A_∞ est une résolution cofibrante de l'opérade associative. Cependant, la mise au jour d'une formule particulière est un problème qui présente une difficulté certaine. Il a été résolu pour la première fois par S. Saneblidze et R. Umble il y a presque vingt ans [SU04]. Une autre formule a été trouvée peu après par M. Markl et S. Schnider [MS06]. Les méthodes des deux travaux sont très différentes, mais des calculs numériques suggèrent que les deux diagonales en fait coïncident ! Il n'existe à ce jour pas de preuve que c'est bien le cas.

Revenons sur la construction d'une algèbre A_∞. En inspectant les figures obtenues à chaque étape, on est amené à se demander si elles représentent des polytopes.

Définition 1.5. *Un* polytope $P \subset \mathbb{R}^n$ *est l'enveloppe convexe d'un nombre fini de points dans* $\mathbb{R}^n$*.*

Définition 1.6. *Une* face *d'un polytope* $P \subset \mathbb{R}^n$ *est un ensemble de la forme* $P \cap \{x \in \mathbb{R}^n \mid \langle d, x\rangle = a\}$ *pour* $d, a \in \mathbb{R}^n$ *tels que* $\langle d, y\rangle \leq a$ *pour tout* $y \in P$*.*

L'ensemble des faces de P forme un treillis pour l'inclusion, noté $\mathcal{L}(P)$. Une face de codimension 1 est appelée *facette*. Une face d'un polytope est elle-même un polytope. La dimension d'un polytope (ou d'une de ses faces) est donnée par la dimension de son enveloppe affine. Les dessins que nous avons tracés précédemment pour illustrer l'associativité "à homotopie près" des algèbres A_∞ représentent bien des polytopes, nommés "associaèdres".

Définition 1.7. *Un* associaèdre *de dimension* $n \geq 0$ *est un polytope dont le treillis des faces est isomorphe au treillis formé par les arbres planaires à* $n + 2$ *feuilles munis de la contraction de leurs arêtes internes.*

En particulier, les sommets d'un associaèdre de dimension n sont en bijection avec les arbres binaires planaires à $n+2$ feuilles. Dans sa thèse où il introduit pour la première fois la notion d'algèbre A_∞ [Sta63], J. Stasheff décrit les associaèdres comme une famille de complexes cellulaires topologiques ; la riche et fascinante histoire de leur réalisation subséquente en tant que polytopes est racontée par C. Ceballos et G. M. Ziegler dans [CZ12]. Pour toute famille de réalisations des associaèdres $\{K_n\}_{n\geq 1}$ on a la relation

$$C_\bullet(K_n) \cong \mathrm{A}_\infty(n) \ , \tag{1.6}$$

où $C_\bullet$ est le foncteur des chaînes cellulaires. Un fait capital concernant ce foncteur est qu'il est symétrique monoidal : il envoie donc toute opérade en polytopes sur une opérade en complexes de chaînes. Sachant que l'on peut munir les complexes de chaînes de (1.6) d'une structure d'opérade (Définition 1.3), trois questions viennent à l'esprit.

1. Est-il possible de munir une famille de réalisations des associaèdres $\{K_n\}$ d'une structure d'opérade topologique et cellulaire, dont l'image par le foncteur des chaînes soit précisément l'opérade A_∞ ?

2. Peut-on définir une famille d'applications cellulaires $\triangle_n : K_n \to K_n \times K_n$ qui forment un morphisme d'opérades topologiques, dont l'image par le foncteur des chaînes serait donc une diagonale pour l'opérade A_∞ ?

3. Peut-on retrouver de cette manière les formules de Saneblidze–Umble [SU04] ou de Markl–Schnider [MS06] ?

1.2 Une nouvelle méthode

Une réponse affirmative à ces trois questions vient d'être apportée par N. Masuda, A. Tonks, H. Thomas et B. Vallette dans [MTTV21]. Les auteurs y introduisent une nouvelle méthode, basée sur la théorie des polytopes de fibres de L. J. Billera et B. Sturmfels [BS92], permettant de définir une approximation cellulaire de la diagonale d'un polytope. En effet, le candidat naturel, la diagonale ensembliste $\triangle_{K_n} : K_n \to K_n \times K_n, x \mapsto (x, x)$ a le défaut rédhibitoire de ne pas être une application cellulaire, c'est-à-dire que son image n'est pas une union de cellules de $K_n \times K_n$. Cette situation n'est pas propre aux associaèdres, elle est commune à tous les polytopes[4].

Définition 1.8. *Pour un polytope P, une* approximation cellulaire de la diagonale *est une application cellulaire $\triangle_P^{cell} : P \to P \times P$ qui soit homotope à la diagonale ensembliste $\triangle_P$ et qui coincide avec elle sur les sommets de P.*

L'idée-force du travail de Masuda–Tonks–Thomas–Vallette est la suivante : pour tout polytope P, chaque sommet du polytope de fibre $\Sigma(P \times P, P)$ associé à la projection

$$\begin{array}{rccc} \pi & : & P \times P & \to & P \\ & & (x, y) & \mapsto & \frac{1}{2}(x+y) \end{array} \tag{1.7}$$

définit une approximation cellulaire de la diagonale de P.

Définition 1.9. *Le* polytope des diagonales *de P est le polytope de fibre $\Sigma P := \Sigma(P \times P, P)$ associé à la projection (1.7).*

Le polytope des diagonales ΣP a la même dimension que P, et sous un choix d'isomorphisme on peut voir les deux polytopes dans le même espace. Un choix de sommet dans ΣP, et donc un choix d'approximation cellulaire de la diagonale de P, revient à un choix de vecteur $\vec{v}$ à l'intérieur d'un cône normal d'un sommet de ΣP.

Définition 1.10. *Le* cône normal $\mathcal{N}_P(F)$ *d'une face $F \in \mathcal{L}(P)$, est défini par*

$$\mathcal{N}_P(F) := \left\{ c \in (\mathbb{R}^n)^* \;\middle|\; \forall x \in F \ , \ cx = \max_{y \in P} cy \right\} .$$

[4] À l'exception du point.

Définition 1.11. L'éventail normal $\mathcal{N}_P$ *de* P *est l'ensemble de tous les cônes normaux des faces de* P*, c'est-à-dire que* $\mathcal{N}_P := \{\mathcal{N}_P(F) \mid F \in \mathcal{L}(P) \setminus \emptyset\}$.

Sous l'identification canonique de $\mathbb{R}^n$ avec son dual, on peut voir ce vecteur $\vec{v}$ dans le même espace que P. Aussi, Masuda–Tonks–Thomas–Vallette donnent une condition suffisante sur $\vec{v}$ pour qu'il soit dans le cône normal d'un sommet de ΣP.

Définition 1.12. *Un vecteur* $\vec{v}$ oriente P *s'il n'est perpendiculaire à aucune arête de* P*. Dans ce cas, un unique sommet* $\operatorname{bot} P$ *(resp.* $\operatorname{top} P$*) minimise (resp. maximise) le produit scalaire avec* $\vec{v}$*.*

Définition 1.13. *Un polytope* P *est* positivement orienté *par un vecteur* $\vec{v}$ *si ce dernier oriente* $P \cap (2z - P)$*, pour tout* $z \in P$*.*

Proposition 1.14 ([MTTV21, Proposition 5])**.** *Si* $(P, \vec{v})$ *est positivement orienté, alors* $\vec{v}$ *définit une approximation cellulaire de la diagonale de* P*, donnée explicitement par la formule*

$$\begin{array}{rcccl} \triangle_{(P,\vec{v})} & : & P & \to & P \times P \\ & & z & \mapsto & \big(\operatorname{bot}(P \cap \rho_z P), \operatorname{top}(P \cap \rho_z P)\big) \ , \end{array}$$

où $\rho_z P := 2z - P$ *dénote la réflexion de* P *par rapport au point* z*.*

Dans [MTTV21], les auteurs choisissent d'étudier les réalisations de Loday des associaèdres. Les sommets de ces réalisations, introduites par J.-L. Loday [Lod04], sont données par un algorithme particulièrement élégant. Soit t un arbre binaire planaire à n feuilles, dont on a numéroté les sommets internes de 1 à $n-1$ en allant de gauche à droite. On le voit comme un graphe orienté depuis les feuilles jusqu'à la racine. Chaque sommet interne i définit deux sous-arbres de t, donnés par l'ensemble des prédécesseurs de l'une ou l'autre des deux arêtes entrantes à i. On définit l'entier naturel $M(t)_i$ comme le produit du nombre de feuilles contenues dans chacun de ces deux sous-arbres. On obtient un point $M(t) := (M(t)_1, \ldots, M(t)_{n-1}) \in \mathbb{R}^{n-1}$.

Définition 1.15. *La* réalisation de Loday K_n *de l'associaèdre de dimension* $n-2$ *est obtenue en prenant l'enveloppe convexe des points* $M(t)$*, pour tous les arbres binaires planaires* t *à* n *feuilles.*

Après avoir fait un choix de vecteur d'orientation $\vec{v}$ pour chaque associaèdre de Loday, les auteurs de [MTTV21] munissent cette famille de polytopes d'une structure d'opérade topologique et cellulaire, qui commute avec les diagonales $\triangle_{(\mathrm{K}_n,\vec{v})}$. Cette étape mérite notre attention : la structure d'opérade obtenue est *unique* [MTTV21, Proposition 7] et sa nature "fractale" l'éloigne irrémédiablement de la catégorie des polytopes munis des applications affines. Cette opérade en polytopes vit en effet dans une nouvelle catégorie dont les morphismes sont beaucoup plus souples que les

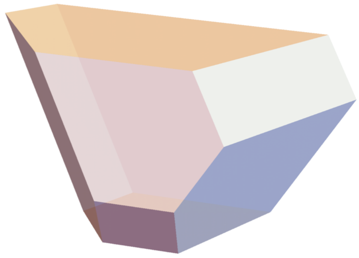


FIGURE 1.1 : La réalisation de Loday de l'associaèdre de dimension trois.

applications affines [MTTV21, Définition 7]. En prenant les chaînes cellulaires, on obtient alors une diagonale pour l'opérade A_∞.

Le dernier résultat de [MTTV21] est la démonstration du fait que l'on retrouve alors la "formule magique" de Markl–Schnider [MS06, Formule (19)]. C'est la simplicité insoupçonnée de cette formule qui avait poussé J.-L. Loday à l'affubler du qualificatif "magique", repris par [MTTV21].

Proposition 1.16 ([MTTV21, Théorème 2])**.** *L'image cellulaire de la diagonale* $\triangle_n : \mathrm{K}_n \to \mathrm{K}_n \times \mathrm{K}_n$ *est donnée par les paires de faces* $F, G \in \mathcal{L}(P)$ *vérifiant* $\operatorname{top} F \leq \operatorname{bot} G$*, c'est-à-dire que*

$$\operatorname{Im} \triangle_n = \bigcup_{\operatorname{top} F \leq \operatorname{bot} G} F \times G \ . \tag{1.8}$$

En mots, la diagonale de l'associaèdre est donnée par toutes les paires de faces comparables (en un certain sens) pour l'ordre de Tamari. Remarquons que l'ensemble des paires de sommets dans l'image de $\triangle_n$ décrit l'ensemble des intervalles du treillis de Tamari, dénombrés par F. Chapoton dans [Cha06]. C'est l'ordre partiel sur ces intervalles donné par la formule (1.8) qui a mené ce même auteur à la découverte de nouvelles structures combinatoires sur les chemins de Dyck dans [Cha20]. La lecture de l'introduction de ce dernier article suggère que les liens entre les diagonales de polytopes et la combinatoire n'ont pas encore révélés tous leurs secrets.

La formule (1.8) est propre à l'associaèdre et à d'autres familles de polytopes dont la combinatoire présente un degré de complexité analogue ou inférieur, comme c'est le cas pour les "freehedra" [San09, Pol21], les simplexes et les cubes [MTTV21, Exemple 1], par exemple. On verra que celle-ci n'est plus valide dès que l'on considère des polytopes de plus grande complexité.

1.3 Résultats

Une formule universelle

On est mené à la question de savoir s'il existe une formule générale permettant de décrire l'image cellulaire de la diagonale d'un polytope de manière combinatoire. C'est là la première contribution de cette thèse. Pour l'obtenir, on introduit un nouvel objet conceptuel associé à un polytope : son arrangement d'hyperplans fondamental.

Définition 1.17 (Définition 2.17)**.** L'arrangement d'hyperplans fondamental $\mathcal{H}_P$ *d'un polytope* $P \subset \mathbb{R}^n$ *est l'ensemble des hyperplans perpendiculaires aux arêtes de* $P \cap \rho_z P$*, pour tout* $z \in P$*.*

Cet arrangement, considéré comme un éventail, raffine l'éventail normal de ΣP. Chaque chambre de $\mathcal{H}_P$ définit donc une approximation de la diagonale de P, deux chambres pouvant parfois définir la même diagonale. La description de l'image cellulaire d'une telle diagonale est alors donnée par le théorème suivant.

Théorème 1.18 (Théorème 2.25)**.** *Soit* P *un polytope positivement orienté par* $\vec{v}$*. Pour chaque* $H \in \mathcal{H}_P$*, on choisit un vecteur normal* $\vec{d}_H$ *tel que* $\langle \vec{d}_H, \vec{v} \rangle > 0$*. Alors, on a*

$$(F, G) \in \operatorname{Im} \triangle_{(P,\vec{v})} \iff \forall H \in \mathcal{H}_P,\ \exists i,\ \langle \vec{F}_i, \vec{d}_H \rangle < 0 \text{ ou } \exists j,\ \langle \vec{G}_j, \vec{d}_H \rangle > 0 \ .$$

Ici, les $\vec{F}_i$ et les $\vec{G}_j$ sont des vecteurs qui définissent les cônes normaux de F et G (voir la Définition 2.24). Cette formule ne dépend que de l'éventail normal du polytope que l'on considère. Il s'avère qu'en plus d'être valide pour P, elle est également valide pour tout polytope Q dont l'éventail normal est raffiné par celui de P.

Proposition 1.19 (Section 2.2.6)**.** *Soit* P *un polytope positivement orienté par* $\vec{v}$*, et soit* Q *un polytope dont l'éventail normal est raffiné par celui de* P*. Alors, on a*

$$(F, G) \in \operatorname{Im} \triangle_{(Q,\vec{v})} \iff \forall H \in \mathcal{H}_P,\ \exists i,\ \langle \vec{F}_i, \vec{d}_H \rangle < 0 \text{ ou } \exists j,\ \langle \vec{G}_j, \vec{d}_H \rangle > 0 \ .$$

La deuxième contribution de cette thèse est le calcul de l'arrangement d'hyperplans fondamental du permutoèdre, ce qui permet via les deux résultats précédents d'obtenir une description combinatoire explicite d'une approximation cellulaire de la diagonale de tout permutoèdre généralisé.

Définition 1.20. *Le* permutoèdre *de dimension* $n-1$ *dans est l'enveloppe convexe des points* $\sum_{i=1}^{n} i e_{\sigma(i)} \in \mathbb{R}^n$ *pour toutes les permutations* $\sigma \in \mathbb{S}_n$*.*

Le permutoèdre de dimension trois est le polytope tout à droite de la Figure 1.2. Son éventail normal, aussi appelé arrangement de Coxeter ou éventail de tresses, est constitué des hyperplans $x_i = x_j$ pour tous $1 \leq i < j \leq n$. Il est représenté à gauche de la Figure 1.3.

Définition 1.21. *Un* permutoèdre généralisé *est un polytope dont l'éventail normal est raffiné par celui du permutoèdre.*

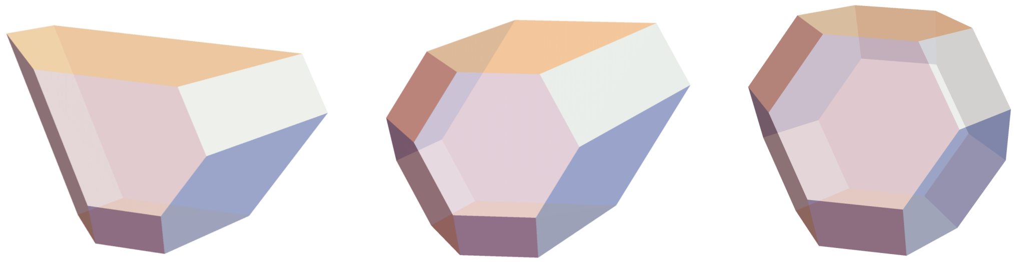


FIGURE 1.2 : Quelques permutoèdres généralisés de dimension trois, faisant partie de la sous-famille des opéraèdres.

Théorème 1.22 (Théorème 3.22)**.** *Pour tout $n \geq 1$, on écrit*

$$D(n) := \{(I, J) \mid I, J \subset \{1, \ldots, n\}, |I| = |J|, I \cap J = \emptyset, \min(I \cup J) \in I\}.$$

L'arrangement d'hyperplans fondamental du permutoèdre de dimension $n-1$ dans $\mathbb{R}^n$ est l'ensemble d'hyperplans

$$\sum_{i \in I} x_i = \sum_{j \in J} x_j \quad \text{pour tous } (I, J) \in D(n) \ .$$

Cet arrangement d'hyperplans raffine l'arrangement de tresses, comme le montre la Figure 1.3. Un choix de chambre dans cet arrangement nous donne une diagonale pour tout permutoèdre généralisé. Les n-ièmes contributions de cette thèse pour $n \geq 3$ sont issues d'un certain choix de diagonales pour deux sous-familles de polytopes "opéradiques" : les opéraèdres et les multiplièdres.

Les facettes cachées des opéraèdres et des multiplièdres*

Comme nous l'avons mentionné plus tôt, les associaèdres encodent la notion d'algèbre A_∞, ou algèbre associative à homotopie près. Les opérades à homotopie près ont été définies par P. Van der Laan dans sa thèse [VdL03] comme généralisations

*À lire en diagonale.

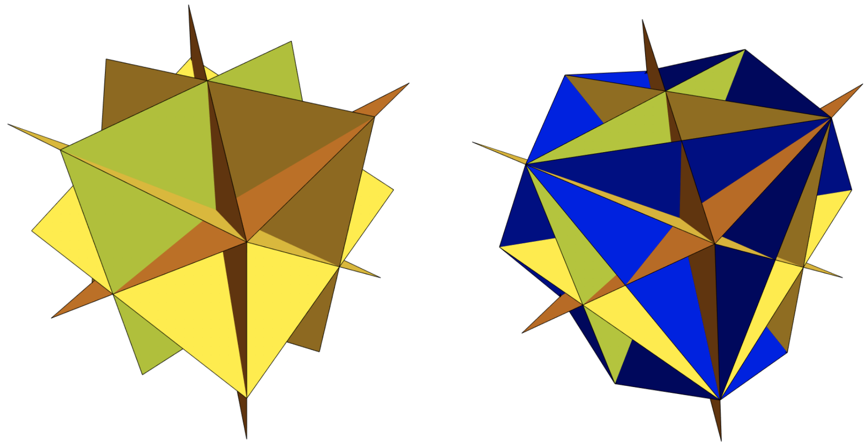


FIGURE 1.3 : L'arrangement de tresses et l'arrangement fondamental du permutoèdre dans $\mathbb{R}^4$, projeté dans $\mathbb{R}^3$.

multi-linéaires des algèbres A_∞. On peut alors se demander s'il existe une généralisation multi-linéaire des associaèdres, encodant la notion d'opérade à homotopie près. Une telle généralisation a effectivement été mise au jour par J. Obradović [Obr19] ; par analogie avec les associaèdres, nous avons décidé de nommer "opéraèdres" cette famille de polytopes. On peut associer à tout arbre planaire un opéraèdre, dont les faces sont en bijection avec le treillis des "nichages", ou "rameaux" de cet arbre (voir la Définition 3.2)[5]. Quelques opéraèdres sont représentés sur la Figure 1.2.

$$\begin{array}{ccc}
\text{Algèbres associatives} & \dashrightarrow & \text{Opérades} \\
\downarrow & & \downarrow \\
\text{Algèbres assoc. à homotopie près} & \dashrightarrow & \text{Opérades à homotopie près} \\
\updownarrow & & \updownarrow \\
\text{Associaèdre} & & \textit{Opéraèdres}
\end{array}$$

Le multiplièdre de dimension $n-1$, quant à lui, est un polytope dont les faces sont en bijection avec les arbres "bicolorés" ou encore "jaugés" à n feuilles (voir la Définition 4.2). Il a, tout comme les associaèdres, été introduit par J. Stasheff en tant que complexe cellulaire topologique, dans le but cette fois de décrire les morphismes infinis entre algèbres A_∞ [Sta70]. Sa réalisation en tant que polytope est récente, due au travaux de S. Forcey [For08a], S. Forcey et S. Devadoss [DF08], F. Ardila et J. Doker [AD13], ou encore F. Chapoton et V. Pilaud [CP22].

[5]Une traduction adéquate des termes "nest" et "nesting" pour un arbre reste à trouver. Une troisième idée serait d'utiliser "boîte" et "emboîtement".

Pour chacune de ces deux familles de polytopes, les opéraèdres au Chapitre 3 et les multiplièdres au Chapitre 4,

1. on choisit une famille de réalisations qui généralisent les réalisations de Loday des associaèdres (Propositions 3.15 et 4.9),

2. on fait un choix cohérent de diagonales pour ces réalisations (Propositions 3.52 et 4.24),

3. on les munit d'une structure opéradique topologique et cellulaire compatible (Théorèmes 3.55 et 4.25), et

4. on obtient un produit tensoriel universel pour les structures algébriques correspondantes, défini par une formule explicite (Propositions 3.64 et 4.43).

1.4 Perspectives

Notons que le choix de diagonales pour ces deux familles de polytopes est contraint, mais n'est pas forcé comme dans le cas des associaèdres : plusieurs choix différents permettent de définir une structure opéradique (voir par exemple la Remarque 3.56 à ce sujet). Il serait intéressant d'étudier ces différents choix et leurs relations.

On montre (c'est la Proposition 2.16) que toute paire de faces (F, G) dans l'image cellulaire d'une diagonale obtenue par la présente méthode vérifie la propriété $\operatorname{top} F \leq \operatorname{bot} G$, mais que cette dernière ne caractérise pas l'image cellulaire en général, comme c'est le cas pour les associaèdres. Ce fait, dont la démonstration résulte d'une analyse de la relation entre le vecteur d'orientation et l'éventail normal de P (Proposition 2.3), simplifie considérablement la démonstration d'une direction du Théorème 2 dans [MTTV21].

Comme il a été mentionné plus haut, le calcul de l'arrangement d'hyperplans fondamental du permutoèdre permet d'envisager des applications directes à tous les permutoèdres généralisés. Cela comprend la sous-famille encodant les opérades modulaires à homotopie près [War21] ainsi que toutes les sous-familles décrivant des structures opéradiques à homotopie près étudiées dans l'article [BMO20]. On peut également envisager d'appliquer la théorie développée ici

- aux 2-associaèdres, qui interviennent en topologie symplectique [Bot19],

- aux "freehedra", qui encodent les représentations à homotopie près d'un groupe algébrique dérivé [ACD11, AC13, Pol20], et

- aux assocoipièdres, qui interviennent en topologie des cordes [DCPR15, PT18, PT19].

Les opéraèdres

Il y a déjà plusieurs exemples importants d'opérades à homotopie près dans la littérature. L'un d'eux est donné par les chaînes singulières des espaces de configurations de points dans le plan [VdL03, Section 5], qui sont quasi-isomorphes aux chaînes singulières sur l'opérade des petits disques. Un autre est donné par l'opérade des cactus normalisés [BCL$^+$], qui intervient dans l'étude des espaces de modules des surfaces de Riemann. Le produit tensoriel défini ici s'applique à ces deux structures.

Du point de vue combinatoire, les opéraèdres font partie de plusieurs familles intéressantes de polytopes. Ils peuvent donc être étudiés selon ces différents points de vue. Ils correspondent

- aux associaèdres de graphes où le graphe sous-jacent est le graphe dual ("line graph") d'un arbre, c'est-à-dire un graphe par blocs sans fourche [Har69, Theorem 8.5],

- aux nestoèdres [FS05] et aux permutoèdres généralisés [Pos09] qui peuvent être obtenus en retirant des équations définissant les facettes du permutoèdre [Pil14],

- à une sous-famille des polytopes d'hypergraphes [DP11, CIO19, Obr19],

- à une sous-famille des associaèdres d'ensembles ordonnés [Gal21].

Les réalisations de Loday de poids standard des opéraèdres ont déjà été définis, de manière différente, par V. Pilaud dans [Pil13], en tant que sous-ensemble d'une famille plus grande de polytopes qui généralisent les réalisations de C. Hohlweg et C. Lange de l'associaèdre [HL07]. On peut se demander si les techniques de [Pil13] peuvent être étendues à tous les associaèdres de graphes par blocs, ce qui fait l'objet d'un travail en cours avec Vincent Pilaud.

Les multiplièdres

Les structures tensorielles que l'on définit dans cette thèse ne forment pas des structures monoidales au sens strict du terme. Cela n'est pas un défaut de notre construction : dans le cas du multiplièdre, on montre qu'il *n'existe pas* de structure tensorielle universelle qui soit compatible à la composition des morphismes infinis (Théorème 4.46). Ce fait ouvre la porte à une myriade de questions intéressantes, que l'on aborde en partie dans la Section 4.4.3. Nous esquissons ensuite plusieurs applications potentielles de notre produit tensoriel à la topologie symplectique dans la Section 4.4.4.

1.5 Contenu de la thèse

Le Chapitre 2 développe la théorie générale des approximations cellulaires, qui est ensuite appliquée aux opéraèdres au Chapitre 3 et aux multiplièdres au Chapitre 4. Les Chapitres 2 et 3 forment ensemble un article intitulé "La diagonale des opéraèdres", qui sera publié dans le volume 405 de la revue *Advances in Mathematics*. Il est également disponible sur arXiv au numéro 2110.14062. Le Chapitre 4 est un travail en commun avec Thibaut Mazuir, et forme un article intitulé "La diagonale des multiplièdres et le produit tensoriel de morphismes A-infinis", qui est maintenant disponible sur arXiv au numéro 2206.05566.

1.6 Notations et conventions

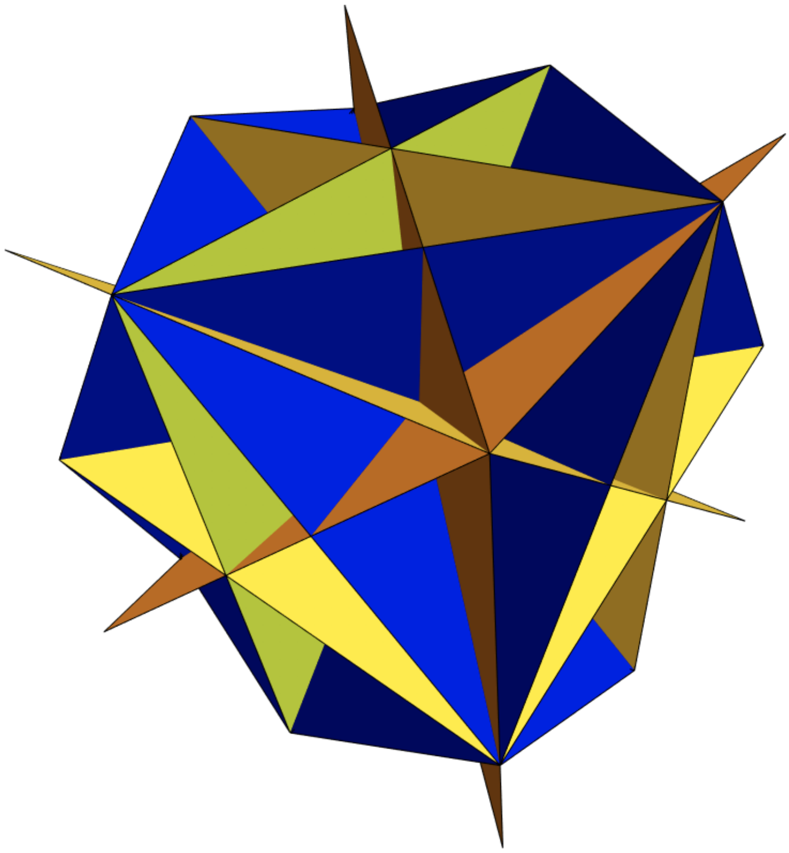
On adopte les conventions du livre de J.-L. Loday et B. Vallette [LV12] pour les opérades et celles du livre de G. M. Ziegler [Zie95] pour les polytopes. Dans la suite on ne considérera que des arbres planaires, on notera $[n]$ l'ensemble $\{1, \ldots n\}$ et on notera $\{e_i\}_{i\in[n]}$ la base canonique de $\mathbb{R}^n$. On considérera l'espace euclidien $\mathbb{R}^n$ muni de la structure euclidienne standard, dont le produit scalaire sera noté $\langle -, - \rangle$.

1.7 Remerciements spécifiques

L'observation à l'origine de la Proposition 3.33 est due à Christian Gaetz, et l'argument de la Proposition 2.16 a été suggéré par Arnau Padrol. La section 4.4.4 doit beaucoup à Lino Amorim et Robert Lipshitz, qui ont accepté de discuter avec générosité de leurs résultats et des applications possibles des nôtres en topologie symplectique.

Chapter 2

General theory



2.1 Introduction

The present work lies at the intersection of the theory of polytopes and the operadic calculus. The starting point is the following observation: for a non-trivial polytope P, the image of the set-theoretic diagonal $\triangle_P : P \to P \times P, x \mapsto (x, x)$ is not a union of faces of $P \times P$. One is led to the problem of finding a cellular approximation to $\triangle_P$, that is finding a cellular map $\triangle_P^{\text{cell}} : P \to P \times P$ which is homotopic to $\triangle_P$ and which agrees with $\triangle_P$ on the vertices of P, see Figure 2.1.

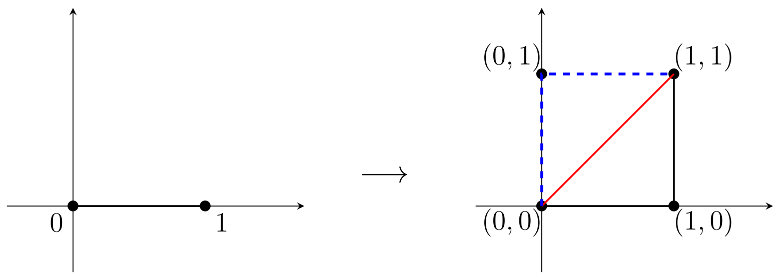


Figure 2.1: The set-theoretic diagonal of the unit interval (in red) is not cellular. One needs to find a cellular approximation (in blue, dashed).

One can always find such an approximation, however, a problem of fundamental importance in algebraic topology is to find coherent cellular approximations of the diagonal for families of polytopes. For instance, the cup product on singular and cubical cohomology comes from coherent cellular approximations of the diagonal of the standard simplices and cubes, the Alexander–Whitney [EZ53, EML54] and Serre [Ser51] maps, respectively.

More recently, families of greater combinatorial complexity have appeared in operad theory. The first seminal example is the family of associahedra. In contrast with the standard simplices and cubes, each face of an associahedron is not itself an associahedron, but a product of lower-dimensional associahedra, which underlies the algebraic structure of an operad. More precisely, the cellular chains on the associahedra are naturally endowed with an operad structure which encodes associative algebras up to homotopy [Sta63]. In light of this fact, finding a family of coherent cellular approximations of the diagonal of the associahedra becomes a very desirable objective, as it defines a functorial tensor product of A_∞-algebras [SU04, MS06]. Such a universal formula has applications in different fields of mathematics, for instance the homology of fibered spaces [Pro86], string field theories [GZ97] and Fukaya categories [Sei08].

N. Masuda, A. Tonks, H. Thomas and B. Vallette introduced in [MTTV21] a method for finding coherent cellular approximations of the diagonal for families of polytopes, using the theory of fiber polytopes of L. J. Billera and B. Sturmfels [BS92]. They applied it to the associahedra and obtained a coherent family of approximations, which led to a topological cellular operad structure on them. This provided a model for topological A_∞-algebras and an explicit functorial formula for their tensor product. Applying the cellular chains functor, it is possible to recover the formula of M. Markl and S. Shnider [MS06], which should coincide with the one of [SU04]. The key feature, which makes this problem highly constrained, is requiring the operadic composition maps to be compatible with the approximation of the diagonal. Such composition maps are in fact unique [MTTV21, Proposition 7], and this uniqueness property is precisely the one allowing for the operad structure [MTTV21, Theorem 1].

The diagonal of the associahedra admits a particularly simple description of its cellular image in terms of the Tamari order, so unexpectedly simple that J.-L. Loday was led to the name "magical formula". However, one cannot expect a similar formula for other families of polytopes, and an explicit combinatorial description for the cellular image of the approximation of the diagonal of an arbitrary polytope is missing in the work of [MTTV21].

The first contribution of the present thesis is to give such a universal formula, which can be applied to any polytope (Theorem 2.25), and which is expressed in terms of a new conceptual object: its *fundamental hyperplane arrangement* (Definition 2.17). In the case of the simplices, the theory developed here allows one to recover conceptually a perturbative formula due to M. Abouzaid [Abo09] for the intersection pairing on cellular chains on a manifold, see Remark 2.5. This suggests deeper connections with combinatorial algebraic topology [RS19, KMM21], higher category theory [KV91, MM20], discrete and continuous Morse theory [For98, FMMS21] and physics [Tho18, Tat20].

2.2 Cellular approximation of the diagonal

In this section, we study the method introduced in [MTTV21] for finding a cellular approximation of the diagonal of a polytope and establish its general properties. We associate to any polytope P its *fundamental hyperplane arrangement* $\mathcal{H}_P$, where each chamber defines an approximation of the diagonal. Two different chambers can define the same approximation, and bringing down the walls between them leads to a new notion of "quasi-positively oriented" polytope.

An approximation of the diagonal of P always exists and it depends only on the normal fan of the polytope. Its image admits a description in terms of the poset structure on the vertices of P induced by the choice of a chamber in $\mathcal{H}_P$. In the case of the associahedra, one recovers the Tamari order. The condition $\mathrm{top}(F) \leq \mathrm{bot}(G)$ in M. Markl and S. Schnider's "magical formula" for the associahedra [MS06] turns out to be present in any approximation of the diagonal, but it is not sufficient to characterize its image in general, as shows the case of the permutahedra treated in the next chapter.

A careful study of the fundamental hyperplane arrangements leads to a universal formula describing combinatorially the cellular image of the approximation of the diagonal for any polytope P. Once one has established the universal formula for P, one has in fact established the formula for any polytope Q whose normal fan coarsens the one of P.

2.2.1 General method

In the following, we adopt notations and conventions of the monograph of G. M. Ziegler [Zie95] on the theory of polytopes. Let $P \subset \mathbb{R}^n$ be a polytope. Except for the case where P is the trivial polytope, the diagonal map

$$\begin{array}{cccc} \triangle_P & : & P & \to & P \times P \\ & & z & \mapsto & (z, z) \end{array}$$

is not cellular, that is, its image is not a union of cells of $P \times P$.

Problem. *Find a* cellular approximation of the diagonal of P, *that is, a cellular map which is homotopic to* $\triangle_P$ *and which coincides with* $\triangle_P$ *on the vertices of* P.

We consider a special case of the fiber polytope construction of L. J. Billera and B. Sturmfels [BS92], see also [Zie95, Chapter 9] for more details. Let $\mathcal{L}(P)$ denote the lattice of faces of P and let $(e_i)_{1 \leq i \leq n+1}$ denote the standard basis of $\mathbb{R}^{n+1}$. For a polytope $P \subset \mathbb{R}^n$ and a vector $\vec{v} \in \mathbb{R}^n$, we consider the projection π and the linear form ϕ defined respectively by

$$\begin{array}{ccccc} \pi & : & P \times P & \to & P \\ & & (x, y) & \mapsto & \frac{1}{2}(x + y) \end{array} \quad \text{and} \quad \begin{array}{ccccc} \phi & : & \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R} \\ & & (x, y) & \mapsto & \langle x - y, \vec{v} \rangle \end{array} .$$

The linear map

$$\begin{array}{ccccc} \pi^\phi & : & P \times P & \to & P \times \mathbb{R} \\ & & (x, y) & \mapsto & (\pi(x, y), \phi(x, y)) \end{array}$$

defines a polytope $P^\phi \coloneqq \mathrm{Im}(\pi^\phi) \subset \mathbb{R}^{n+1}$. Let

$$\mathcal{L}^\downarrow(P^\phi) \coloneqq \{F \in \mathcal{L}(P^\phi) \mid \forall x \in F, \lambda > 0, x - \lambda e_{n+1} \notin P^\phi\} \subset \mathcal{L}(P^\phi)$$

be the family of lower faces of P^ϕ. Then, the set of faces

$$\mathcal{F}^\phi \coloneqq (\pi^\phi)^{-1} \mathcal{L}^\downarrow(P^\phi) \subset \mathcal{L}(P \times P) \cong \mathcal{L}(P) \times \mathcal{L}(P)$$

induces a subdivision $\pi(\mathcal{F}^\phi)$ of P that is called *coherent*. As indicated by the last isomorphism, the faces of $P \times P$ are pairs of faces of P. Depending on the context, we will denote such a pair by $F \times G$ or (F, G).

One always has $\dim(F \times G) \geq \dim(\pi(F \times G))$ for all $F \times G \in \mathcal{F}^\phi$. The coherent subdivision $\pi(\mathcal{F}^\phi)$ is said to be *tight* if $\dim(F \times G) = \dim(\pi(F \times G))$ for all $F \times G \in \mathcal{F}^\phi$.

To any tight coherent subdivision $\pi(\mathcal{F}^\phi)$ of P one can associate the unique section $\triangle_{(P,\vec{v})} : P \to P \times P$ of π which minimizes ϕ in each fiber, see [Zie95, Lemma 9.5].

Proposition 2.1. *Let $P \subset \mathbb{R}^n$ be a polytope. Suppose that $\vec{v} \in \mathbb{R}^n$ induces a tight coherent subdivision of P. Then, the associated section $\triangle_{(P,\vec{v})}$ is a cellular approximation of the diagonal of P.*

Proof. If z is a vertex of P, then the fiber $\pi^{-1}(z)$ is the point (z, z), so $\triangle_{(P,\vec{v})}$ agrees with the set-theoretic diagonal on vertices. An explicit homotopy between the two maps is given by

$$\begin{array}{rcccc} H & : & P \times [0,1] & \longrightarrow & P \times P \\ & & (z,t) & \longmapsto & (1-t)(z,z) + t(x,y) \end{array}$$

where $(x, y) \in \pi^{-1}(z)$ is such that $\langle x - y, \vec{v}\rangle = \min \left\{\phi|_{\pi^{-1}(z)}\right\}$. □

2.2.2 Cellular description of the diagonal

Given a cellular approximation $\triangle_{(P,\vec{v})}$ of the diagonal of a polytope P, one key problem is to describe combinatorially its image. For more clarity, let us first recall some standard notations. Let $P \subset \mathbb{R}^n$ be a polytope. Codimension 1 faces of P are called *facets*. For a face $F \in \mathcal{L}(P)$, the *normal cone* of F is the cone

$$\mathcal{N}_P(F) := \left\{ c \in (\mathbb{R}^n)^* \;\middle|\; F \subseteq \{x \in P \mid cx = \max_{y \in P} cy\} \right\} .$$

The codimension of $\mathcal{N}_P(F)$ is equal to the dimension of F. The *normal fan* of P is the collection of the normal cones $\mathcal{N}_P := \{\mathcal{N}_P(F) \mid F \in \mathcal{L}(P) \setminus \emptyset\}$. This fan is complete, i.e. it is a partition of $(\mathbb{R}^n)^*$. From now on we see $\mathcal{N}_P$ as a subset of $\mathbb{R}^n$ via the canonical identification $(\mathbb{R}^n)^* \cong \mathbb{R}^n$.

Remark 2.2. When $P \subset \mathbb{R}^n$ is full-dimensional, i.e. when $\dim P = n$, one can alternatively define the normal fan of P via the lines generated by a family of normal vectors to the facets of P, called rays. Given a polytope P which is not full-dimensional, one can then consider the restriction to the affine hull of P, and define the normal fan in this space. We have decided not to follow this approach, and consider always the normal fan to be full-dimensional.

For X a subset of $\mathbb{R}^n$, the *cone* of X is defined by $\mathrm{Cone}(X) := \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid \{x_1, \ldots, x_n\} \subseteq X, \lambda_i \geq 0\}$ and its *polar cone* is defined by $X^* := \{y \in \mathbb{R}^n \mid \forall x \in X, \langle x, y\rangle \leq 0\}$. The following result, which will be at the heart of further developments, applies to *any* coherent subdivision of P.

Proposition 2.3. *Let P be a polytope in $\mathbb{R}^n$, let $\vec{v} \in \mathbb{R}^n$ and let $F, G \in \mathcal{L}(P)$ be two faces of P. Then,*

$$(F,G) \in \mathcal{F}^\phi \iff \vec{v} \in \mathrm{Cone}(-\mathcal{N}_P(F) \cup \mathcal{N}_P(G)) \ .$$

Proof. Elaborating on the proof of [MTTV21, Proposition 6], we have that

$$\begin{aligned}
(F,G) \in \mathcal{F}^\phi \quad &\iff \quad \nexists x \in F, y \in G, \lambda > 0 \text{ such that } \left(\tfrac{1}{2}(x+y), \langle x-y, \vec{v}\rangle - \lambda\right) \in P^\phi \\
&\iff \quad \nexists x \in F, y \in G, \vec{w} \in \mathbb{R}^n, \varepsilon > 0 \text{ such that } \langle \vec{v}, \vec{w}\rangle > 0 \text{ and} \\
&\qquad\quad (x - \varepsilon\vec{w}, y + \varepsilon\vec{w}) \in P \times P \\
&\iff \quad \nexists \vec{w} \in \mathbb{R}^n \text{ such that } \langle \vec{v}, \vec{w}\rangle > 0 \text{ and } \vec{w} \in -\mathcal{N}_P(F)^* \cap \mathcal{N}_P(G)^* \\
&\iff \quad \forall \vec{w} \in \mathrm{Cone}(-\mathcal{N}_P(F) \cup \mathcal{N}_P(G))^* \text{ we have } \langle \vec{v}, \vec{w}\rangle \leq 0 \\
&\iff \quad \mathrm{Cone}(-\mathcal{N}_P(F) \cup \mathcal{N}_P(G))^* \subset \mathrm{Cone}(\vec{v})^* \\
&\iff \quad \mathrm{Cone}(\vec{v}) \subset \mathrm{Cone}(-\mathcal{N}_P(F) \cup \mathcal{N}_P(G)) \\
&\iff \quad \vec{v} \in \mathrm{Cone}(-\mathcal{N}_P(F) \cup \mathcal{N}_P(G)) \ ,
\end{aligned}$$

where we used that for X, Y two subsets of $\mathbb{R}^n$, we have $\mathrm{Cone}(X)^* \cap \mathrm{Cone}(Y)^* = \mathrm{Cone}(X \cup Y)^*$ and $\mathrm{Cone}(X)^* \subset \mathrm{Cone}(Y)^* \iff \mathrm{Cone}(Y) \subset \mathrm{Cone}(X)$. □

Corollary 2.4. *For all $\varepsilon > 0$, we have*

$$\begin{aligned}
(F,G) \in \mathcal{F}^\phi \quad &\iff \quad (\mathcal{N}_P(F) + \varepsilon\vec{v}) \cap \mathcal{N}_P(G) \neq \emptyset \\
&\iff \quad \mathcal{N}_P(F) \cap (\mathcal{N}_P(G) - \varepsilon\vec{v}) \neq \emptyset \ .
\end{aligned}$$

Moreover, if P is full-dimensional and if the coherent subdivision $\pi(\mathcal{F}^\phi)$ is tight, then the pairs $(F,G) \in \mathcal{F}^\phi$ which satisfy $\dim F + \dim G = \dim P$ are in bijection with the dimension zero cells of $(\mathcal{N}_P + \varepsilon\vec{v}) \cap \mathcal{N}_P$ or $(\mathcal{N}_P - \varepsilon\vec{v}) \cap \mathcal{N}_P$.

Proof. The first part of the statement follows directly from Proposition 2.3 : for $\varepsilon > 0$, we have by definition of a cone that the inclusion $\mathrm{Cone}(\vec{v}) \subset \mathrm{Cone}(-\mathcal{N}_P(F) \cup \mathcal{N}_P(G))$ holds if and only if $\varepsilon\vec{v} \in \mathrm{Cone}(-\mathcal{N}_P(F) \cup \mathcal{N}_P(G))$. This is equivalent to the existence of $f \in \mathcal{N}_P(F)$ and $g \in \mathcal{N}_P(G)$ such that $-f + g = \varepsilon\vec{v}$, which proves the claim. For the second part of the statement, if a pair of faces $(F,G) \in \mathcal{F}^\phi$ verifies $\dim((\mathcal{N}_P(F) + \varepsilon\vec{v}) \cap \mathcal{N}_P(G)) = 0$, then we have $\dim \mathcal{N}_P(F) + \dim \mathcal{N}_P(G) \leq \dim P$ since P is full-dimensional, so we have $\dim F + \dim G \geq \dim P$. In the case where the subdivision is tight, we must have $\dim F + \dim G = \dim P$, otherwise we would have $\dim(\pi(F \times G)) = \dim(F \times G) = \dim F + \dim G > \dim P$, which is impossible since $\mathrm{Im}(\pi) = P$. □

Corollary 2.4 is a "perturbative" way of seeing Proposition 2.3 : the pairs of $\mathcal{F}^\phi$ arise as intersections of the normal fan of P with a translated copy of itself in the direction of $\vec{v}$, see Figure 2.2.

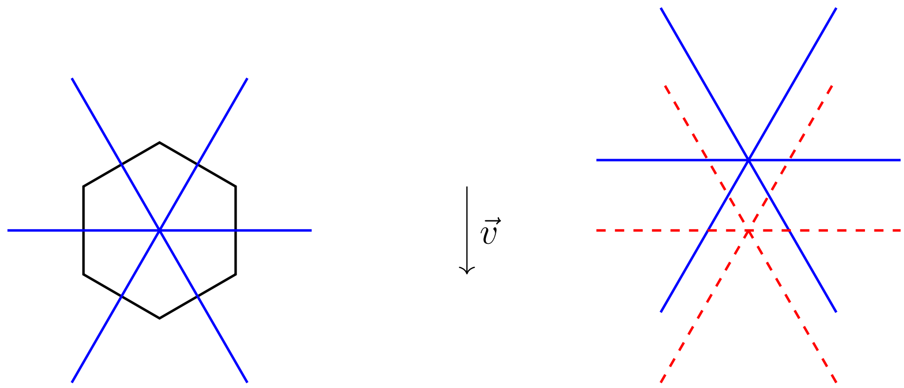


Figure 2.2: The normal fan $\mathcal{N}_P$ of the 2-dimensional permutahedron (in blue), and its perturbed copy $\mathcal{N}_P + \varepsilon\vec{v}$ (in red, dashed).

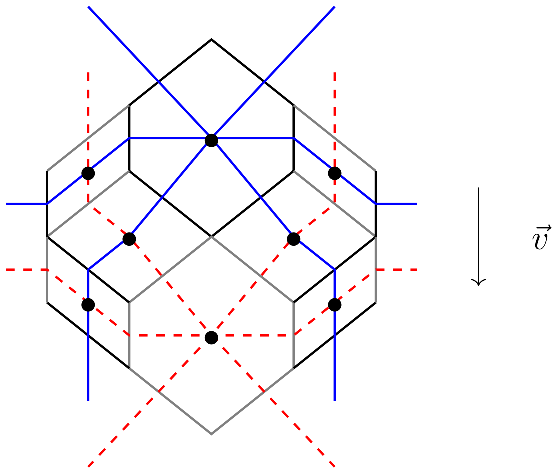


Figure 2.3: A tight coherent subdivision of the 2-dimensional permutahedron and its dual cell decomposition.

By definition, the coherent subdivision $\pi(\mathcal{F}^\phi)$ of P is given by union of the polytopes $(F+G)/2$, for all the pairs of faces $(F,G) \in \mathcal{F}^\phi$. In the case where P is full-dimensional, the dual cell decomposition of $\pi(\mathcal{F}^\phi)$ is then isomorphic to $(\mathcal{N}_P + \varepsilon\vec{v}) \cap \mathcal{N}_P$, see Figure 2.3.

Remark 2.5. In the case of the simplices, one recovers via Corollary 2.4 the classical equivalence between the cup product on the simplicial cochains of a triangulation of a manifold and the intersection pairing on cellular chains, as described by M. Abouzaid in [Abo09, Appendix E]. We denote by $(e_i)_{1\leq i\leq n}$ the standard basis of $\mathbb{R}^n$ and we set $e_0 \coloneqq 0$. Let us consider the following full-dimensional realization of the n-simplex

$$\Delta^n \coloneqq \mathrm{conv}(\{e_i \mid 0 \leq i \leq n\}) \ .$$

The rays of the normal fan $\mathcal{N}_{\triangle^n}$ are generated by the vectors $\{-e_i \mid 1 \leq i \leq n\}$ and $(1, \ldots, 1)$. We fix some $\varepsilon > 0$, and define $\vec{v} = (\varepsilon/n, \ldots, \varepsilon/2, \varepsilon)$. Then, [Abo09, Lemma E.4] shows that the non-empty dimension 0 intersections of $\mathcal{N}_{\Delta^n} \cap (\mathcal{N}_{\triangle^n} - \varepsilon\vec{v})$ coincide with the terms in the formula for the Alexander-Whitney map. By means of Corollary 2.4, this is exactly what we would obtain by proving that $\vec{v}$ induces a tight coherent subdivision of Δ^n.

We aim now at giving a geometric meaning to the cone $\mathrm{Cone}(-\mathcal{N}_P(F) \cup \mathcal{N}_P(G))$ that appears in Proposition 2.3. We denote by $\mathring{P}$ the *relative interior* of a polytope P, that is the interior of P with respect to its embedding into its affine hull.

Lemma 2.6. *Let $P, Q \subset \mathbb{R}^n$ be two polytopes. There is a bijection*

$$\mathcal{L}(P \cap Q) \cong \{(F, G) \in \mathcal{L}(P) \times \mathcal{L}(Q) \mid \mathring{F} \cap \mathring{G} \neq \emptyset\} .$$

Moreover, for any face $F \cap G \in \mathcal{L}(P \cap Q)$, we have

$$\mathcal{N}_{P \cap Q}(F \cap G) = \mathrm{Cone}(\mathcal{N}_P(F) \cup \mathcal{N}_Q(G)) .$$

Proof. Any polytope $P \subset \mathbb{R}^n$ is a bounded intersection of facet-defining closed halfspaces, one for each facet of P, and of the affine hull of P. Each halfspace has a support hyperplane. Let x be a point in the interior of a face of $P \cap Q$. Then x is in the support hyperplanes H_i of P for a certain subset I and also in the support hyperplanes H_j of Q for a certain subset J. Thus x is in the face F of P defined by the H_i and in the face G of Q defined by the H_j. For the second part of the statement, we observe that the normal cone $\mathcal{N}_P(F)$ of a face F is spanned by the normal vectors of the support hyperplanes defining that face and any basis of the orthogonal complement of P in $\mathbb{R}^n$, and the result follows. □

Definition 2.7. *Let $P \subset \mathbb{R}^n$ be a polytope. For $z \in P$, we denote by $\rho_z P \coloneqq 2z - P$ the reflection of P with respect to z, see Figure 2.4.*

Proposition 2.8. *Let $P \subset \mathbb{R}^n$ be a polytope, and let F, G be two faces of P. For any $z, z' \in (\mathring{F} + \mathring{G})/2$, we have*

$$\mathcal{N}_{P \cap \rho_z P}(G \cap \rho_z F) = \mathcal{N}_{P \cap \rho_{z'} P}(G \cap \rho_{z'} F) = \mathrm{Cone}(-\mathcal{N}_P(F) \cup \mathcal{N}_P(G)) .$$

Proof. The result follows directly from the application of Lemma 2.6 to the intersection $P \cap \rho_z P$, and the fact that for any face F of P and any $z \in P$ we have $\mathcal{N}_{\rho_z P}(\rho_z F) = -\mathcal{N}_P(F)$. □

Corollary 2.9. *Let $P \subset \mathbb{R}^n$ be a polytope, let $\vec{v} \in \mathbb{R}^n$. For two faces F, G of P, we have*

$$\begin{aligned} (F, G) \in \mathcal{F}^\phi \quad &\Longleftrightarrow \quad \forall z \in \tfrac{\mathring{F} + \mathring{G}}{2}, \ \vec{v} \in \mathcal{N}_{P \cap \rho_z P}(G \cap \rho_z F) \\ &\Longleftrightarrow \quad \exists z \in \tfrac{\mathring{F} + \mathring{G}}{2}, \ \vec{v} \in \mathcal{N}_{P \cap \rho_z P}(G \cap \rho_z F) . \end{aligned}$$

Proof. The result is obtained by combining Propositions 2.3 and 2.8. □

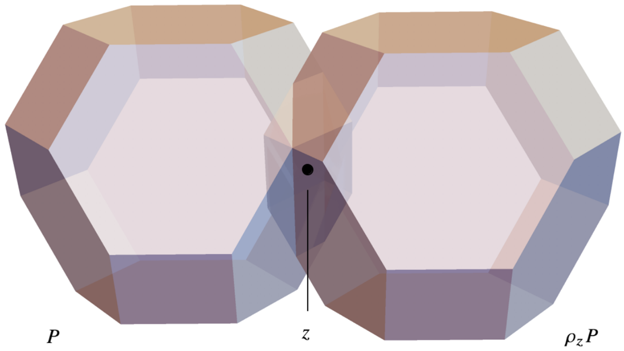


Figure 2.4: A 3-dimensional permutahedron P, its reflection $\rho_z P$ and the intersection $P \cap \rho_z P$.

2.2.3 Pointwise description of the diagonal

We are interested in answering the following question: which choice of vector $\vec{v}$ gives a tight coherent subdivision of P?

Definition 2.10 (Quasi-oriented polytope)**.** *A polytope $P \subset \mathbb{R}^n$ is* quasi-oriented *by $\vec{v} \in \mathbb{R}^n$ if the linear form $\langle -, \vec{v} \rangle$ has a unique minimal element* $\mathrm{bot}(P)$ *and a unique maximal element* $\mathrm{top}(P)$ *in P.*

Definition 2.11 (Oriented polytope)**.** *A polytope $P \subset \mathbb{R}^n$ is* oriented *by $\vec{v} \in \mathbb{R}^n$ if $\vec{v}$ is not perpendicular to any edge of P.*

An orientation vector induces a poset on the vertices of P, for which the oriented 1-skeleton of P is the Hasse diagram. Dually, it corresponds to a poset structure on the maximal cones of the normal fan $\mathcal{N}_P$. We observe that if P is oriented by $\vec{v}$, then so is any face of P. Any oriented polytope is quasi-oriented, but the converse in not true in general. Consider the 3-dimensional cross-polytope $\Diamond_3 := \mathrm{conv}(e_1, -e_1, e_2, -e_2, e_3, -e_3)$, and choose $\vec{v} := e_3$. Then, $(\Diamond_3, \vec{v})$ is quasi-oriented but not oriented, since $\vec{v}$ is perpendicular to the four edges contained in the xy-plane.

Definition 2.12 (Positively and quasi-positively oriented polytope)**.** *A polytope $P \subset \mathbb{R}^n$ is* positively oriented *(resp.* quasi-positively oriented*) by $\vec{v} \in \mathbb{R}^n$ if for any $z \in P$, the intersection $P \cap \rho_z P$ is oriented (resp. quasi-oriented) by $\vec{v}$.*

Any positively oriented polytope is quasi-positively oriented, but the converse is not true in general, see Example 2.15. We note that any quasi-positively oriented polytope is also oriented. To see this, let e be an edge from a vertex x to a vertex y in P, and set $z := (x + y)/2$. Then $P \cap \rho_z P = e$ is quasi-oriented by $\vec{v}$, so $\vec{v}$ is not perpendicular to e.

Proposition 2.13. *Let P be a polytope. Then,*

$$(P,\vec{v}) \text{ is quasi-positively oriented } \iff \pi(\mathcal{F}^\phi) \text{ is tight.}$$

Proof. We read the proof of [MTTV21, Proposition 5] with a new perspective. We have that $\pi(\mathcal{F}^\phi)$ is tight if and only if for any $z \in P$, the fiber $\pi^{-1}(z) = \{(x,y) \in P \times P \mid x+y = 2z\}$ admits a unique minimal element with respect to ϕ. Since the sum of $x+y$ is constant, $\phi(x,y)$ is minimized in $\pi^{-1}(z)$ if and only if $\langle x, \vec{v}\rangle$ is minimized and $\langle y, \vec{v}\rangle$ is maximized. On both coordinates, $\pi^{-1}(z)$ projects down to the intersection $P \cap \rho_z P$. So, the fiber $\pi^{-1}(z)$ admits a unique minimal element with respect to ϕ if and only if $P \cap \rho_z P$ admits a unique pair of minimal and maximal elements with respect to $\langle -, \vec{v}\rangle$. □

In summary, we have the chain of implications showed in Figure 2.5.

pos. oriented	$\implies$	quasi-pos. oriented	$\implies$	oriented	$\implies$	quasi oriented
hyperplane arrangement		*tight coherent subdivision*		*poset on vertices*		*bot and top*

Figure 2.5: The different notions of orientation and their associated properties.

In the case where $(P,\vec{v})$ is quasi-positively oriented, the proof of Proposition 2.13 gives the following pointwise description of $\triangle_{(P,\vec{v})}$.

Proposition 2.14 (Bot-top diagonal)**.** *The map $\triangle_{(P,\vec{v})}$ associated to a quasi-positively oriented polytope $(P,\vec{v})$ admits the following pointwise description*

$$\begin{array}{rccl} \triangle_{(P,\vec{v})} & : & P & \to & P \times P \\ & & z & \mapsto & \big(\mathrm{bot}(P \cap \rho_z P), \mathrm{top}(P \cap \rho_z P)\big) \ . \end{array}$$

We call it the bot-top diagonal *of* $(P,\vec{v})$.

Example 2.15. We consider the pyramid

$$P = \mathrm{conv}((0,0,1), (-1,0,0), (0,1.5,-0.5), (0,-1.5,-0.5), (3,0,-2)) \subset \mathbb{R}^3 \ ,$$

shown in Figure 2.6, and we set $\vec{v} = (0,0,1)$. We claim that $\pi(\mathcal{F}^\phi)$ is tight while $(P,\vec{v})$ is not positively oriented. For the second assertion, we let $z = 0$ and we observe that four edges of $P \cap \rho_z P$ lie in the xy plane and are thus perpendicular to $\vec{v}$. For the first assertion, we first observe that directions of the rays of $\mathcal{N}_P$ are given by $(1,1,1), (-1,1,1), (-1,-1,1), (1,-1,1)$ and $(-0.5,0,-1)$. Then, one can compute that for any pair of faces (F,G) with $\dim F + \dim G > \dim P$, we have $\vec{v} \notin \mathrm{Cone}(-\mathcal{N}_P(F) \cup \mathcal{N}_P(G))$. We conclude with Proposition 2.3.

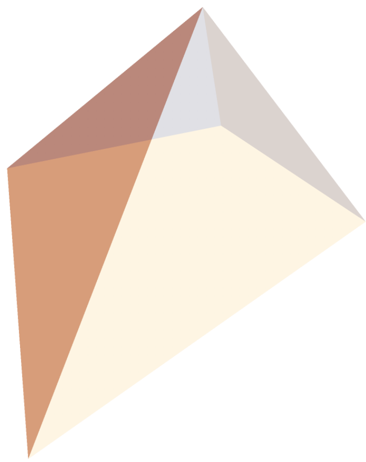


Figure 2.6: The pyramid P described in Example 2.15 is quasi-positively oriented by $\vec{v} = (0, 0, 1)$ but not positively oriented.

2.2.4 Poset description of the diagonal

Proposition 2.16. *Let $(P, \vec{v})$ be an oriented polytope. Then,*

$$(F, G) \in \mathcal{F}^\phi \implies \text{top}(F) \leq \text{bot}(G) .$$

Proof. Corollary 2.4 asserts that $(F, G) \in \mathcal{F}^\phi \iff \forall \varepsilon > 0\ \exists x \in \mathcal{N}_P(F)$ such that $x + \varepsilon\vec{v} \in \mathcal{N}_P(G)$.

1. Suppose that F and G are vertices of P. Let $\varepsilon > 0$ and choose $x \in \mathcal{N}_P(F)$ such that $x + \varepsilon\vec{v} \in \mathcal{N}_P(G)$. We consider the segment $\ell = \{x + t\vec{v} \mid t \in [0, \varepsilon]\}$ and the linearly ordered set of maximal cones of $\mathcal{N}_P$ crossed by ℓ. They determine a sequence of vertices $F = F_1, F_2, \ldots, F_k = G$. We claim that $F_i \leq F_{i+1}$ for all i. Indeed, when ℓ goes from $\mathcal{N}_P(F_i)$ to $\mathcal{N}_P(F_{i+1})$, it intersects the interior of a cone $\mathcal{N}_P(E)$, where E is a face with $\dim(E) \geq 1$. Since $\vec{v}$ orients P, this intersection is a point. So we must have $F_i = \text{bot}(E)$ and $F_{i+1} = \text{top}(E)$.

2. For general faces F and G, we have $(F, G) \in \mathcal{F}^\phi \implies (\text{top}(F), \text{bot}(G)) \in \mathcal{F}^\phi$ and we can apply the preceding point.

□

Applying the present method to the cubes, the standard simplices and the associahedra as in [MTTV21, Example 1 and Theorem 2], one obtains a characterization of the form $(F, G) \in \mathcal{F}^\phi \iff \text{top}(F) \leq \text{bot}(G)$. Proposition 2.16 shows that these are the "simplest" possible formulas. In Section 3.3 we will study a family of examples where these formulas are no longer sufficient to characterize the image of the diagonal.

2.2.5 Fundamental hyperplane arrangement and universal formula

We now restrict our attention to a positively oriented polytope $(P, \vec{v})$. For such a polytope, the orientation vector $\vec{v}$ does not live in any linear space orthogonal to an edge of $P \cap \rho_z P$, for any $z \in P$. That is, $\vec{v}$ lives in a chamber of the following hyperplane arrangement.

Definition 2.17 (Fundamental hyperplane arrangement)**.** *Let $P \subset \mathbb{R}^n$ be a polytope. An* edge hyperplane *of P is an hyperplane in $\mathbb{R}^n$ which is orthogonal to the direction of an edge of $P \cap \rho_z P$ for some $z \in P$. The* fundamental hyperplane arrangement *$\mathcal{H}_P$ of P is the collection of all edge hyperplanes of P.*

Here, a *direction* of an edge is an arbitrary choice of vector spanning its affine hull. The fundamental hyperplane arrangement is *central*, i.e. every hyperplane $H \in \mathcal{H}_P$ contains the origin. We call the interior of a maximal cone in $\mathcal{H}_P$ a *chamber*. We observe that $\mathcal{H}_P$, considered as a fan, refines both the normal fan $\mathcal{N}_P$ of P and its opposite $-\mathcal{N}_P$.

Example 2.18 (The cubes). The fundamental hyperplane arrangement of the n-dimensional cube $C^n = [0,1]^n \subset \mathbb{R}^n$ is the set of coordinate hyperplanes $\mathcal{H}_{C^n} = \{x_i = 0 \mid 1 \leq i \leq n\}$. In this case, for any $z \in C^n$, the edges of $C^n \cap \rho_z C^n$ are all parallel to some edges of C^n, so $\mathcal{H}_{C^n}$ is just the set of hyperplanes perpendicular to the directions of the edges of C^n.

Example 2.19 (The simplices). The fundamental hyperplane arrangement of the n-dimensional standard simplex $\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1\}$ (which is distinct from the realization considered in Remark 2.5) is the braid arrangement $\mathcal{H}_{\Delta^n} = \{x_i = x_j \mid 0 \leq i < j \leq n\}$. Here again, it corresponds to the set of hyperplanes perpendicular to the directions of the edges of Δ^n.

Example 2.20 (The associahedra). The fundamental hyperplane arrangement of the Loday realization of the n-dimensional associahedron $K^n \subset \mathbb{R}^{n+1}$ is the refinement of the braid arrangement made of the hyperplanes defined by $x_{i_1} + x_{i_3} + \cdots + x_{i_{2k-1}} = x_{i_2} + x_{i_4} + \cdots + x_{i_{2k}}$, where $1 \leq i_1 < i_2 < \cdots < i_{2k} \leq n+1$ and $1 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$. In contrast with the preceding examples, new directions of edges appear when considering $K^n \cap \rho_z K^n$ for some $z \in K^n$, see [MTTV21, Proposition 2].

The following proposition is useful for computing the fundamental hyperplane arrangement of a polytope.

Proposition 2.21. *Let P be a polytope. There is a surjection*

$$\left\{\begin{array}{c} \textit{pair of faces } (F,G) \textit{ of } P \\ \textit{with } \operatorname{codim}(\operatorname{Cone}(-\mathcal{N}_P(F)\cup\mathcal{N}_P(G))) = 1 \end{array}\right\} \twoheadrightarrow \left\{\begin{array}{c} \textit{direction } \vec{d} \textit{ of an edge of } P\cap\rho_z P \\ \textit{for some } z\in P \end{array}\right\}_{/\sim},$$

where we identify in the target two directions which are scalar multiples of each other.

Proof. Let (F,G) be a pair of faces of P such that $\operatorname{codim}(\operatorname{Cone}(-\mathcal{N}_P(F)\cup\mathcal{N}_P(G)) = 1$. By Proposition 2.8, this condition is equivalent to $\operatorname{codim}(\mathcal{N}_{P\cap\rho_z P}(G\cap\rho_z F)) = 1$, where $z\in(\mathring{F}+\mathring{G})/2$. Thus, a pair of faces satisfying this codimension 1 condition defines an edge $G\cap\rho_z F$ of $P\cap\rho_z P$ and the application above is well-defined. Now by Lemma 2.6, any edge of $P\cap\rho_z P$ arises as the intersection $G\cap\rho_z F$ for some pair of faces $(F,G)\in P\times P$, so the application is also surjective. □

Now we aim at extracting a combinatorial formula for the cellular image of the diagonal from the geometry of the fundamental hyperplane arrangement.

Proposition 2.22 (Chamber invariance)**.** *Let $P\subset\mathbb{R}^n$ be a polytope. Two vectors $\vec{v}$ and $\vec{w}$ belonging to the same chamber of $\mathcal{H}_P$ define the same bot-top diagonal, that is*

$$\triangle_{(P,\vec{v})} = \triangle_{(P,\vec{w})}\ .$$

Proof. Suppose that $\vec{v}$ and $\vec{w}$ are such that $\triangle_{(P,\vec{v})}\neq\triangle_{(P,\vec{w})}$. This means that there is a point $z\in P$ for which $\operatorname{bot}_{\vec{v}}(P\cap\rho_z P)\neq\operatorname{bot}_{\vec{w}}(P\cap\rho_z P)$ or $\operatorname{top}_{\vec{v}}(P\cap\rho_z P)\neq\operatorname{top}_{\vec{w}}(P\cap\rho_z P)$. So, there is an edge e of $P\cap\rho_z P$ such that $\vec{v}$ and $\vec{w}$ determine two different orientations of e. If e has direction $\vec{d}$, this means that $\langle\vec{d},\vec{v}\rangle$ and $\langle\vec{d},\vec{w}\rangle$ have opposite signs. Thus $\vec{v}$ and $\vec{w}$ lie on opposite sides of the hyperplane $H=\{x\in\mathbb{R}^n|\ \langle\vec{d},x\rangle=0\}\in\mathcal{H}_P$. □

Remark 2.23. We note that the converse of Proposition 2.22 does not hold in general, that is, two distinct chambers in $\mathcal{H}_P$ can determine the same bot-top diagonal. This is due to the fact that the condition of being positively oriented is strictly stronger than being quasi-positively oriented.

For a vector $\vec{d}\in\mathbb{R}^n$ defining an hyperplane $H=\{x\in\mathbb{R}^n \mid \langle\vec{d},x\rangle=c\in\mathbb{R}\}$, we set the notation $H^+ := \{x\in\mathbb{R}^n \mid \langle\vec{d},x\rangle > c\}$.

Definition 2.24 (Outward pointing normal vector)**.** *Let F be a facet of a polytope $P\subset\mathbb{R}^n$. A vector $\vec{d}\in\mathbb{R}^n$ is said to be an* outward pointing normal vector *for F if it defines an hyperplane H such that $P\cap H=F$ and $P\cap H^+=\emptyset$.*

Recall that a face F of a polytope P is equal to the intersection of a family of facets $\{F_i\}$. If we choose an outward pointing normal vector $\vec{F_i}$ for each facet F_i, then the normal cone of F is spanned by these normal vectors together with a basis $\{b_k\}$ of the orthogonal complement of the affine hull of P in $\mathbb{R}^n$, i.e. we have $\mathcal{N}_P(F) = \mathrm{Cone}(\{\vec{F_i}\} \cup \{b_k, -b_k\})$.

For a pair of faces F, G of P, let $\mathcal{H}_P(F,G)$ denote the set of hyperplanes $H \in \mathcal{H}_P$ such that H intersects the interior of a codimension 1 face of $\mathrm{Cone}(-\mathcal{N}_P(F) \cup \mathcal{N}_P(G))$.

Theorem 2.25 (Universal formula for the bot-top diagonal)**.** *Let $(P, \vec{v})$ be a positively oriented polytope in $\mathbb{R}^n$. For each $H \in \mathcal{H}_P$, we choose a normal vector $\vec{d}_H$ such that $\langle \vec{d}_H, \vec{v}\rangle > 0$. We have*

$$
\begin{aligned}
& (F,G) \in \mathrm{Im}\,\triangle_{(P,\vec{v})} \\
\iff \quad & \forall H \in \mathcal{H}_P(F,G),\ \exists i,\ \langle \vec{F_i}, \vec{d}_H\rangle < 0 \ \textit{or}\ \exists j,\ \langle \vec{G_j}, \vec{d}_H\rangle > 0 \qquad (2.1)\\
\iff \quad & \forall H \in \mathcal{H}_P,\ \exists i,\ \langle \vec{F_i}, \vec{d}_H\rangle < 0 \ \textit{or}\ \exists j,\ \langle \vec{G_j}, \vec{d}_H\rangle > 0\ . \qquad (2.2)
\end{aligned}
$$

Proof. Let us write $\mathrm{Cone}(-F,G) := \mathrm{Cone}(-\mathcal{N}_P(F) \cup \mathcal{N}_P(G)) = \mathrm{Cone}(\{-\vec{F_i}\} \cup \{\vec{G_j}\} \cup \{b_k, -b_k\})$ and let us denote by C the chamber of $\mathcal{H}_P$ containing $\vec{v}$. Combining Propositions 2.3 and 2.22 we have that $(F,G) \in \mathrm{Im}\,\triangle_{(P,\vec{v})} \iff \vec{v} \in \mathrm{Cone}(-F,G) \iff C \subset \mathrm{Cone}(-F,G)$. Moreover, we recall from Proposition 2.8 that we have

$$\mathrm{Cone}(-F,G) = \mathcal{N}_{P\cap\rho_z P}(G \cap \rho_z F)$$

for any $z \in (\mathring{F} + \mathring{G})/2$. We observe that since every $\vec{d}_H$ lives in the affine hull of P, we have $\langle \vec{d}_H, b_k\rangle = 0$ for all H and for all k, so we can focus on the families of $\vec{F_i}$'s and $\vec{G_j}$'s and distinguish two cases.

If $\dim(G \cap \rho_z F) \geq 1$, both sides of (1) are false and thus equivalent. Indeed, in this case $\mathrm{Cone}(-F,G)$ is not full-dimensional, so it cannot contain C, which is full-dimensional. Moreover, $\mathrm{Cone}(-F,G)$ belongs to all hyperplanes $H \in \mathcal{H}_P(F,G)$, which implies $\langle \vec{F_i}, \vec{d}_H\rangle = \langle \vec{G_j}, \vec{d}_H\rangle = 0$ for all i, j. The same argument applies to (2), since $\mathcal{H}_P(F,G) \subset \mathcal{H}_P$.

Suppose now that $\dim(G\cap\rho_z F) = 0$. In this case $\mathrm{Cone}(-F,G)$ is full-dimensional and its bounding hyperplanes are precisely the hyperplanes perpendicular to the edges of $P\cap\rho_z P$ which are adjacent to $G\cap\rho_z F$, that is, the hyperplanes of $\mathcal{H}_P(F,G)$. By definition, we have $C \subset H^+$ for all $H \in \mathcal{H}_P$. We examine the first implication ($\Longrightarrow$) of (2). Suppose that $(F,G) \in \mathrm{Im}\,\triangle_{(P,\vec{v})}$, and let $H \in \mathcal{H}_P$. Since C is full-dimensional, we have $C \subset \mathrm{Cone}(-F,G) \implies \mathrm{Cone}(-F,G) \cap H^+ \neq \emptyset$. In particular, there exists an outward pointing normal vector in $\mathrm{Cone}(-F,G)$ which has a strictly positive scalar product with $\vec{d}_H$, hence the right hand side of (2). This implies the right hand side of (1). Now we prove the reverse implication ($\Longleftarrow$) of (1) by contraposition. If $C \not\subset \mathrm{Cone}(-F,G)$, then Proposition 2.22 implies that

$C \cap \mathrm{Cone}(-F, G) = \emptyset$. In this case, there exists an $H \in \mathcal{H}_P(F, G)$ such that $\mathrm{Cone}(-F, G) \subset \mathbb{R}^n \setminus H^+$. Indeed, if we had $\mathrm{Cone}(-F, G) = \cap_{H \in \mathcal{H}_P(F,G)} \overline{H^+}$, where $\overline{H^+} := \{x \in \mathbb{R}^n | \ \langle \vec{d}_H, x \rangle \geq 0\}$, then we would have $C = \cap_{H \in \mathcal{H}_P} H^+ \subset \mathrm{Cone}(-F, G)$ which is impossible. So the scalar product of the spanning vector of any outward pointing normal vector in $\mathrm{Cone}(-F, G)$ with $\vec{d}_H$ has a nonpositive value. □

In practice, one uses Theorem 2.25 by first computing the directions $\vec{d}_H$ from the normal fan of P, and then applying (2.2). The equivalence (2.1) is of a more conceptual nature: it says that strictly speaking, all the hyperplanes of $\mathcal{H}_P$ are not needed in the computation of $\mathrm{Im} \triangle_{(P,\vec{v})}$. However, computing the set of hyperplanes $\mathcal{H}_P(F, G)$ for a given pair of faces seems to be more complicated than applying (2.2), both from the combinatorial and the computational points of view.

Example 2.26 (The cubes). The n-dimensional cube $C^n = [0, 1]^n \subset \mathbb{R}^n$ is positively oriented by the vector $\vec{v} = (1, \ldots, 1)$. We choose as normal vectors $\vec{d}_H$ the family $\{e_i \mid 1 \leq i \leq n\}$. Any pair of subsets $K, L \subset \{1, \ldots, n\}$ with $K \cap L = \emptyset$ defines a face F such that $\mathcal{N}_{C^n}(F) = \mathrm{Cone}(\{e_k \mid k \in K\} \cup \{-e_l \mid l \in L\})$. Theorem 2.25 says that $(F, G) \in \mathrm{Im} \triangle_{(C^n,\vec{v})}$ if and only if for each $1 \leq i \leq n$, either $-e_i \in \mathcal{N}_{C^n}(F)$ or $e_i \in \mathcal{N}_{C^n}(G)$. Restricting our attention to pairs with $\dim F + \dim G = n$, we obtain directly the families $\{\vec{F}_i = -e_i \mid i \in I\}$ and $\{\vec{G}_j = e_j \mid j \in J\}$ for partitions $I \cup J = \{1, \ldots, n\}$, which define J.-P. Serre's approximation of the diagonal.

Example 2.27 (The simplices). The n-dimensional standard simplex $\Delta^n \subset \mathbb{R}^{n+1}$ is positively oriented by any vector $\vec{v}$ with strictly increasing coordinates. We write $[n + 1] = \{1, \ldots, n + 1\}$. We choose as normal vectors $\vec{d}_H$ the family $\{e_j - e_i \mid 1 \leq i < j \leq n + 1\}$ and we set $\vec{n} = (1, \ldots, 1)$. Any subset $I \subset [n + 1]$ defines a face F such that $\mathcal{N}_{\Delta^n}(F) = \mathrm{Cone}(\{-e_j \mid j \in [n + 1] \setminus I\} \cup \{\vec{n}, -\vec{n}\})$. Theorem 2.25 says that $(F, G) \in \mathrm{Im} \triangle_{(\Delta^n,\vec{v})}$ if and only if for each pair $1 \leq i < j \leq n + 1$, either $-e_j \in \mathcal{N}_{\Delta^n}(F)$ or $-e_i \in \mathcal{N}_{\Delta^n}(G)$. Restricting our attention to pairs with $\dim F + \dim G = n$, we obtain directly the families $\{\vec{F}_i = -e_i \mid 1 \leq i \leq k\}$ and $\{\vec{G}_j = -e_j \mid k \leq j \leq n + 1\}$ for $k \in [n + 1]$, which define the Alexander–Whitney map.

The case of the associahedra will be treated in the same fashion in Section 3.3, as a special case of Theorem 3.32.

2.2.6 Universal formula and refinement of normal fans

We consider polytopes related by refinement of their normal fans. We have in mind applications to the operahedra in Section 3.3, and to generalized permutahedra in forthcoming work. We recall that a fan $\mathcal{G}'$ *refines* a fan $\mathcal{G}$, or that $\mathcal{G}$ *coarsens* $\mathcal{G}'$, if every cone of $\mathcal{G}$ is the union of cones of $\mathcal{G}'$ and $\cup_{C \in \mathcal{G}} C = \cup_{C' \in \mathcal{G}'} C'$, see [Zie95, Lecture 7] for more details and examples.

Definition 2.28 (Coarsening projection)**.** *Let P and Q be two polytopes in $\mathbb{R}^n$ such that the normal fan of P refines the normal fan of Q. The* coarsening projection *from P to Q is the application $\theta : \mathcal{L}(P) \to \mathcal{L}(Q)$ which sends a face F of P to the face $\theta(F)$ of Q whose normal cone $\mathcal{N}_Q(\theta(F))$ is the minimal cone with respect to inclusion which contains $\mathcal{N}_P(F)$.*

Proposition 2.29. *Let P and Q be two polytopes in $\mathbb{R}^n$ such that the normal fan of P refines the normal fan of Q. Then, their fundamental hyperplane arrangements satisfy $\mathcal{H}_Q \subset \mathcal{H}_P$.*

Proof. Let $H \in \mathcal{H}_Q$. Let F, G be two faces of Q such that the intersection $G \cap \rho_z F$ is an edge of $Q \cap \rho_z Q$ perpendicular to H, for any $z \in (\mathring{F} + \mathring{G})/2$. If we write $\mathcal{N}_Q(F) = \mathrm{Cone}(\{\vec{F}_i\}_{i \in I})$ and $\mathcal{N}_Q(G) = \mathrm{Cone}(\{\vec{G}_j\}_{j \in J})$, this means that a direction $\vec{d}$ of this edge is solution to the system of equations $\langle \vec{F}_i, \vec{d} \rangle = 0$ and $\langle \vec{G}_j, \vec{d} \rangle = 0$. Now we choose any $F' \in \theta^{-1}(F)$ and $G' \in \theta^{-1}(G)$ such that $\dim(F') = \dim(F)$ and $\dim(G') = \dim(G)$. We can write the normal cones of F' and G' as $\mathcal{N}_P(F') = \mathrm{Cone}(\{\vec{F}'_k\}_{k \in K})$ and $\mathcal{N}_P(G') = \mathrm{Cone}(\{\vec{G}'_l\}_{l \in L})$ where for each k and l, we have $\vec{F}'_k = \sum_{i \in I} \alpha_i \vec{F}_i$ and $\vec{G}'_l = \sum_{j \in J} \beta_j \vec{G}_j$ with $\alpha_i, \beta_j \geq 0$ for all i, j. So, the direction $\vec{d}$ is also solution to the system of equations $\langle \vec{F}'_k, \vec{d} \rangle = 0$ and $\langle \vec{G}'_l, \vec{d} \rangle = 0$. Then, the dimension assumption shows that for any $w \in (\mathring{F}' + \mathring{G}')/2$ the intersection $G' \cap \rho_w F'$ is an edge of $P \cap \rho_w P$ with direction $\vec{d}$, and thus $H \in \mathcal{H}_P$. □

Corollary 2.30. *Suppose that the normal fan of P refines the normal fan of Q. If P is positively oriented by $\vec{v}$, then so is Q.*

Proof. This is an immediate consequence of Proposition 2.29. □

Proposition 2.31. *Let P and Q be two polytopes in $\mathbb{R}^n$ such that the normal fan of P refines the normal fan of Q, and suppose that they are both positively oriented by the same vector $\vec{v} \in \mathbb{R}^n$. Then, the coarsening projection θ commutes with the cellular maps $\triangle_{(P,\vec{v})}$ and $\triangle_{(Q,\vec{v})}$.*

Proof. Let F be a face of Q. By definition of the coarsening projection θ, we have that $\mathcal{N}_Q(F) = \cup_{F' \in \theta^{-1}(F)} \mathcal{N}_P(F')$. It follows that

$$\bigcup_{\substack{F' \in \theta^{-1}(F) \\ G' \in \theta^{-1}(G)}} \mathrm{Cone}(-\mathcal{N}_P(F') \cup \mathcal{N}_P(G')) = \mathrm{Cone}(-\mathcal{N}_Q(F) \cup \mathcal{N}_Q(G)) \ ,$$

from which we conclude by using Proposition 2.3. □

Proposition 2.32. *Let P and Q be two polytopes such that $\mathcal{H}_Q \subset \mathcal{H}_P$, and suppose that they are both positively oriented by the same vector $\vec{v}$. For each $H \in \mathcal{H}_P$, we choose a normal vector $\vec{d}_H$ such that $\langle \vec{d}_H, \vec{v} \rangle > 0$. We have*

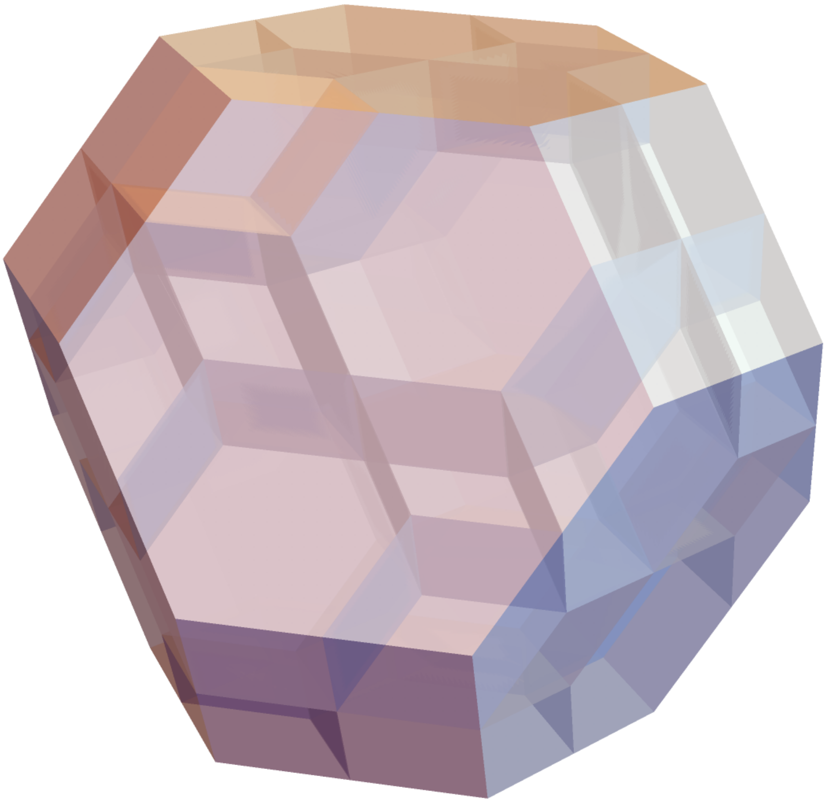
$$(F, G) \in \mathrm{Im} \triangle_{(Q,\vec{v})} \iff \forall H \in \mathcal{H}_P, \ \exists i, \ \langle \vec{F}_i, \vec{d}_H \rangle < 0 \ or \ \exists j, \ \langle \vec{G}_j, \vec{d}_H \rangle > 0 \ .$$

Proof. We denote by C_P (resp. C_Q) the chamber of $\mathcal{H}_P$ (resp. $\mathcal{H}_Q$) containing $\vec{v}$. Since $\mathcal{H}_Q \subset \mathcal{H}_P$, we have $C_P \subset C_Q$. As in the proof of Theorem 2.25, we write $\mathrm{Cone}(-F, G) \coloneqq \mathrm{Cone}(-\mathcal{N}_Q(F) \cup \mathcal{N}_Q(G))$ and we have $(F, G) \in \operatorname{Im} \triangle_{(Q,\vec{v})} \iff C_Q \cap \mathrm{Cone}(-F, G) \neq \emptyset \iff C_Q \subset \mathrm{Cone}(-F, G)$. For the first implication ($\implies$), suppose that there exists an $H \in \mathcal{H}_P$ such that $\mathrm{Cone}(-F, G) \subset \overline{H^-} =: \{x \in \mathbb{R}^n \mid \langle \vec{d}_H, x\rangle \leq 0\}$. Since $C_P \subset C_Q \subset \mathrm{Cone}(-F, G)$, we have $C_P \subset \overline{H^-}$, which is impossible. The reverse implication ($\impliedby$) follows immediately from Theorem 2.25 since $\mathcal{H}_Q \subset \mathcal{H}_P$. □

One can thus compute the universal formula for a polytope P and apply it *mutatis mutandis* to any polytope Q whose normal fans coarsens the one of P. Alternatively, one can apply the coarsening projection via Proposition 2.31. Depending on the polytopes under consideration, one approach or the other might give a simpler combinatorial description.

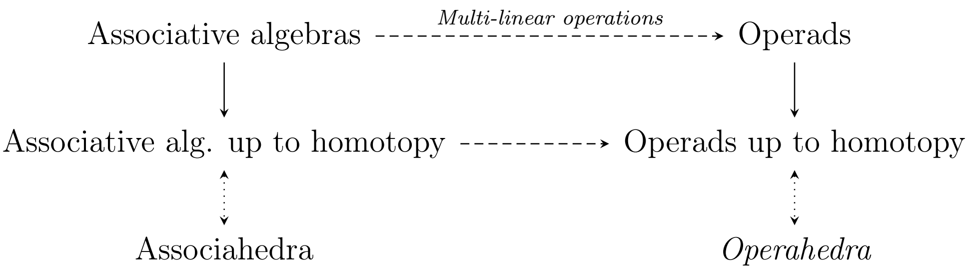
Chapter 3

The diagonal of the operahedra



3.1 Introduction

As already mentioned the associahedra encode homotopy associative algebras, and homotopy operads were defined by P. Van der Laan in [VdL03] as a multi-linear generalization of homotopy associative algebras. This leads to thinking about a multi-linear generalization of the associahedra, which encodes homotopy operads. Such a generalization was provided by J. Obradović in [Obr19].



The second contribution of this thesis is to define Loday realizations of the operahedra, the family of polytopes encoding homotopy operads, and to apply to it the general theory developed in the first part. In contrast with existing realizations of the operahedra, these integer-coordinates realizations, which generalize J.-L. Loday's realizations of the associahedra [Lod04], present simple geometric properties that ease calculations. They allow for the definition of a coherent family of cellular approximations of the diagonal, which lead to a compatible topological cellular operad structure on the family of Loday realizations of the operahedra. This is the first topological cellular operad structure for this family of polytopes, which provides a model of topological and algebraic operads up to homotopy (Theorem 3.55) and an explicit functorial formula for their tensor product (Corollary 3.61). This formula presents interesting combinatorial properties, and agrees with the magical formula for the associahedra [MTTV21, Theorem 2].

In addition to the associahedra, the operahedra contain yet another important family of polytopes: the permutahedra. The $(n-1)$-dimensional standard permutahedron is defined as the convex hull of all the permutations of $\{1, \ldots, n\}$. Closely related to various properties of the symmetric group, it has important applications in algebraic topology, appearing in the study of iterated loop spaces [Mil66], E_n-operads [Ber97] and topological Hochschild cohomology [MS03, KZ17].

In order to define a cellular approximation of the diagonal of the permutahedron, we first have to compute its fundamental hyperplane arrangement (Theorem 3.22). This new hyperplane arrangement refines the braid arrangement and deserves further study. In contrast with the cases of the simplices, the cubes and the associahedra, there are many distinct diagonals that agree with the natural order on the vertices, in this case the weak Bruhat order. So, for the first time, one has to make a choice of approximation. In the case of the operahedra, this choice is further restricted, but not completely determined, by requiring coherence with operadic composition, see Proposition 3.51.

General geometric arguments show that a choice of approximation of the diagonal for a polytope P gives a choice of approximation for any polytope Q whose normal fan coarsens the one of P (Corollary 2.30). Moreover, the universal formula for the diagonal of P applies *mutatis mutandis* to Q (Proposition 2.32). Since the normal fan of any operahedron is refined by the normal fan of the permutahedron, we restrict our attention to the latter. In fact, the preceding argument shows that the formula obtained here applies immediately to all generalized permutahedra [Pos09], which precisely are the polytopes whose normal fans coarsen the one of the permutahedron.

3.2 Realizations of the operahedra

In this section we define the operahedra, the family of polytopes that will be the center of attention for the rest of the chapter. These polytopes range from the associahedra to the permutohedra. Their face lattices are described by the combinatorics of planar nested trees. Via the line graph construction, these correspond to tubed clawfree block graphs, and the operahedra are thus instances of graph-associahedra [CD06]. We define integer-coordinate realizations of the operahedra by the same procedure as for J.-L. Loday's realizations of the associahedra [Lod04]. Their fundamental geometric properties are described in Proposition 3.15. The standard weight realizations were already studied by V. Pilaud in [Pil13], but the construction given here is different.

3.2.1 What is an operahedron?

Let us consider the set PT_n of reduced planar rooted trees with n internal vertices, for $n \geq 1$, that is trees where each internal vertex is at least bivalent. We denote the set of internal vertices of a tree $t \in \mathrm{PT}_n$ by $V(t)$ and its set of internal edges by $E(t)$, and we label them as pictured in Figure 3.1: starting from the root, we traverse around the tree in clockwise direction, numbering a vertex (resp. an edge) only the first time we see it.

The leaves and root (resp. the leaf edges and root edge) are not considered part of the set $V(t)$ (resp. $E(t)$), and from now on, the word "vertex" (resp. "edge") will refer exclusively to internal vertices (resp. edges). Moreover, we will abuse terminology and use the terms *leaves* and *root*, as well as the terms *inputs* and *output*, to designate the leaf edges and root edge, respectively.

Any subset of edges $N \subset E(t)$ defines a subgraph of t whose edges are N and whose vertices are all the vertices adjacent to an edge in N. We call this graph the *closure* of N.

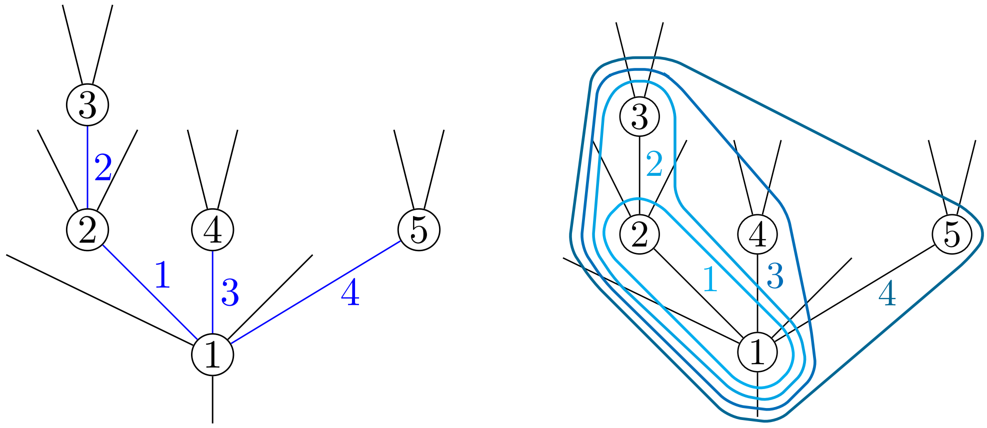


Figure 3.1: A tree $t \in \mathrm{PT}_5$ with our labeling conventions and one of its maximal nestings.

Definition 3.1 (Nest). *A* nest *of a tree $t \in \mathrm{PT}_n$ is a non-empty set of edges $N \subset E(t)$ whose closure is a connected subgraph of t.*

Every nest N thus defines a subtree $t(N)$ of t by adjoining to its closure all the edges, leaves or root adjacent to its vertices. We call it the *induced subtree of N*.

Definition 3.2 (Nesting). *A* nesting *of a tree $t \in \mathrm{PT}_n$ is a set $\mathcal{N} = \{N_i\}_{i \in I}$ of nests such that*

1. *the* trivial nest *$E(t)$ is in $\mathcal{N}$,*
2. *for every pair of nests $N_i \neq N_j$, we have either $N_i \subsetneq N_j$, $N_j \subsetneq N_i$ or $N_i \cap N_j = \emptyset$, and*
3. *if $N_i \cap N_j = \emptyset$ then no edge of N_i is adjacent to an edge of N_j.*

Two nests that satisfy Conditions (2) and (3) are said to be *compatible*. We naturally represent a nesting by circling the closure of each nest as in Figure 3.1. We denote by $\mathcal{N}(t)$ the set of nestings of a tree t. We notice that for a *corolla* $t \in \mathrm{PT}_1$ we have $\mathcal{N}(t) = \{\emptyset\}$. We call *nested tree* a pair $(t, \mathcal{N})$ made up of a tree and a nesting.

Definition 3.3 (Lattice of nestings). *We denote by $(\mathcal{N}(t), \subset)$ the poset of nestings of a tree $t \in \mathrm{PT}_n$ ordered by inclusion, together with a maximal element.*

Remark 3.4. As explained in detail in [War21, Section 3.4], a nesting of a tree t is the same as a tubing [CD06, Definition 2.2] of the line graph [Har69, Chapter 8] of the closure of $E(t)$. Note that the leaves and the root are not taken into account, so any tree t' with the same internal structure as t has the same line graph.

Definition 3.5 (Edge contraction). *The* contraction of an edge e *connecting vertices* v_1 *and* v_2 *consists in deleting* e *and collapsing* v_1 *and* v_2 *to a new vertex* v *having as inputs the union of the inputs of* v_1 *and* v_2*.*

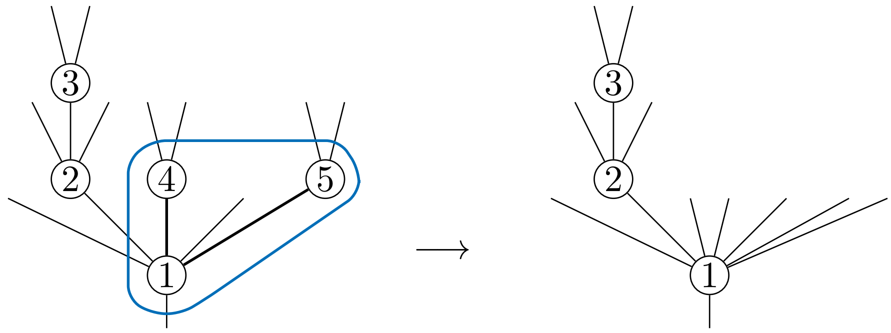


Figure 3.2: Contraction of the edges in a nest.

We observe that trees are stable under edge contraction. Given a nested tree $(t, \mathcal{N})$, the *contraction of a nest* $N \in \mathcal{N}$ consists in the contraction of all the edges of $t(N)$, as pictured in Figure 3.2.

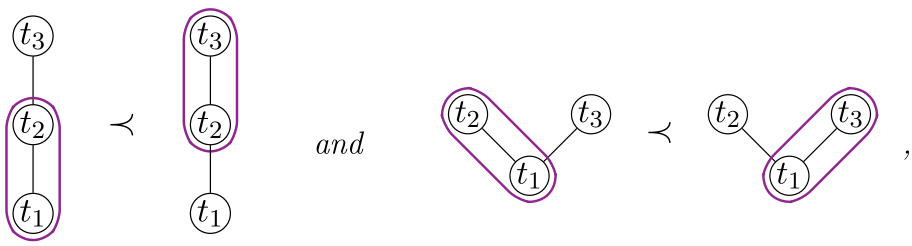
Definition 3.6 (Maximal nesting). *A nesting is* maximal *if it has maximal cardinality.*

We denote by $\mathcal{MN}(t) \subset \mathcal{N}(t)$ the set of maximal nestings. If $t \in \mathrm{PT}_n$ and $\mathcal{N} \in \mathcal{MN}(t)$, we have $|\mathcal{N}| = |E(t)| = n - 1$. We call *fully nested tree* a nested tree $(t, \mathcal{N})$ where $\mathcal{N}$ is maximal.

Definition 3.7. *For any subset of edges* $S \subset E(t)$ *of a nested tree* $(t, \mathcal{N})$*, we denote by* $\mathcal{N}(S)$ *the set of nests of* $\mathcal{N}$ *containing* S*.*

By definition of a nesting, the set $\mathcal{N}(e)$ is totally ordered by inclusion, for any edge $e \in E(t)$. For a maximal nesting $\mathcal{N} \in \mathcal{MN}(t)$, the assignment $e \mapsto \min \mathcal{N}(e)$ defines a bijection between $E(t)$ and $\mathcal{N}$.

Definition 3.8 (Poset of maximal nestings). *We denote by* $(\mathcal{MN}(t), <)$ *the poset generated by the transitive closure of the covering relations*



where t_1, t_2 *and* t_3 *are trees.*

On the set of *linear trees*, i.e. trees where each vertex is connected to at most two edges, this order relation specializes to the *Tamari order* [Tam51]. This can be seen via the bijection between the set of maximal nestings of a linear tree with n vertices and the set of planar binary trees with n leaves shown in Figure 3.3.

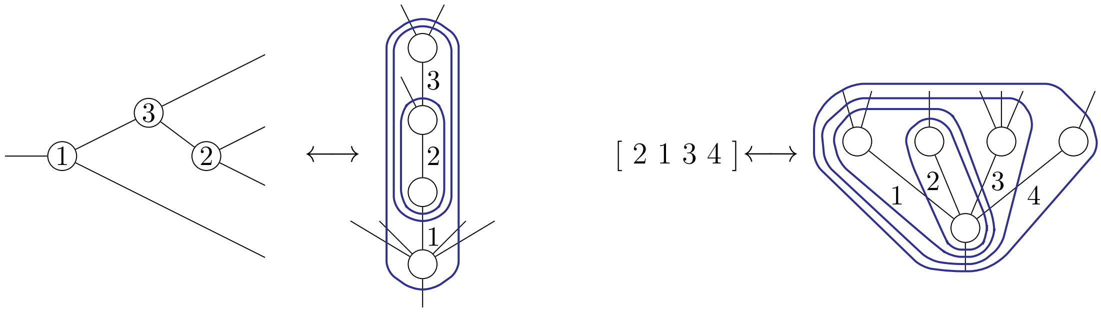


Figure 3.3: Bijections between maximal nestings of a linear tree and planar binary trees and between maximal nestings of a 2-leveled tree and permutations.

On the set of *2-leveled trees*, i.e. trees where all the edges are adjacent to the same vertex, this order specializes to the weak Bruhat order. This can be seen via the bijection between the set of maximal nestings of a 2-leveled tree with $n+1$ vertices and the elements of the symmetric group of order n shown in Figure 3.3. For a 2-leveled tree $t \in \mathrm{PT}_{n+1}$ and a maximal nesting $\mathcal{N} \in \mathcal{MN}(t)$, we construct a permutation $\sigma \in \mathbb{S}_n$ in the following way. First, for each $i \in E(t)$ we write $N_i := \min \mathcal{N}(i) \in \mathcal{N}$. Then, the image of the permutation σ is the unique ordered sequence $(\sigma(1), \ldots, \sigma(n))$ such that $|N_{\sigma(j)}| < |N_{\sigma(j+1)}|$ for all $j \in \{1, \ldots, n-1\}$. A covering relation in Definition 3.8 between two maximal nestings $\mathcal{N}$ and $\mathcal{N}'$ then corresponds precisely to a covering relation of the weak Bruhat order between the associated permutations σ and σ'.

Definition 3.9 (Operahedron)**.** *An* operahedron *is a polytope whose face lattice is isomorphic to the dual* $(\mathcal{N}(t), \subset^{\mathrm{op}})$ *of the lattice of nestings of a planar tree* $t \in \mathrm{PT}_n$*, for any* $n \geq 1$*.*

The operahedron corresponding to a tree $t \in \mathrm{PT}_n$ is of dimension $n-2$ (by convention, the empty set has dimension -1). The face corresponding to a nested tree $(t, \mathcal{N})$ has codimension $|\mathcal{N}|-1$, the number of non-trivial nests of $\mathcal{N}$. The oriented 1-skeleton of an operahedron gives the Hasse diagram of the poset of maximal nestings $(\mathcal{MN}(t), <)$.

Remark 3.10. Following Remark 3.4, one can see that the operahedra are a special class of graph-associahedra, as defined in [CD06]: they are associated to line graphs of trees, that is, clawfree block graphs [Har69, Theorem 8.5]. Hence they are also a special class of hypergraph polytopes as defined in [DP11].

Remark 3.11. It would be interesting to know whether or not the posets $(\mathcal{MN}(t), <)$ are lattices. As studied in [BM21], the poset of maximal tubings of a graph do not form a lattice in general. For linear and 2-leveled trees, we have lattices isomorphic to the Tamari and weak Bruhat order lattices, respectively. For the other operahedra, comparison with the calculations of [BM21, Section 6.1] shows that we indeed have lattices up to dimension 3.

3.2.2 Loday realizations of the operahedra

Definition 3.12 (Weighted fully nested tree)**.** *A* weighted fully nested tree *is a triple $(t, \mathcal{N}, \omega)$ made up of a fully nested tree with n vertices together with a weight $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}^n_{>0}$. We say that the weight ω has* length *n.*

Let us fix a weighted fully nested tree $(t, \mathcal{N}, \omega)$. For any edge $i \in E(t)$, we consider the two subtrees t_1 and t_2 of $t(\min\mathcal{N}(i))$ such that i is the root of t_1 and a leaf of t_2. In other words, t_1 and t_2 are obtained by cutting the tree $t(\min\mathcal{N}(i))$ at the edge i. We define the two sums

$$\alpha_i := \sum_{j \in V(t_1)} \omega_j \quad \text{and} \quad \beta_i := \sum_{j \in V(t_2)} \omega_j \ .$$

Multiplying these two numbers together for each edge of t, we obtain the following point

$$M(t, \mathcal{N}, \omega) := \big(\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_{n-1}\beta_{n-1}\big) \in \mathbb{Z}^{n-1} \ .$$

Remark 3.13. We will use the notations α_i and β_i for brevity, even though these numbers depend on the tree t, the nesting $\mathcal{N}$ and the weight ω. This dependence will be implicit but should be clear from the context.

Definition 3.14 (Loday realization of the operahedra)**.** *For any $n \geq 1$, and for any tree $t \in \mathrm{PT}_n$, the* Loday realization of weight ω of the operahedron *is the polytope*

$$P_{(t,\omega)} := \mathrm{conv}\left\{M(t, \mathcal{N}, \omega) \mid \mathcal{N} \in \mathcal{MN}(t)\right\} \subset \mathbb{R}^{n-1} \ .$$

The Loday realization of the operahedron associated to the standard weight $(1, \ldots, 1)$ is simply denoted by P_t. Some three-dimensional examples are shown in Figure 3.4. For any corolla $t \in \mathrm{PT}_1$, we adopt the following convention: the polytope $P_{(t,\omega)}$, with weight $\omega = (\omega_1)$ of length 1, is made up of one point in 0-dimensional space.

The following proposition summarizes the fundamental properties of Loday realizations of the operahedra and show in particular that they are indeed realizations of the operahedra. In the case of standard weight realizations, it should be compared with [Pil13, Theorem 56].

Proposition 3.15. *For any tree $t \in \mathrm{PT}_n$ and for any weight ω of length n, the Loday realization of the operahedron $P_{(t,\omega)}$ satisfies the following properties.*

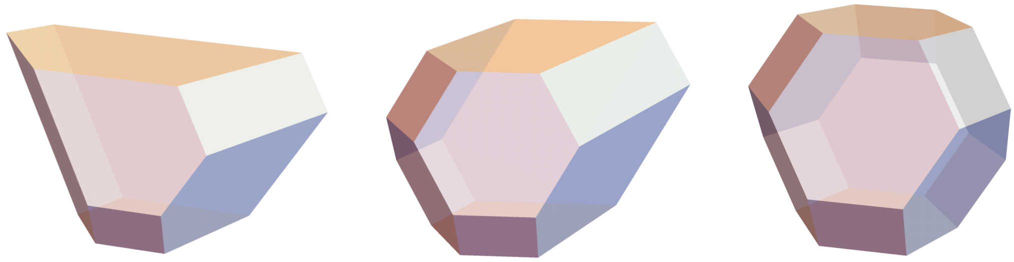


Figure 3.4: Standard weight Loday realizations of some 3-dimensional operahedra, from the associahedron (left) to the permutahedron (right).

1. *It is contained in the hyperplane with equation*

$$\sum_{i \in E(t)} x_i = \sum_{\substack{k,\ell \in V(t) \\ k<\ell}} \omega_k \omega_l \ .$$

2. *Let N be a non-trivial nest of t. For any maximal nesting $\mathcal{N}$, the point $M(t, \mathcal{N}, \omega)$ is contained in the half-space defined by the inequality*

$$\sum_{i \in E(t(N))} x_i \geq \sum_{\substack{k,\ell \in V(t(N)) \\ k<\ell}} \omega_k \omega_l \ ,$$

with equality if and only if $N \in \mathcal{N}$.

3. *The polytope $P_{(t,\omega)}$ is the intersection of the hyperplane of* (1) *and the half-spaces of* (2)*.*

4. *The face lattice $(\mathcal{L}(P_{(t,\omega)}) \ , \subset)$ is isomorphic to the dual of the lattice of nestings $(\mathcal{N}(t) \ , \subset^{\mathrm{op}})$.*

5. *Any face of a Loday realization of an operahedron is isomorphic to a product of Loday realizations of operahedra of lower dimension, via a permutation of coordinates.*

Proof.

1. We show that every nest N of a maximal nesting $\mathcal{N}$ satisfies the equation

$$\sum_{i \in E(t(N))} \alpha_i \beta_i = \sum_{\substack{k,\ell \in V(t(N)) \\ k<\ell}} \omega_k \omega_l \ ,$$

by induction on the cardinality of N. The case when $|N| = 1$ is clear. We suppose that every nest $N \in \mathcal{N}$ with $1 \leq |N| \leq m-1$ satisfies the equation above. We consider now a nest N with $|N| = m \geq 2$. We select $j \in N$ the unique edge such that $N = \min \mathcal{N}(j)$. Denoting by t_1 and t_2 the two subtrees of $t(N)$ having j respectively as a root and a leaf, we have

$$\begin{aligned}
\sum_{i \in E(t(N))} \alpha_i \beta_i &= \alpha_j \beta_j + \sum_{i \in E(t_1)} \alpha_i \beta_i + \sum_{i \in E(t_2)} \alpha_i \beta_i \\
&= \left(\sum_{k \in V(t_1)} \omega_k \right) \left(\sum_{\ell \in V(t_2)} \omega_\ell \right) + \sum_{\substack{k,\ell \in V(t_1) \\ k<\ell}} \omega_k \omega_l + \sum_{\substack{k,\ell \in V(t_2) \\ k<\ell}} \omega_k \omega_l \\
&= \sum_{\substack{k,\ell \in V(t(N)) \\ k<\ell}} \omega_k \omega_l \ .
\end{aligned}$$

Taking the trivial nest $N = E(t)$, which is contained in every maximal nesting, we obtain that every point $M(t, \mathcal{N}, \omega)$ is contained in the hyperplane of (1). By convexity, the same is true for the entire polytope.

2. The proof of Point (1) shows that if the nest N is in $\mathcal{N}$, then

$$\sum_{i \in E(t(N))} \alpha_i \beta_i = \sum_{\substack{k,\ell \in V(t(N)) \\ k<\ell}} \omega_k \omega_l \ .$$

Let us show that every nest $N \notin \mathcal{N}$ satisfies the strict inequality

$$\sum_{i \in E(t(N))} \alpha_i \beta_i > \sum_{\substack{k,\ell \in V(t(N)) \\ k<\ell}} \omega_k \omega_l \ ,$$

by induction on the cardinality of N. The case when $|N| = 1$ is clear. We suppose that every nest $N \notin \mathcal{N}$ with $1 \leq |N| \leq m-1$ satisfies the strict inequality above. We consider now a nest $N \notin \mathcal{N}$ with $|N| = m \geq 2$. We select j the unique edge such that $\min \mathcal{N}(j) = \min \mathcal{N}(N)$. It is clear that this edge exists and is unique. We denote by t_1 and t_2 the two subtrees of $t(\min \mathcal{N}(j))$ having j respectively as a root and a leaf.

If we suppose that $j \notin N$, then $N \subset E(t_1)$ or $N \subset E(t_2)$ which contradicts the assumption that $\min \mathcal{N}(j) = \min \mathcal{N}(N)$. So we have $j \in N$. We denote by t'_1 and t'_2 the two subtrees of $t(N)$ having j respectively as a root and a leaf. At least one of the inclusions $E(t'_1) \subset E(t_1)$ or $E(t'_2) \subset E(t_2)$ has to be strict, otherwise we would have $N = \min \mathcal{N}(N) \in \mathcal{N}$. Thus we have

$$\begin{aligned}
\sum_{i\in E(t(N))} \alpha_i\beta_i &= \alpha_j\beta_j + \sum_{i\in E(t'_1)} \alpha_i\beta_i + \sum_{i\in E(t'_2)} \alpha_i\beta_i \\
&\geq \left(\sum_{k\in V(t_1)} \omega_k\right)\left(\sum_{\ell\in V(t_2)} \omega_\ell\right) + \sum_{\substack{k,\ell\in V(t'_1)\\ k<\ell}} \omega_k\omega_l + \sum_{\substack{k,\ell\in V(t'_2)\\ k<\ell}} \omega_k\omega_l \\
&> \left(\sum_{k\in V(t'_1)} \omega_k\right)\left(\sum_{\ell\in V(t'_2)} \omega_\ell\right) + \sum_{\substack{k,\ell\in V(t'_1)\\ k<\ell}} \omega_k\omega_l + \sum_{\substack{k,\ell\in V(t'_2)\\ k<\ell}} \omega_k\omega_l \\
&= \sum_{\substack{k,\ell\in V(t(N))\\ k<\ell}} \omega_k\omega_l \ .
\end{aligned}$$

3. Let us denote by P the polytope defined by the intersection of the hyperplane of (1) and the half-spaces of (2). We show that $P_{(t,\omega)} = P$. The first inclusion ($\subset$) is obvious. For the reverse inclusion, we observe first that the equations of Point (2), with equality, define the facets of P. Let $x = (x_1, \ldots, x_{n-1})$ be a point in the intersection of two facets F_1 and F_2 of P. We claim that the associated nests N_1 and N_2 are compatible. We suppose to the contrary that the nests N_1 and N_2 are not compatible. We are in one of the following two situations. First, we suppose that $N_1 \cap N_2 = \emptyset$. We have by the proof of Point (1) that

$$\begin{aligned}
\sum_{i\in E(t(N_1\cup N_2))} x_i &= \sum_{i\in E(t(N_1))} x_i + \sum_{i\in E(t(N_2))} x_i \\
&= \sum_{\substack{k,\ell\in V(t(N_1))\\ k<\ell}} \omega_k\omega_l + \sum_{\substack{k,\ell\in V(t(N_2))\\ k<\ell}} \omega_k\omega_l \\
&< \sum_{\substack{k,\ell\in V(t(N_1))\\ k<\ell}} \omega_k\omega_l + \sum_{\substack{k,\ell\in V(t(N_2))\\ k<\ell}} \omega_k\omega_l + \sum_{\substack{k\in V(t(N_1))\setminus V(t(N_2))\\ \ell\in V(t(N_2))\setminus V(t(N_1))}} \omega_k\omega_l \\
&= \sum_{\substack{k,\ell\in V(t(N_1\cup N_2))\\ k<\ell}} \omega_k\omega_l \ ,
\end{aligned}$$

which contradicts the inequality of Point (2) associated to the nest $N_1 \cup N_2$. Second, we suppose that $N_1 \cap N_2 \neq \emptyset$. In particular, since N_1 and N_2 are not compatible, we have $N_1 \not\subseteq N_2$ and $N_2 \not\subseteq N_1$. So, we have

$$\begin{aligned}
\sum_{i\in E(t(N_1\cap N_2))} x_i &= \sum_{i\in E(t(N_1))} x_i + \sum_{i\in E(t(N_2))} x_i - \sum_{i\in E(t(N_1\cup N_2))} x_i \\
&\leq \sum_{\substack{k,\ell\in V(t(N_1))\\ k<\ell}} \omega_k\omega_l + \sum_{\substack{k,\ell\in V(t(N_2))\\ k<\ell}} \omega_k\omega_l - \sum_{\substack{k,\ell\in V(t(N_1\cup N_2))\\ k<\ell}} \omega_k\omega_l \\
&= \sum_{\substack{k,\ell\in V(t(N_1\cap N_2))\\ k<\ell}} \omega_k\omega_l - \sum_{\substack{k\in V(t(N_1))\setminus V(t(N_2))\\ \ell\in V(t(N_2))\setminus V(t(N_1))}} \omega_k\omega_l \\
&< \sum_{\substack{k,\ell\in V(t(N_1\cap N_2))\\ k<\ell}} \omega_k\omega_l \ ,
\end{aligned}$$

which contradicts the inequality of Point (2) associated to the nest $N_1 \cap N_2$. Thus, N_1 and N_2 must be compatible.

A vertex M of P is solution to a system of $n-1$ independent linear equations, one of type (1) and $n-2$ of type (2). By the preceding argument, the associated nests are compatible and assemble into a maximal nesting $\mathcal{N}$ of t. Also the point $M(t,\mathcal{N},\omega)$ is solution to this system of equations, in virtue of Point (1) and Point(2). Since the solution is unique, this implies $M = M(t,\mathcal{N},\omega)$ and therefore $P = P_{(t,\omega)}$.

4. Point (2) shows that the facets of $P_{(t,\omega)}$ correspond bijectively to nestings with only one non-trivial nest: the facet labeled by the non-trivial nest N is the convex hull of the points $M(t,\mathcal{N},\omega)$ such that $N \in \mathcal{N}$. Any face of $P_{(t,\omega)}$ of codimension k, with $0 \leq k \leq n-2$ is defined as the intersection of k facets. The preceding description of facets gives that the set of faces of codimension k is bijectively labeled by nestings with k non-trivial nests: the face corresponding to such a nesting $\mathcal{N}$ is the convex hull of the points $M(t,\mathcal{N}',\omega)$ such that $\mathcal{N}' \subset \mathcal{N}$. With the top dimensional face labeled by the trivial nest, the statement is proved.

5. The proof of the preceding point shows that it is enough to treat the case of the facets. Let $\mathcal{N}$ be a nesting of t with only one non-trivial nest N. We contract the nest N to obtain a new tree $\bar{t}$. We define a weight $\overline{\omega}$ on $\bar{t}$ as follows. As a result of the contraction of N, the set $V(t(N))$ is reduced in $\bar{t}$ to a single vertex j. We assign to this vertex the sum of the weights of the vertices of N, that is,
$$\overline{\omega_j} := \sum_{k\in V(t(N))} \omega_k \ .$$

Each of the other vertices keeps its weight, only the label changes: for each $i \neq j$ in $V(\bar{t})$, we define $\overline{\omega_i} := \omega_\ell$ for the corresponding vertex ℓ in $V(t)$. We also define a weight $\widetilde{\omega}$ on $t(N)$, considered as an independent tree that we denote by $\widetilde{t}$. This weight is simply the restriction of ω to the vertices of $t(N)$: for each $i \in V(\widetilde{t})$, we define $\widetilde{\omega}_i := \omega_\ell$ for the corresponding vertex ℓ in $V(t)$.

We write $|E(\bar{t})| = p$ and $|E(\widetilde{t})| = q$ and we renumber the edges of $\widetilde{t}$ from $p+1$ to $p+q$. We denote by $\sigma : E(\bar{t}) \sqcup E(\widetilde{t}) \to E(t)$ the permutation mapping each (just renumbered) edge of $\bar{t}$ and $\widetilde{t}$ to its label in t. We obtain a (p,q)-shuffle. We claim that the image of $P_{(\bar{t},\overline{\omega})} \times P_{(\widetilde{t},\widetilde{\omega})} \hookrightarrow P_{(t,\omega)}$ under the isomorphism

$$\begin{array}{cccc} \Theta \ : & \mathbb{R}^p \times \mathbb{R}^q & \xrightarrow{\cong} & \mathbb{R}^{n-1} \\ & (x_1, \ldots, x_p) \times (x_{p+1}, \ldots, x_{p+q}) & \mapsto & (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n-1)}) \end{array}$$

is equal to the facet defined by the weighted nested tree $(t, \mathcal{N}, \omega)$. To see this, we recall that the two polytopes $P_{(\bar{t},\overline{\omega})}$ and $P_{(\widetilde{t},\widetilde{\omega})}$ are defined by the equations

$$\sum_{i \in E(\bar{t})} x_i \overset{(a)}{=} \sum_{\substack{k,\ell \in V(\bar{t}) \\ k<\ell}} \overline{\omega}_k \overline{\omega}_l \quad \text{and} \quad \sum_{i \in E(\widetilde{t})} x_i \overset{(b)}{=} \sum_{\substack{k,\ell \in V(\widetilde{t}) \\ k<\ell}} \widetilde{\omega}_k \widetilde{\omega}_l \ ,$$

respectively, and observe that the image under Θ of the pair of equations $(a) + (b)$ and (b) consists exactly in the equations defining the facet labelled by $(t, \mathcal{N}, \omega)$.

□

Restricting to linear trees, we recover the weighted Loday realizations of the associahedra of [MTTV21, Proposition 1]. Restricting to 2-leveled trees, we obtain weighted realizations of the permutahedron. To end this section, let us point out some geometric properties of the Loday realizations of the operahedra. They can be visualized on the examples of Figure 3.4.

Corollary 3.16. *For any tree $t \in \mathrm{PT}_n$ and for any weight ω of length n, the Loday realizations of the operahedron P_t and $P_{(t,\omega)}$ satisfy the following geometric properties.*

1. *The polytope $P_{(t,\omega)}$ is obtained by successive truncations of a simplex.*
2. *The polytope $P_{(t,\omega)}$ is obtained from the classical permutahedron by parallel translation of its facets, i.e. it is a* generalized permutahedron *in the sense of [Pos09].*
3. *The polytope P_t is obtained by deleting inequalities from the facet description of the classical permutahedron, i.e. it is a* removahedron *in the sense of [Pil14].*

Proof. One can read off the normal fan of the operahedron $P = P_{(t,\omega)}$ in Points (1) and (2) of Proposition 3.15 as follows. A face F of codimension k of P is determined by a nesting $\mathcal{N} = \{N_j\}_{1\leq j\leq k+1}$ of t, where N_{k+1} is the trivial nest. For any nest $N_j \in \mathcal{N}$, we define its associated *characteristic vector* $\vec{N}_j$ which has a 1 in position i if $i \in N_j$ and 0 otherwise. The normal cone of F is then given by $\mathcal{N}_P(F) = \mathrm{Cone}(-\vec{N}_1, \ldots, -\vec{N}_{k+1}, \vec{N}_{k+1})$. If t is a 2-leveled tree, all the subsets of edges define nests, and we have the normal fan of the permutahedron. If t is a tree which is not a 2-leveled tree, only some subsets of edges define nests. Thus, we have Points (2) and (3) above. For Point (1), we observe that the nests containing only one edge of t define the normal fan of the standard simplex. □

3.3 The diagonal of the operahedra

The main goal of this section is to compute the fundamental hyperplane arrangement of the permutahedron which, as we shall see, turns out to be a refinement of the braid arrangement. By the general theory of Section 2.2, any choice of a chamber in this arrangement then gives a cellular approximation of the diagonal of the permutahedron. Moreover, such a choice gives a cellular approximation of the diagonal for every operahedron (in fact, any generalized permutahedra), as well as an explicit combinatorial formula describing its cellular image. In contrast with the cases of the simplices, the cubes, and the associahedra, the combinatorics of the permutahedron are less constrained: many choices of chambers agree with the weak Bruhat order on the vertices, and the condition $\mathrm{top}(F) \leq \mathrm{bot}(G)$ is no longer sufficient to characterize the image of the diagonal. We make a choice, motivated by the operadic structure that will appear in the next section. The formula thus obtained, which consists of complementary pairs of ordered partitions of $\{1, \ldots, n\}$, has interesting combinatorial properties.

3.3.1 The fundamental hyperplane arrangement of the permutahedra

Let us first recall from the proof of Corollary 3.16 above that a face F of codimension k of the operahedron $P = P_t$ is determined by a nesting $\mathcal{N} = \{N_j\}_{1\leq j\leq k+1}$ of t where N_{k+1} is the trivial nest. For any nest $N_j \in \mathcal{N}$, we define its associated *characteristic vector* $\vec{N}_j$ which has a 1 in position i if $i \in N_j$ and 0 otherwise. The vectors $-\vec{N}_j$, $1 \leq j \leq k$ are outward pointing normal vectors for the facets defining $\mathcal{N}$, in the sense of Definition 2.24. Together with the vector $\vec{N}_{k+1}$, which forms a basis of the orthogonal complement of the affine hull of P, they define the normal cone

$$\mathcal{N}_P(F) = \mathrm{Cone}\left(-\vec{N}_1, \ldots, -\vec{N}_k, -\vec{N}_{k+1}, \vec{N}_{k+1}\right) .$$

Definition 3.17 (Trinary and boolean vectors)**.** *We say that a vector $\vec{v} \in \mathbb{R}^{n-1}$ is a* trinary vector *(resp.* boolean*) if its coordinates are 0, 1 or -1 (resp. 0 or 1).*

Let us recall that two nests N_1 and N_2 are said to be *compatible* if they fulfill Conditions (2) and (3) of Definition 3.2. Moreover, we say that they are *linearly independent* if $\vec{N}_1$ and $\vec{N}_2$ are.

Proposition 3.18. *Let $t \in \mathrm{PT}_n$ and let us denote by $P = P_t$ the standard weight Loday realization of the operahedron. There is a surjection*

$$\left\{ \begin{array}{c} \textit{a set of } k \textit{ compatible nests,} \\ \textit{a set of } l \textit{ compatible nests,} \\ \textit{with } k+l = n-3 \textit{ and } k,l \geq 0 \text{ ,} \\ \textit{mutually linearly independent and with the trivial nest} \end{array} \right\}$$
$$\twoheadrightarrow \left\{ \begin{array}{c} \textit{direction } \vec{d} \textit{ of an edge} \\ \textit{of } P \cap \rho_z P, \textit{ for some } z \in P \end{array} \right\} \Big/ \sim \quad ,$$

where two directions in the target are identified if they are a scalar multiple of each other.

Proof. This follows from a direct application of Proposition 2.21. □

Definition 3.19 (Support and length)**.** *The set of non-zero entries of a vector $\vec{v} \in \mathbb{R}^n$ is called its* support *and the cardinality of this set is called its* length.

Proposition 3.20 (Direction of the edges of $P \cap \rho_z P$)**.** *Let $t \in \mathrm{PT}_n$ and let $P = P_t$ be the standard weight Loday realization of the operahedron. Then, representatives for the equivalence classes of directions of the edges of $P \cap \rho_z P$, for all $z \in P$, are given by trinary vectors of $\mathbb{R}^{n-1}$ with the same number of 1 and -1 and whose first non-zero coordinate is 1.*

Proof. The space of solutions of the system of linear equations in the left hand side of the surjection in Proposition 3.18 is given by the kernel of the $(n-1) \times (n-1)$ boolean matrix

$$\begin{pmatrix} — & \vec{N}_1 & — \\ — & \vdots & — \\ — & \vec{N}_{k+1} & — \\ — & \vec{N}'_1 & — \\ — & \vdots & — \\ — & \vec{N}'_{l+1} & — \end{pmatrix} , \tag{3.1}$$

where the vectors are written horizontally. The $k+1$ first (resp. $l+1$ last) lines are included in one another as elements of the boolean lattice $\{0,1\}^{n-1}$. We can thus substract the line of minimal length to the k (resp. l) others, then the line with second minimal length to the $k-1$ (resp. $l-1$) others, and so on until we obtain a family of $k+1$ (resp. $l+1$) lines with disjoint support, whose sum is $(1, \ldots, 1)$.

We claim that the system of linear equations obtained in this way is equivalent to a system where the length of each line is at most 2, that is, where each line has a 1 in at most two places. We proceed by induction on n. The case $n-1=2$ is clear. Let $n-1 \geq 3$ and suppose that the result holds for every matrix of size $k \leq n-2$. Let M be a matrix of size $n-1$ filling the hypothesis.

1. Suppose that M contains a line of length 1, that is a line i with zeros everywhere except in place j. We can then reduce every non-zero element of the jth column to 0 and apply the induction hypothesis to the $(n-2) \times (n-2)$ matrix M' obtained from M by suppressing its ith line and jth column.

2. Suppose that no line of M has length 1.

   (a) Suppose that $k > l$. The length of the sum of the $k+1$ lines of the first group is at least $2k+2 > k+l+2 = n-1$, which is impossible.

   (b) Suppose that $k = l$. The length of the sum of the $k+1$ lines of the first group is, as for the $l+1$ lines of the second group, exactly $2k+2$, which means that every line has length 2. This finishes the proof of the claim.

The kernel of (3.1) has dimension 1. Since the vector $(1, \ldots, 1)$ is in the system, the coordinates of any non-zero vector in it sum to zero. By the preceding claim, it is a scalar multiple of a trinary vector with the same number of 1 and -1, and whose first non-zero coordinate is 1. □

Corollary 3.21. *Let $t \in \mathrm{PT}_n$ be a 2-leveled tree, and let us denote by $P = P_t$ the standard weight Loday realization of the permutahedron. There is a bijection*

$$\left\{ \begin{array}{c} \textit{direction } \vec{d} \textit{ of an edge} \\ \textit{of } P \cap \rho_z P, \textit{ for some } z \in P \end{array} \right\}_{/\sim} \cong \left\{ \begin{array}{c} \textit{trinary vector of } \mathbb{R}^{n-1} \\ \textit{with the same number of } 1 \textit{ and } -1 \\ \textit{whose first non-zero coordinate is } 1 \end{array} \right\},$$

where, in the first set, two linearly dependent directions are identified.

Proof. We prove that every trinary vector on the right-hand side is a representative of some equivalence class of directions on the left-hand side. Let $\vec{d} \in \mathbb{R}^{n-1}$ be a vector having p coordinates equal to 1, p coordinates equal to -1 and q coordinates equal to 0 with $p \geq 1, q \geq 0$ and $2p+q = n-1$. We construct a system of nested boolean vectors $\{\vec{N}_1, \ldots, \vec{N}_p, \vec{N}'_1, \ldots, \vec{N}'_{q+p}\}$ that has $\vec{d}$ as solution. First label the pairs of $\{1, -1\}$ from left to right with $\{1, \ldots, p\}$ and the zeros, if there are any, from left to right with $\{1, \ldots, q\}$. Then,

1. If $p = 1$, go directly to Step (2). If $p \geq 2$, define $p-1$ boolean vectors $\{\vec{N}_1, \vec{N}_2, \ldots, \vec{N}_{p-1}\}$ by the following: the vector $\vec{N}_i$ has as support the columns of the ith first pairs of 1 and -1.

(2) If $p = 1$ and $q = 0$, go directly to Step (3). Otherwise, define $q+p-1$ boolean vectors $\{\vec{N}'_1, \vec{N}'_2, \ldots, \vec{N}'_{q+p-1}\}$ by the following: the vector $\vec{N}'_j$ has as support the columns of the jth first zeros if $j \leq q$; otherwise for $j \geq q+1$ (that is, if $p \geq 2$) it has as support the columns of the q zeros, the 1 of the first pair, the -1 of the $(j-q+1)$th pair, and if they exist (that is, if $p \geq 3$ and $j \geq q+2$) all the pairs from 2 to $j-q$.

(3) Set $\vec{N}_p = \vec{N}'_{q+p} = (1, \ldots, 1)$ and add the two vectors to the system.

Choosing such vectors is possible since the permutahedron has as normal vectors to its facets every possible non-zero boolean vector, see Point (2) of Proposition 3.15. It is clear from the construction that the vector $\vec{d}$ is a basis of the space of solutions to this system, see Figure 3.5. □

$$\vec{d} = \begin{pmatrix} 1 & 0 & -1 & -1 & 1 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} — & \vec{N}_1 & — \\ — & \vec{N}_2 & — \\ — & \vec{N}_3 & — \\ — & \vec{N}'_1 & — \\ — & \vec{N}'_2 & — \\ — & \vec{N}'_3 & — \\ — & \vec{N}'_4 & — \\ — & \vec{N}'_5 & — \\ — & \vec{N}'_6 & — \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 3.5: The result of the procedure described in the proof of Corollary 3.21 for a vector $\vec{d}$ with $p = 3$ pairs of 1 and -1, and $q = 3$ zeros.

Among the fundamental hyperplane arrangements of the operahedra, the one associated to the permutahedron plays a special role.

Theorem 3.22 (Fundamental hyperplane arrangement of the permutahedron)**.** *Let $n \geq 1$, and let us write*

$$D(n) := \{(I, J) \mid I, J \subset \{1, \ldots, n\}, |I| = |J|, I \cap J = \emptyset, \min(I \cup J) \in I\}.$$

The fundamental hyperplane arrangement of the $(n-1)$*-dimensional permutahedron in* $\mathbb{R}^n$ *is the set of hyperplanes defined by*

$$\sum_{i\in I} x_i = \sum_{j\in J} x_j \quad \textit{for all } (I,J) \in D(n) \ .$$

Proof. This follows immediately from Corollary 3.21. □

This hyperplane arrangement is a refinement of the braid arrangement, see Figure 3.6. Computations show that it is not a simplicial arrangement in $\mathbb{R}^5$, and it seems likely that this is the case for all $\mathbb{R}^n$, $n \geq 5$.

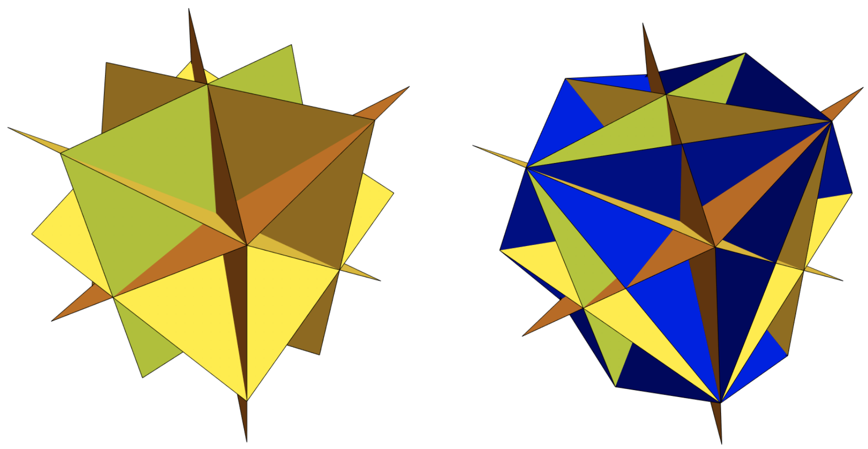


Figure 3.6: The braid arrangement and fundamental hyperplane arrangement of the permutohedron in $\mathbb{R}^4$, projected into $\mathbb{R}^3$.

Remark 3.23. The fundamental hyperplane arrangement appears as the normal fan of a zonotope, which is itself a facet of the zonotope denoted $H_\infty(d,1)$ in [DPR21], the facet contained in the hyperplane $x_1 + \cdots + x_d = 0$. This last zonotope is related to matroid optimization [DMO18] and generalizes L. Billera's *White Whale* [Bil18], which has been the subject of active research in the recent years.

For a tree t, we denote by $\mathcal{H}_t$ the fundamental hyperplane arrangement of P_t.

Proposition 3.24. *Let* $t, t' \in \mathrm{PT}_n$ *such that* t' *is a 2-leveled tree. We have* $\mathcal{H}_t \subset \mathcal{H}_{t'}$*, and if* $P_{t'}$ *is positively oriented by* $\vec{v}$*, then so is* P_t*.*

Proof. This is an immediate consequence of Proposition 3.20 and Corollary 3.21. Alternatively, it is a special case of the general results Proposition 2.29 and Corollary 2.30. □

Theorem 3.25. *Let $t \in \mathrm{PT}_n$ be a tree. Any vector $\vec{v} = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n-1}$ satisfying the equations*

$$\sum_{i \in I} v_i \neq \sum_{j \in J} v_j$$

for all $(I, J) \in D(n-1)$ defines a bot-top diagonal of P_t.

Proof. This follows directly from Proposition 3.24. □

3.3.2 Universal formula for the operahedra

We restrict our attention to a certain class of orientation vectors.

Definition 3.26 (Well-oriented realization of the operahedron)**.** *Let $t \in \mathrm{PT}_n$ be a tree. A* well-oriented realization of the operahedron *is a positively oriented realization which also induces the poset of maximal nestings $(\mathcal{MN}(t), <)$ on the set of vertices.*

Proposition 3.27. *Let $t \in \mathrm{PT}_n$ and let ω be a weight of length n. Any vector $\vec{v} \in \mathbb{R}^{n-1}$ with strictly decreasing coordinates induces the poset of maximal nestings $(\mathcal{MN}(t), <)$ on the set of vertices of $P_{(t,\omega)}$.*

Proof. Let $\mathcal{N}$ and $\mathcal{N}'$ be two maximal nestings of t corresponding to a covering relation $\mathcal{N} \prec \mathcal{N}'$. They differ only by a nest. We show that the corresponding edge in $P_{(t,\omega)}$ is of the form

$$\overrightarrow{M(t, \mathcal{N}, \omega) M(t, \mathcal{N}', \omega)} = (0, \ldots, 0, x, 0, \ldots, 0, -x, 0, \ldots, 0) \qquad (*)$$

for some $x > 0$. We denote by N the unique nest of $\mathcal{N} \setminus \mathcal{N}'$ and by N' the unique nest of $\mathcal{N}' \setminus \mathcal{N}$. Let j and j' be the two edges of t such that $\min \mathcal{N}(j) = N$ and $\min \mathcal{N}'(j') = N'$, see Definition 3.7. Now, by the definition of the order on the edges of t we have $j < j'$, see Figure 3.1 and Definition 3.8. We denote by $\alpha'_i \beta'_i$ the ith coordinate of the point $M(t, \mathcal{N}', \omega)$. The fact that $\min \mathcal{N}(i) = \min \mathcal{N}'(i)$ for all $i \neq j, j'$ implies that $\alpha_i \beta_i = \alpha'_i \beta'_i$ for all $i \neq j, j'$. We show that $\alpha_j \beta_j < \alpha'_j \beta'_j$. Since the nestings $\mathcal{N}$ and $\mathcal{N}'$ are maximal, we have $\min \mathcal{N}(j') = \min \mathcal{N}'(j)$ and

$$\alpha_j \beta_j = \left(\sum_{k \in V(t_1)} \omega_k \right) \left(\sum_{\ell \in V(t_2)} \omega_\ell \right) < \left(\sum_{k \in V(t_1)} \omega_k \right) \left(\sum_{\ell \in V(t_2)} \omega_\ell + \sum_{\ell \in V(t_3)} \omega_\ell \right) = \alpha'_j \beta'_j \ ,$$

where t_1, t_2 and t_3 are trees with possibly only one vertex, see Definition 3.8. Moreover, since $\sum_{i \in E(t)} \alpha_i \beta_i = \sum_{i \in E(t)} \alpha'_i \beta'_i$ is constant, we must have $\alpha_j \beta_j + \alpha_{j'} \beta_{j'} = \alpha'_j \beta'_j + \alpha'_{j'} \beta'_{j'}$. Defining $x := \alpha_j \beta_j - \alpha'_j \beta'_j$ we obtain $\alpha'_{j'} \beta'_{j'} - \alpha_{j'} \beta_{j'} = -x$, which proves $(*)$. So $\left\langle \overrightarrow{M(t, \mathcal{N}, \omega) M(t, \mathcal{N}', \omega)}, \vec{v} \right\rangle = x(v_j - v_{j'}) > 0$, and $\vec{v}$ induces the poset of maximal nestings on the set of vertices. □

REMARK 3.28. This poset and an orientation vector inducing it were studied in more depth in [Pil13, Section 6]. In particular, it is shown that as soon as t is not a linear tree, the poset $(\mathcal{MN}(t), <)$ is never a quotient of the weak order, see [Pil13, Proposition 86].

Combining the results of Theorem 3.25 and Proposition 3.27, we obtain the following one.

Corollary 3.29. *Let $t \in \mathrm{PT}_n$ be a tree and let ω be a weight of length n. Any vector $\vec{v} = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n-1}$ satisfying $v_1 > v_2 > \cdots > v_{n-1}$ and $\sum_{i\in I} v_i \neq \sum_{j\in J} v_j$, for all sets of indices $(I, J) \in D(n-1)$, induces a well-oriented realization of the operahedron $(P_{(t,\omega)}, \vec{v})$.*

REMARK 3.30. Following Proposition 2.22, one can wonder how many distinct well-oriented realizations of a given operahedron P_t exist, i.e. how many chambers in $\mathcal{H}_t$ induce a well-oriented realization of P_t. For a linear tree, there is only one such chamber [MTTV21, Proposition 3]. In the case of the permutahedra of dimensions 2, 3 and 4 there are respectively 1, 2 and 12 such chambers. It would be interesting to count the number of chambers in higher dimensions.

Now, we make a coherent choice of cellular approximations of the diagonal of the operahedra. By the preceding results, this amounts to a coherent choice of chambers in the fundamental hyperplane arrangement of the permutahedra. We are motivated by the perspective of endowing the family of standard weight Loday realizations of the operahedra with a topological cellular operad structure, which will be done in the next section.

Definition 3.31. *A* principal orientation vector *is a vector $\vec{v} \in \mathbb{R}^{n-1}$ such that $\sum_{i\in I} v_i > \sum_{j\in J} v_j$ for all $(I, J) \in D(n-1)$.*

Theorem 3.32 (Universal formula for the operahedra)**.** *Let t be a tree with n vertices, let ω be a weight of length n, and let $P = P_{(t,\omega)}$ denote the Loday realization of the operahedron. Let $\vec{v}$ be a principal orientation vector. For F, G two faces of P with associated nestings $\mathcal{N}$ and $\mathcal{N}'$, we have*

$$(F, G) \in \operatorname{Im} \triangle_{(P,\vec{v})} \iff \begin{array}{l} \forall (I, J) \in D(n-1), \exists N \in \mathcal{N}, |N \cap I| > |N \cap J| \text{ or} \\ \exists N' \in \mathcal{N}', |N' \cap I| < |N' \cap J| \ . \end{array}$$

Proof. To any $(I, J) \in D(n-1)$, and thus to any hyperplane H in the fundamental hyperplane arrangement of the permutohedron, we associate a normal vector $\vec{d_H} \in \mathbb{R}^{n-1}$ by setting $(d_H)_i = 1$ for $i \in I$, $(d_H)_j = -1$ for $j \in J$ and $(d_H)_k = 0$ otherwise. Let us recall that the normal cone of a face F of P, defined by a nesting $\mathcal{N} = \{N_i\}_{1\leq i\leq l+1}$, is given by $\mathcal{N}_P(F) = \mathrm{Cone}(-\vec{N}_1, \ldots, -\vec{N}_l, -\vec{N}_{l+1}, \vec{N}_{l+1})$, where $\vec{N}_{l+1} = (1, \ldots, 1)$ is the basis of the orthogonal complement of the affine hull of P. Using Proposition 3.24, we can apply Proposition 2.32 and the result follows directly. □

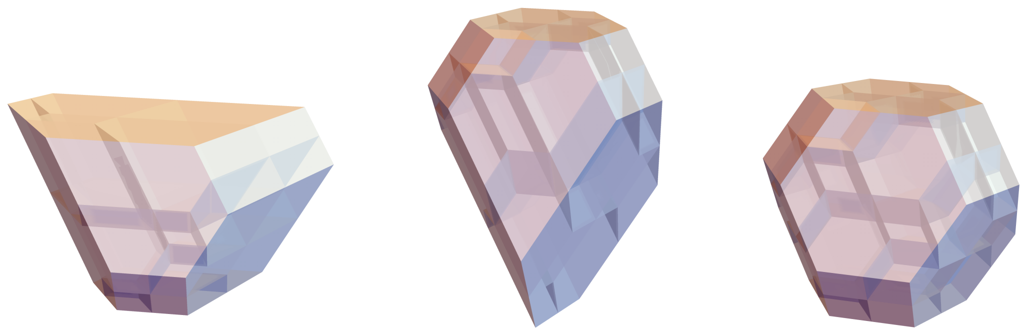


Figure 3.7: Polytopal subdivisions of some 3-dimensional operahedra, from the associahedron (left) to the permutahedron (right), given by the universal formula.

Let us make explicit the image of the diagonal of the permutahedron in low dimensions. The bijection between maximal nestings of a 2-leveled tree and permutations pictured in Figure 3.3 extends in a straightforward manner to all ordered partitions, and we use this more convenient labeling of the faces of the permutahedron to write the image of the diagonal. We also restrict ourselves to the pairs (F, G) such that $\dim F + \dim G = \dim P$, since any other pair can be obtained from these by taking faces. In dimension 1, 2, and 3, we get the following formulas:

$$\triangle_{(P,\vec{v})}(12) = 1|2 \times 12 \cup 12 \times 2|1$$

$$\begin{aligned}\triangle_{(P,\vec{v})}(123) = {} & 1|2|3 \times 123 \cup 123 \times 3|2|1 \cup 12|3 \times 2|13 \\ & \cup 13|2 \times 3|12 \cup 2|13 \times 23|1 \cup 1|23 \times 13|2 \\ & \cup 12|3 \times 23|1 \cup 1|23 \times 3|12\end{aligned}$$

$$
\begin{array}{rcccc}
 & & \triangle_{(P,\vec{v})}(1234) & & \\
= & \mathbf{1|2|3|4} \times \mathbf{1234} & \cup\ \mathbf{1234} \times \mathbf{4|3|2|1} & \cup\ \mathbf{12|3|4} \times \mathbf{2|134} \\
\cup & \mathbf{134|2} \times \mathbf{4|3|12} & \cup\ 12|3|4 \times 23|14 & \cup\ \mathbf{14|23} \times \mathbf{4|3|12} \\
\cup & 2|13|4 \times 23|14 & \cup\ \mathbf{14|23} \times \mathbf{4|13|2} & \cup\ \mathbf{13|2|4} \times \mathbf{3|124} \\
\cup & \mathbf{124|3} \times \mathbf{4|2|13} & \cup\ \mathbf{1|23|4} \times \mathbf{3|124} & \cup\ 124|3 \times 4|23|1 \\
\cup & \mathbf{1|2|34} \times \mathbf{124|3} & \cup\ \mathbf{3|124} \times \mathbf{34|2|1} & \cup\ \mathbf{1|3|24} \times \mathbf{134|2} \\
\cup & 2|134 \times 24|3|1 & \cup\ \mathbf{1|23|4} \times \mathbf{134|2} & \cup\ 2|134 \times 4|23|1 \\
\cup & 2|3|14 \times 234|1 & \cup\ \mathbf{1|234} \times \mathbf{14|3|2} & \cup\ 2|13|4 \times 234|1 \\
\cup & \mathbf{1|234} \times \mathbf{4|13|2} & \cup\ 12|3|4 \times 234|1 & \cup\ \mathbf{1|234} \times \mathbf{4|3|12} \\
\cup & \mathbf{1|24|3} \times \mathbf{14|23} & \cup\ 23|14 \times 3|24|1 & \cup\ \mathbf{1|2|34} \times \mathbf{14|23} \\
\cup & 23|14 \times 34|2|1 & \cup\ \mathbf{1|23|4} \times \mathbf{13|24} & \cup\ 24|13 \times 4|23|1 \\
\cup & \mathbf{14|2|3} \times \mathbf{4|123} & \cup\ \mathbf{123|4} \times \mathbf{3|2|14} & \cup\ \mathbf{1|24|3} \times \mathbf{4|123} \\
\cup & 123|4 \times 3|24|1 & \cup\ \mathbf{1|2|34} \times \mathbf{4|123} & \cup\ \mathbf{123|4} \times \mathbf{34|2|1} \\
\cup & \mathbf{3|14|2} \times \mathbf{34|12} & \cup\ \mathbf{12|34} \times \mathbf{2|14|3} & \cup\ \mathbf{1|3|24} \times \mathbf{34|12} \\
\cup & 12|34 \times 24|3|1 & \cup\ \mathbf{13|4|2} \times \mathbf{34|12} & \cup\ \mathbf{12|34} \times \mathbf{2|4|13} \\
\cup & \mathbf{1|23|4} \times \mathbf{34|12} & \cup\ 12|34 \times 4|23|1 & \cup\ 2|14|3 \times 24|13 \\
\cup & \mathbf{13|24} \times \mathbf{3|14|2} & \cup\ 12|4|3 \times 24|13 & \cup\ \mathbf{13|24} \times \mathbf{3|4|12} \\
\cup & 1|2|34 \times 24|13 & \cup\ \mathbf{13|24} \times \mathbf{34|2|1} &
\end{array}
$$

The pairs in blue describe the image of the diagonal of the associahedron and the pairs in bold belong to an operahedron which sits between the associahedron and the permutahedron. The associated subdivisions of the three 3-dimensional polytopes are shown in Figure 3.7. The number of pairs in the image of the diagonal of the permutahedra of dimensions 0 to 7 are given by the sequence 1, 2, 8, 50, 432, 4802, 65536, which coincides with the beginning of the integer sequence A007334 in [OEI21].

The condition $\mathrm{top}(F) \leq \mathrm{bot}(G)$ of Proposition 2.16 characterizes completely all the pairs in dimension 3, except eight of them: $12|34 \times 2|4|13$, $12|34 \times 24|3|1$, $1|2|34 \times 24|13$, $12|4|3 \times 24|13$, $13|24 \times 3|4|12$, $13|24 \times 34|2|1$, $1|3|24 \times 34|12$ and $13|4|2 \times 34|12$. In fact, the condition $\mathrm{top}(F) \leq \mathrm{bot}(G)$ is equivalent to the conditions in Theorem 3.32 for the pairs (I, J) of size $|I| = |J| = 1$.

Proposition 3.33. *For any pair of faces F, G of an $(n-1)$-dimensional permutahedron $P \subset \mathbb{R}^n$, we have that*

$$(*) \qquad \mathrm{top}(F) \leq \mathrm{bot}(G) \iff \forall 1 \leq i < j \leq n-1, \exists N \in \mathcal{N}, i \in N, j \notin N \text{ or } \exists N' \in \mathcal{N}', i \notin N', j \in N' \ .$$

Proof. We proceed in two steps.

1. Let us first prove the statement for F and G two vertices of P. They are associated to ordered partitions of $\{1, \ldots, n-1\}$, from which one can extract the nestings by reading the connected unions of blocs containing the leftmost element. The condition on the right hand side of $(*)$ is then exactly the condition that the set of inversions of F is contained in the set of inversions of G, and thus that $F \leq G$ with respect to the weak Bruhat order.

2. Now let F and G be two faces of P that are not necessarily vertices. If they satisfy the condition on the right hand side of $(*)$, then $\mathrm{top}(F)$ and $\mathrm{bot}(G)$ certainly do, so by the preceding point we have $\mathrm{top}(F) \leq \mathrm{bot}(G)$. For the reverse implication, suppose that we have $\mathrm{top}(F) \leq \mathrm{bot}(G)$. Observe that $\mathrm{top}(F)$ is obtained from the ordered partition defining F by refining each bloc in the following way: putting the elements of the bloc in strictly decreasing order, and then making each of the elements into a new bloc. From this description, it is clear that if i is to the left of j in $\mathrm{top}(F)$ and $i < j$, then the two must be in distinct blocs of F. Similarly, if i is to the right of j in $\mathrm{bot}(G)$ and $i < j$, then the two must be in distinct blocs of G. Thus, F and G satisfy the condition on the right hand side of $(*)$, which concludes the proof.

□

Remark 3.34. When computing $\mathrm{Im} \triangle_{(P,\vec{v})}$, it appears that only a certain proportion of the pairs (I, J) for $|I| = |J| \geq 2$ are necessary. Is there a more "efficient" description of the diagonal? In particular, is there a "purely combinatorial" description in terms of ordered partitions?

Remark 3.35. The diagonal of the permutahedron considered here differs from that of [SU04], see Example 4 therein. In dimension 3 the two diagonals correspond to the two chambers of the fundamental hyperplane arrangement respecting the weak order, see Remark 3.30. It would be interesting to know if there is a choice of chambers in all dimensions that recovers the diagonal of [SU04]. According to [VJ07, Table 1], both diagonals have the same number of pairs in dimensions up to 7.

From the description of the diagonal of the permutohedron in terms of ordered partitions or nestings of a 2-leveled tree, one obtains the description of the diagonal of any operahedron by applying the coarsening projection of Definition 2.28.

Proposition 3.36 (Coarsening projection for the operahedra)**.** *Let $t, t' \in \mathrm{PT}_n$ be such that t' is a 2-leveled tree but t is not. The coarsening projection $\theta : \mathcal{N}(t') \to \mathcal{N}(t)$ admits the following description. To each nest $N \subset E(t')$, we associate the minimal collection of disjoint nests $N_1, \ldots, N_r \subset E(t)$ such that $\cup_{1 \leq i \leq r} N_i = N$. To obtains $\theta(\mathcal{N})$, for any nesting $\mathcal{N}$ of t', we apply this procedure to every nest of $\mathcal{N}$ and then take the union of the resulting nests.*

Proof. This follows from a direct translation of Definition 2.28 in terms of nestings. □

Proposition 2.31 ensures that the coarsening projection for the operahedra is surjective and commutes with the diagonal maps. We have $|\mathcal{N}| \geq |\theta(\mathcal{N})|$, so the dimension of a face stays the same or diminishes. To obtain the image of the diagonal of P_t, one has to apply θ to the pairs (F, G) with $\dim F + \dim G = \dim P$ and keep only those for which $\dim \theta(F) = \dim F$ and $\dim \theta(G) = \dim G$. In the case where t is a linear tree, one recovers A. Tonks' projection [Ton97].

Remark 3.37. Restricting to linear trees, we recover with a different combinatorial description the "magical formula" of [MS06, MTTV21] and conjecturally [SU04] for the associahedra.

3.4 Tensor product of homotopy operads

In this section, we show that there exists a topological cellular colored operad structure on the Loday realizations of the operahedra compatible with (in fact, forced by) the above choices of diagonals. Applying the cellular chains functor, we obtain a "Hopf" operad in chain complexes (this operad is not quite a Hopf operad since the diagonal is not strictly coassociative) describing non-symmetric operads up to homotopy, that is non-symmetric operads where the parallel and sequential axioms are relaxed up to a coherent tower of homotopies. The formula for the image of the diagonal obtained in Section 3.3 allows us to define explicitly, for the first time, the tensor product of two non-symmetric operads up to homotopy.

3.4.1 The colored operad encoding the space of non-symmetric operads

We work over a field $\mathbb{K}$ of characteristic 0. Note that since the operads that we consider here come from set-theoretic ones, we could as well work over $\mathbb{Z}$.

Definition 3.38 (Tree substitution)**.** *For any trees $t' \in \mathrm{PT}_k$ and $t'' \in \mathrm{PT}_l$, for any vertex $i \in V(t')$ having the same number of inputs as t'', we define the tree $t' \circ_i t'' \in \mathrm{PT}_{k+l-1}$ obtained by replacing the induced subtree of the vertex i in t' by* *the tree t''. More precisely, the tree $t' \circ_i t''$ has vertices $(V(t') \setminus \{i\}) \sqcup V(t'')$ and edges $E(t') \sqcup E(t'')$, see Figure 3.8.*

Definition 3.39 (The colored operad $\mathcal{O}$)**.** *We denote by $\mathcal{O}$ the $\mathbb{N}$-colored operad whose $\mathbb{N}$-colored collection is defined by*

$$\mathcal{O}(n_1, \ldots, n_k; n) = \mathbb{K}\left\{ \begin{array}{c} \textit{Planar tree } t \in \mathrm{PT}_k \textit{ with a bijection } \sigma : \{1, \ldots, k\} \to V(t) \\ \textit{such that } \sigma(i) \textit{ has } n_i \textit{ inputs for all } i \end{array} \right\}$$

if $n_1 + \cdots + n_k - k + 1 = n$, and the trivial vector space otherwise. Let $t' \in \mathrm{PT}_k$ and $t'' \in \mathrm{PT}_l$ be trees with bijections σ' and σ'', respectively. For any $i \in \{1, \ldots, k\}$ such that $\sigma'(i) \in V(t')$ has the same number of inputs as t'', we define a partial composition map via tree substitution

$$(t', \sigma') \circ_i (t'', \sigma'') := (t' \circ_{\sigma'(i)} t'', \sigma' \circ_i \sigma'') \ ,$$

where the permutation $\sigma' \circ_i \sigma''$ is defined by

$$\sigma' \circ_i \sigma''(j) := \begin{cases} \sigma'(j) & \textit{if } j < i \\ \sigma''(j - i + 1) & \textit{if } i \leq j \leq i + l - 1 \\ \sigma'(j - l + 1) & \textit{if } i + l \leq j \leq k + l - 1 \ . \end{cases}$$

The symmetric groups action is given by precomposition and the corollas $\mathrm{PT}_1 = \{\mathcal{O}(n; n) \mid n \in \mathbb{N}\}$ give the unit elements.

The basis elements of $\mathcal{O}$ are called *operadic trees.* We represent an operadic tree (t, σ) by labeling every vertex $j \in V(t)$ with the number $\sigma^{-1}(j)$. If this labeling coincides with the canonical labeling defined in Section 3.2.1, we say that t is a *left-recursive* operadic tree.

When we restrict $\mathcal{O}$ to the linear trees where each vertex has only one input (that is, we restrict the set of colors from $\mathbb{N}$ to 1), we obtain the symmetric operad Ass encoding associative algebras.

When we restrict $\mathcal{O}$ to the two-leveled trees where each vertex of the second level has only one input, and identify all the trees with the same number of vertices and vertices labels, we obtain the permutad permAs^{sh} encoding associative permutadic algebras [LR13, Section 7.6]. Here, substitution is restricted to the vertex of the first level only, in such a way that we stay in 2-leveled trees.

Proposition 3.40. *Algebras over the colored operad $\mathcal{O}$ are non-unital non-symmetric operads.*

Proof. We refer to [VdL03, Section 4], [DV15, Section 1] or [Obr19, Section 2] for details. □

The operation of tree substitution naturally extends to nested trees. For $n \geq 2$, let us denote by NPT_n the set of nested trees with n vertices. By convention, we define $\mathrm{NPT}_1 := \mathrm{PT}_1$.

Definition 3.41 (Nested tree substitution). *For any nested trees $(t', \mathcal{N}') \in \mathrm{NPT}_k$ and $(t'', \mathcal{N}'') \in \mathrm{NPT}_l$, for any $i \in V(t')$ having the same number of inputs as t'', we define the nested tree*

$$(t', \mathcal{N}') \circ_i (t'', \mathcal{N}'') := (t' \circ_i t'', \mathcal{N}' \circ_i \mathcal{N}'') \in \mathrm{NPT}_{k+l-1} ,$$

where $\mathcal{N}' \circ_i \mathcal{N}'' = \{(N' \setminus \{i\}) \sqcup V(t'') \mid N' \in \mathcal{N}'\} \sqcup \mathcal{N}''$.

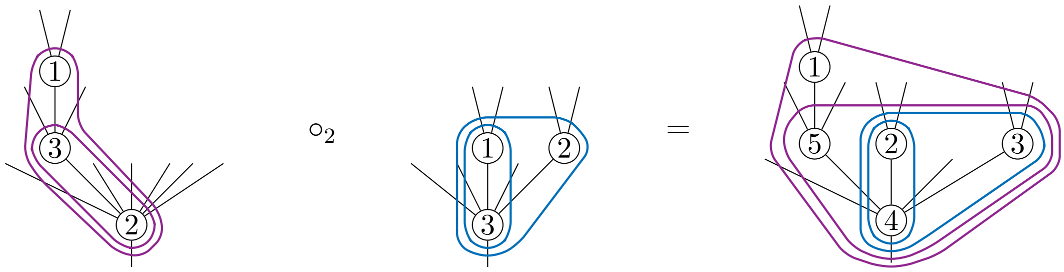


Figure 3.8: Substitution of nested operadic trees.

We note that any nested tree can be obtained from a family of trivially nested trees by successive substitutions. In general, these substitutions can be performed in different orders without changing the resulting nested tree.

Definition 3.42 (Increasing order on nestings). *For a nested tree $(t, \mathcal{N})$, we order the nests of $\mathcal{N}$ by decreasing order of cardinality, and further order nests of the same cardinality according to the increasing order on their minimal elements. We obtain a total order on the set $\mathcal{N}$ and a corresponding unique sequence of substitution of trivially nested trees $(\cdots(t_1 \circ_{i_1} t_2) \circ_{i_2} t_3) \cdots \circ_{i_k} t_{k+1}) = (t, \mathcal{N})$.*

Let us recall the permutation introduced in Point (5) of Proposition 3.15.

Definition 3.43. *Let t be a tree, and let $\mathcal{N}$ be a nesting of t with only one non-trivial nest N. We contract the nest N to obtain a new tree t'. We write $t'' = t(N)$ for the subtree induced by N, considered as an independent tree. We write $|E(t')| = p$ and $|E(t'')| = q$ and we renumber the edges of t'' from $p+1$ to $p+q$. We denote by $\sigma_N : E(t') \sqcup E(t'') \to E(t)$ the (p,q)-shuffle mapping each just renumbered edge of t' and t'' to its label in t.*

Definition 3.44 (The graded colored operad $\mathcal{O}_\infty$). *We denote by $\mathcal{O}_\infty$ the graded $\mathbb{N}$-colored operad whose space of operations $\mathcal{O}_\infty(n_1, \ldots, n_k; n)$ is given by*

$$\mathbb{K}\left\{ \begin{array}{c} \textit{Planar nested tree } (t, \mathcal{N}) \in \mathrm{NPT}_k \textit{ with a bijection } \sigma : \{1, \ldots, k\} \to V(t) \\ \textit{such that } \sigma(i) \textit{ has } n_i \textit{ inputs for all } i \end{array} \right\}$$

if $n_1 + \cdots + n_k - k + 1 = n$*, and the trivial vector space otherwise. The homological degree of a basis element* $(t, \mathcal{N}, \sigma)$ *is given by* $|E(t)| - |\mathcal{N}|$*. Let* $(t', \mathcal{N}') \in \mathrm{NPT}_k$ *and* $(t'', \mathcal{N}'') \in \mathrm{NPT}_l$ *be two nested trees with bijections* σ' *and* σ''*. For any* $i \in \{1, \ldots, k\}$ *such that* $\sigma'(i) \in V(t')$ *has the same number of inputs as* t''*, partial composition is defined via substitution of nested trees*

$$(t', \mathcal{N}', \sigma') \circ_i (t'', \mathcal{N}'', \sigma'') \coloneqq \pm((t', \mathcal{N}') \circ_{\sigma'(i)} (t'', \mathcal{N}''), \sigma' \circ_i \sigma'')$$

where the permutation $\sigma' \circ_i \sigma''$*, the symmetric groups action and the units are defined exactly as in* $\mathcal{O}$*, and the sign is induced by the choice of increasing order on nestings.*

An example of partial composition in $\mathcal{O}_\infty$ is pictured in Figure 3.8. The degree 0 part of $\mathcal{O}_\infty$ forms a suboperad made up of fully nested trees.

Proposition 3.45. *The* $\mathbb{N}$*-colored operad* $\mathcal{O}_\infty$ *is free on operadic trees.*

Proof. Substituting operadic trees produces nested trees. There is a unique way to write a nested tree by iterating this process, up to the parallel and sequential axioms. So, this operad is free. □

Now we turn $\mathcal{O}_\infty$ into a differential graded colored operad. The differential is the unique derivation extending the map ∂, defined on trivially nested operadic trees by

$$\partial(t, \mathcal{N}, \sigma) \coloneqq - \sum_{N \in \mathcal{N}(t)} (-1)^{|E(t) \setminus N|} \mathrm{sgn}(\sigma_N)(t, \mathcal{N} \cup \{N\}, \sigma) \ ,$$

where the sum runs over all nests of t.

Proposition 3.46. *The dg* $\mathbb{N}$*-colored operad* $\mathcal{O}_\infty$ *is the minimal model* $\Omega\mathcal{O}^{\mathrm{i}}$ *of* $\mathcal{O}$*.*

Proof. One can compute the operad $\Omega\mathcal{O}^{\mathrm{i}}$ by using the binary quadratic presentation of $\mathcal{O}$ [DV15, Definition 5]. Using the fact that the colored operad $\mathcal{O}$ is Koszul self-dual [VdL03, Theorem 4.3] and the bijection between composite and operadic trees [DV15, Section 1.3], one obtains $\mathcal{O}_\infty$ as defined above. The sign in the differential comes from the choice of the left-levelwise order on composite trees and application of the Koszul sign rule thereafter. The term $\mathrm{sgn}(\sigma_N)$ comes from the decomposition map of $\mathcal{O}^{\mathrm{i}}$ and the term $(-1)^{|E(t) \setminus N|}$ comes from the desuspension in the definition of the differential in the cobar construction. □

The part of $\mathcal{O}_\infty$ made up of the linear trees where each vertex has only one input gives the minimal model Ass_∞ of the operad Ass. The part made up of the equivalences classes two-leveled trees with restricted substitution gives the minimal model $\mathrm{permAs}^{sh}_\infty$ of the permutad permAs^{sh} [LR13]. Considering the associated non-symmetric operad and permutad, one recovers the minimal models A_∞ [LV12, Section 9.2.4] and permAs_∞ [LR13, Section 5.2] -see also [Mar20], of As and permAs, respectively.

REMARK 3.47. Under the bijection between nested trees and their tubed line graphs mentioned in Remark 3.4, we recover the operation of substitution of tubings defined by S. Forcey and M. Ronco in [FR19], and we observe that the family of clawfree block graphs is stable under this operation.

Algebras over the operad $\mathcal{O}_\infty$ are non-symmetric operads up to homotopy, as introduced by P. Van der Laan in [VdL03].

Definition 3.48 (Non-symmetric operad up to homotopy). *A* non-symmetric non-unital operad up to homotopy *is a family of graded vector spaces $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ together with operations*

$$\mu_t : \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) \to \mathcal{P}(n_1 + \cdots + n_k - k + 1)$$

of degree $|E(t)| - 1$ for each $t \in \mathrm{PT}_k$ and all $k \geq 1$, where $n_1, \ldots, n_k$ are the number of inputs of the vertices of t, which satisfy the relations

$$\sum_{t' \circ_i t'' = t} (-1)^{|E(t) \setminus E(t'')|} \mathrm{sgn}(\sigma_N)\ \mu_{t'} \circ_i \mu_{t''} = 0\ ,$$

where the sum runs over all the subtrees t'' of t.

The operations μ_t for the corollas $t \in \mathrm{PT}_1$ satisfy the relations $\mu_t \circ_1 \mu_t = 0$, so they make the spaces $\{\mathcal{P}(n)\}_{n\geq 1}$ into chain complexes. The operations for trees with 2 vertices correspond to partial composition operations $\circ_i$ as in the definition of a non-symmetric operad. The presence of the operations μ_t for trees $t \in \mathrm{PT}_3$ indicates that these partial compositions verify the parallel and sequential axioms only up to homotopy. The operations μ_t for trees $t \in \mathrm{PT}_4$ are homotopies between these homotopies, and so on.

EXAMPLE 3.49. As proved by P. Van der Laan in [VdL03, Theorem 5.7], the singular $\mathbb{Q}$-chains on configuration spaces of points in the plane form an operad up to homotopy quasi-isomorphic to the operad of singular $\mathbb{Q}$-chains on the little discs operad.

3.4.2 Topological colored operad structure on the operahedra

In order to contemplate polytopal $\mathbb{N}$-colored operads, we need a suitable symmetric monoidal category of polytopes. We consider the following category, which is a slight modification of the symmetric monoidal category defined in [MTTV21, Section 2.1].

Definition 3.50 (The category Poly).

1. *The objects are the disjoint unions $\coprod_{i=1,\ldots,r} P_i$ of non-necessarily distinct polytopes.*

2. *Morphisms are disjoint union* $\coprod_{i=1,\ldots,r} f_i$ *of continuous maps* $f_i : P_i \to Q_i$ *where for each* i, f_i *sends* P_i *homeomorphically to the underlying set* $|\mathcal{D}_i|$ *of a polytopal subcomplex* $\mathcal{D}_i \subset \mathcal{L}(Q_i)$ *of* Q_i *such that* $f_i^{-1}(\mathcal{D}_i)$ *defines a polytopal subdivision of* P_i.

The results of [MTTV21] extend in a straightforward manner to this new category. The symmetric monoidal structure is given by the cartesian product of polytopes, and the unit is the trivial polytope made up of one point in $\mathbb{R}^0$.

We want to endow the Loday realizations of the operahedra of standard weight with a colored operad structure in the category Poly. The underlying set-theoretic operad structure is given on the set of face lattices by substitution of trees. The geometric avatar of this operation is the isomorphism $\Theta : \mathbb{R}^{k-1} \times \mathbb{R}^{l-1} \cong \mathbb{R}^{n-1}$ introduced in the proof of Point (5) of Proposition 3.15.

Problem. *For each operahedron* P_t*, make a choice of an orientation vector* $\vec{v}$ *such that the family of diagonal maps* $\triangle_{(t,\vec{v})}$ *commutes with the maps* Θ.

Suppose that we have made a choice of an orientation vector for each operahedron. We fix $t \in \mathrm{PT}_n$ with chosen orientation vector $\vec{v}$. We let $t' \in \mathrm{PT}_k$ and $t'' \in \mathrm{PT}_l$ be two trees, with chosen orientation vectors $\vec{v}'$ and $\vec{v}''$ respectively, such that $t' \circ_i t'' = t$ for some $i \in V(t')$. We denote by ω the weight $(1,\ldots,1,l,1,\ldots,1)$ of length k, where l is in position i. We want the following diagram to commute

$$\begin{array}{ccc} P_{(t',\omega)} \times P_{t''} & \xrightarrow{\quad\Theta\quad} & P_t \\ \Big\downarrow{\scriptstyle \triangle_{(t',\vec{v}')} \times \triangle_{(t'',\vec{v}'')}} & & \Big\downarrow{\scriptstyle \triangle_{(t,\vec{v})}} \\ P_{(t',\omega)} \times P_{(t',\omega)} \times P_{t''} \times P_{t''} & \xrightarrow{(\Theta\times\Theta)(\mathrm{id}\times\sigma_2\times\mathrm{id})} & P_t \times P_t \ , \end{array}$$

where σ_2 is the permutation of the two middle blocks of coordinates. The preimage of $\vec{v}$ under Θ determines two orientation vectors $\vec{w}' \in \mathbb{R}^{k-1}$ and $\vec{w}'' \in \mathbb{R}^{l-1}$ of $P_{(t',\omega)}$ and $P_{t''}$ explicitly given by

$$\vec{w}' = (v_{\sigma(1)}, \ldots, v_{\sigma(k-1)}) \quad \text{and} \quad \vec{w}'' = (v_{\sigma(k)}, \ldots, v_{\sigma(k+l-1)}) \ ,$$

where σ is a $(k-1, l-1)$-shuffle.

Proposition 3.51. *Suppose that for each map* Θ*, the two orientation vectors* $\vec{w}'$ *and* $\vec{w}''$ *in the preimage* $\Theta^{-1}(\vec{v})$ *are in the same chambers of* $\mathcal{H}_{t'}$ *and* $\mathcal{H}_{t''}$ *as* $\vec{v}'$ *and* $\vec{v}''$ *respectively. Then, the family of diagonal maps* $\triangle_{(t,\vec{v})}$ *commutes with the maps* Θ.

Proof. Proposition 2.22 shows that $\triangle_{(t',\vec{w}')} = \triangle_{(t',\vec{v}')}$ and $\triangle_{(t'',\vec{w}'')} = \triangle_{(t'',\vec{v}'')}$. The fact that the above diagram commutes is then straightforward to verify, using the pointwise definition of $\triangle_{(t,\vec{v})}$. ☐

Recall from Definition 3.31 that a *principal orientation vector* $\vec{v} \in \mathbb{R}^{n-1}$ is such that $\sum_{i\in I} v_i > \sum_{j\in J} v_j$ for all $(I, J) \in D(n-1)$.

Proposition 3.52. *For any choice of principal orientation vector $\vec{v}$ for every Loday realization of operahedron P_t of standard weight, the family of diagonal maps $\triangle_{(t,\vec{v})}$ commutes with the maps Θ.*

Proof. Since σ is a $(k-1, l-1)$-shuffle, the two vectors $\vec{w}'$ and $\vec{w}''$ are again principal orientation vectors. We conclude with Proposition 3.51. □

Proposition 3.53 (Transition map [MTTV21, Proposition 7])**.** *Let $(P, \vec{v})$ and $(Q, \vec{w})$ be two positively oriented polytopes, with a combinatorial equivalence $\Phi : \mathcal{L}(P) \xrightarrow{\cong} \mathcal{L}(Q)$. Suppose that tight coherent subdivisions $\mathcal{F}_{(P,\vec{v})}$ and $\mathcal{F}_{(Q,\vec{w})}$ are combinatorially equivalent under $\Phi \times \Phi$.*

1. *There exists a unique continuous map*
$$\mathrm{tr} = \mathrm{tr}_P^Q : P \to Q \ ,$$
which extends the restriction of Φ to the set of vertices and which commutes with the respective diagonal maps.
2. *The map* tr *is an isomorphism in the category* Poly*, whose correspondence of faces agrees with Φ.*

The map tr constructed explicitly in [MTTV21] and has a strong "fractal" character. We fix a tree $t \in \mathrm{PT}_n$, a weight ω of the form $(1, \ldots, 1, l, 1, \ldots, 1)$ with $l \geq 1$ and a principal orientation vector $\vec{v} \in \mathbb{R}^{n-1}$. We apply Proposition 3.53 to the polytopes P_t and $P_{(t,\omega)}$ and obtain a map $\mathrm{tr} : P_t \longrightarrow P_{(t,\omega)}$.

Definition 3.54 (Partial composition and symmetric group action)**.** *We consider the $\mathbb{N}$-colored collection*
$$O_\infty(n_1, \ldots, n_k; n) \coloneqq \coprod_{(t,\sigma)\in\mathcal{O}(n_1,\ldots,n_k;n)} P_t \ .$$
Let (t', σ') and (t'', σ'') be two composable operadic trees with k and l vertices, respectively. We denote by $(t, \sigma) \coloneqq (t', \sigma')\circ_i(t'', \sigma'')$ their composition at vertex $\sigma'(i) \in V(t')$ in $\mathcal{O}$. We denote by ω the weight $(1, \ldots, 1, l, 1, \ldots, 1)$ of length k, where l is in position i. We define the partial composition map *by*
$$\circ_i \ : \ P_{t'} \times P_{t''} \xrightarrow{\mathrm{tr}\times\mathrm{id}} P_{(t',\omega)} \times P_{t''} \xhookrightarrow{\Theta} P_t \ .$$
For $\kappa \in \mathbb{S}_n$, we define the symmetric group action *on the polytopes associated to (t, σ) and $(t, \kappa \circ \sigma)$ by the identity map $P_t \to P_t$.*

Theorem 3.55.

1. *The* $\mathbb{N}$*-colored collection* $\{O_\infty(n_1,\ldots,n_k;n) \mid n_1,\ldots,n_k,n\in\mathbb{N}\}$*, together with the partial composition maps* $\circ_i$*, the symmmetric group actions and the 0-dimensional unit elements* $\{O_\infty(n;n) \mid n\in\mathbb{N}\}$*, forms a symmetric colored operad in the category* Poly*.*

2. *This colored operad structure extends the topological operad structure on the vertices of the operahedra, that is, the fully nested trees.*

3. *The maps* $\{\triangle_{(t,\vec{v})} : P_t \to P_t \times P_t\}_{(t,\sigma)\in\mathcal{O}}$*, where* $\vec{v}$ *are principal orientation vectors, form a morphism of symmetric colored operads in the category* Poly*.*

Proof. Once we have in hand Proposition 3.52 asserting that the diagonal maps commute with the maps Θ, we can apply the proof of [MTTV21, Theorem 1] *mutatis mutandis*. The additional facts involving the symmetric groups action and units are straightforward to verify. □

Remark 3.56. The proof of Theorem 3.55 shows that any family of orientation vectors satisfying Proposition 3.51 induces a colored operad structure on the operahedra. There is more than one such family: consider for instance the vectors $\vec{v}$ with strictly descreasing coordinates which satisfy $\sum_{i\in I} v_i > \sum_{j\in J} v_j$ for all $I,J\subset\{1,\ldots,n\}$ such that $I\cap J=\emptyset$, $|I|=|J|\geq 2$ and $\max(I\cup J)\in I$. It would be interesting to know how many such families exist, and how they are related to each other.

3.4.3 Tensor product of operads up to homotopy

We consider the set of all ordered basis of a finite-dimensional vector space V. We declare two basis *equivalent* if the unique linear endomorphism of V sending one basis to the other has positive determinant. In this way, we obtain two equivalence classes of ordered basis.

Definition 3.57. *An* orientation *of* V *is a bijection between the equivalence classes of ordered basis and the set* $\{+1,-1\}$*. Any basis in the first equivalence class is called a* positively oriented basis*.*

So there are exactly two distinct orientations of V.

Definition 3.58 (Cellular orientation of a polytope)**.** *Let* $P\subset\mathbb{R}^n$ *be a polytope, and let* F *be a face of* P*. A* cellular orientation of F *is a choice of orientation of its linear span. A* cellular orientation of P *is a choice of cellular orientation for each face* F *of* P*.*

An orientation vector of P, in the sense of Definition 2.11, induces a cellular orientation of the 1-skeleton of P.

Proposition 3.59. *A cellular orientation of a polytope P makes it into a regular CW complex. Moreover, the choice of a cellular orientation for every operahedron promotes the colored operad O_∞ to an operad in CW complexes and Theorem 3.55 holds in this category.*

Proof. The choice of a cellular orientation of a face F is equivalent to the choice of a generator in the top homology group of F relative to its boundary, which is a $\dim F$-sphere. Thus, it makes sense to choose a degree one attaching map from the boundary of the $\dim(F)$-ball to the boundary of F. We endow P with the regular CW structure given by a family of such attaching maps. Now it is clear that the morphisms in the category Poly define cellular maps, and that the proof of Theorem 3.55 can be performed *mutatis mutandis* in the category of CW complexes. □

One can thus apply the cellular chains functor to O_∞ and obtain a colored operad in chain complexes.

Theorem 3.60. *There is a choice of cellular orientation that yields an isomorphism of differential graded symmetric colored operads $C_\bullet^{\text{cell}}(O_\infty) \cong \mathcal{O}_\infty$.*

Proof. By definition, the operadic structure of O_∞ coincides cellularly with the operadic structure of $\mathcal{O}_\infty$, and the boundary map of $C_\bullet^{\text{cell}}(O_\infty)$ coincides up to sign with the differential of $\mathcal{O}_\infty$. We make an explicit choice of orientations and prove that we recover the signs of $\mathcal{O}_\infty$. We build on the work of T. Mazuir who recovered this way in [Maz21, I,Section 4] the signs of the operad A_∞. For a left-recursive operadic tree (t, σ), we choose as orientation of the top dimensional cell of P_t the positively oriented basis

$$e_j = (1, 0, \ldots, 0, -1_{j+1}, 0, \cdots, 0) \ ,$$

where -1 is in position $j+1$ for $j = 1, \ldots, n-2$. For any operadic tree $(t, \kappa \circ \sigma)$ obtained from (t, σ) by the action of an element κ of the symmetric group, we set the orientation of the top-dimensional cell of $P_{(t,\kappa\circ\sigma)}$ to be the orientation of $P_{(t,\sigma)}$ multiplied by $\operatorname{sgn}(\kappa)$. Then, we choose the orientation of any other cell $(t, \mathcal{N})$ of P_t to be the one induced by operadic composition as follows. We consider the unique sequence of substitution of trivially nested trees $(t, \mathcal{N}) = (\cdots(t_1 \circ_{i_1} t_2) \circ_{i_2} t_3) \cdots \circ_{i_k} t_{k+1})$ arising from the increasing order on $\mathcal{N}$ (Definition 3.42), and we set the orientation of $(t, \mathcal{N})$ to be the image of the positively oriented basis of the top cells of the polytopes P_{t_i} under this sequence of operations. Computing the signs amounts to comparing bases where the vectors have been permuted. Since we have chosen the increasing order on the nests, we recover precisely the signs involved in the composition of $\mathcal{O}_\infty$.

We claim that this choice of orientations recovers the signs in the differential of $\mathcal{O}_\infty$. It is enough to consider the boundary map of the top cell of P_t. Let $(t, \mathcal{N})$ be a facet of P_t and let t' and t'' be two composable operadic trees with trivial nestings such that $t' \circ_i t'' = (t, \mathcal{N})$. Let N denote the unique non-trivial nest of $\mathcal{N}$. We denote by $(e'_j)_{1\leq j\leq k-1}$ and $(e''_j)_{1\leq j\leq l-1}$, the positively oriented basis associated to t' and t'', respectively. We recall the application Θ and its associated permutation $\sigma_N : E(t') \sqcup E(t'') \to E(t)$ from Point (5) of Proposition 3.15. We choose an outward pointing normal vector ν to the facet $(t, \mathcal{N})$. The sign associated to this facet in the sum $\partial(t)$ is given by comparing the orientation induced by the operad structure and the orientation of P_t. This amounts to computing the determinant $\det(\nu, \Theta(e'_j), \Theta(e''_j))$ in the basis e_j. We distinguish two cases.

1. If N contains 1, i.e. if $\sigma_N(1) \neq 1$, an outward pointing normal vector ν is given by forgetting the first coordinate of the vector $\vec{N} - (1, \ldots, 1)$. We have in this case

$$\begin{aligned} \Theta(e'_j) &= -e_{\sigma_N(1)-1} + e_{\sigma_N(j+1)-1}\ , \quad 1 \leq j \leq k-1 \\ \Theta(e''_j) &= e_{\sigma_N(j+1+k)-1}\ , \qquad 1 \leq j \leq l-1 \end{aligned}$$

and the value of the determinant is

$$\det(\nu, \Theta(e'_j), \Theta(e''_j)) = -|E(t) \setminus N|(-1)^{|E(t)\setminus N|}\mathrm{sgn}(\sigma_N)\ .$$

2. If N does not contain 1, i.e. if $\sigma_N(1) = 1$, an outward pointing normal vector ν is given by forgetting the first coordinate of the vector $\vec{N}$. We have in this case

$$\begin{aligned} \Theta(e'_j) &= e_{\sigma_N(j+1)-1}\ , \qquad 1 \leq j \leq k-1 \\ \Theta(e''_j) &= -e_{\sigma_N(1+k)-1} + e_{\sigma_N(j+1+k)-1}\ , \quad 1 \leq j \leq l-1 \end{aligned}$$

and the value of the determinant is

$$\det(\nu, \Theta(e'_j), \Theta(e''_j)) = -|N|(-1)^{|E(t)\setminus N|}\mathrm{sgn}(\sigma_N)\ .$$

We thus recover in both cases the sign of the differential of $\mathcal{O}_\infty$. □

Corollary 3.61. *The image of the diagonal maps under the cellular chains functor define a morphism of operads in chain complexes $\mathcal{O}_\infty \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty$, and thus a functorial tensor product of non-symmetric operads up to homotopy.*

Proof. The cellular chains functor is strong symmetric monoidal and sends the operad O_∞ to the operad $\mathcal{O}_\infty$. For $\mathcal{P}$ and $\mathcal{Q}$ two homotopy operads defined by morphisms of $\mathbb{N}$-colored operads $f : \mathcal{O}_\infty \to \mathrm{End}_\mathcal{P}$ and $g : \mathcal{O}_\infty \to \mathrm{End}_\mathcal{Q}$, the composite of morphisms

$$\mathcal{O}_\infty \xrightarrow{C^{cell}_\bullet(\triangle)} \mathcal{O}_\infty \otimes \mathcal{O}_\infty \xrightarrow{f\otimes g} \mathrm{End}_\mathcal{P} \otimes \mathrm{End}_\mathcal{Q} \to \mathrm{End}_{\mathcal{P}\otimes\mathcal{Q}} \ ,$$

where the last arrow is given by permutation of the factors, defines the structure of an operad up to homotopy on the tensor product of $\mathcal{P}$ and $\mathcal{Q}$. □

REMARK 3.62. The diagonal $\triangle_{(t,\vec{v})}$ is neither pointwise nor cellular coassociative and the induced diagonal of the dg colored operad $\mathcal{O}_\infty$ is not coassociative either. M. Markl and S. Schnider have actually shown in [MS06, Section 6] that such a diagonal *cannot* exist. So the newly defined tensor product cannot make the category of homotopy non-symmetric operads (with strict morphisms) into a symmetric monoidal category.

We end by computing the signs associated to our choice of cellular orientation for the approximation of the diagonal $\triangle_{(t,\vec{v})}$.

Definition 3.63. *Let t be a tree and let $\mathcal{N}, \mathcal{N}'$ be a pair of nestings such that $|\mathcal{N}| + |\mathcal{N}'| = |V(t)|$. An edge $i \in E(t)$ is said to be* admissible *in $\mathcal{N}$ if $i \neq \min(\min \mathcal{N}(i)) =: \inf_i(\mathcal{N})$. The set of admissible edges of $\mathcal{N}$ is denoted by $Ad(\mathcal{N})$.*

We give the set $Ad(\mathcal{N}) \sqcup Ad(\mathcal{N}')$ a total order by using the increasing order on the nestings (Definition 3.42) and within a nest by following the numbering of the edges in increasing order. Then, the function $\sigma_{\mathcal{N}\mathcal{N}'} : Ad(F) \sqcup Ad(G) \to (1, \ldots, |Ad(\mathcal{N}) \sqcup Ad(\mathcal{N}')|)$ defined for $i \in Ad(\mathcal{N})$ by

$$\sigma_{\mathcal{N}\mathcal{N}'}(i) = \begin{cases} \inf_i(\mathcal{N}) - 1 & \text{if } i \in Ad(\mathcal{N}) \cap Ad(\mathcal{N}') \text{ and } 1 \neq \inf_i(\mathcal{N}) < \inf_i(\mathcal{N}') \\ i - 1 & \text{otherwise} \end{cases}$$

and similarly on $i \in Ad(\mathcal{N}')$ by inverting the roles of $\mathcal{N}$ and $\mathcal{N}'$, induces a permutation of the set $\{1, \ldots, |Ad(\mathcal{N}) \sqcup Ad(\mathcal{N}')|\}$ that we still denote by $\sigma_{\mathcal{N}\mathcal{N}'}$.

For convenience, let us recall that

$$D(n) := \{(I, J) \mid I, J \subset \{1, \ldots, n\}, |I| = |J|, I \cap J = \emptyset, \min(I \cup J) \in I\} \ .$$

Proposition 3.64 (Tensor product of operads up to homotopy)**.** *Given two non-symmetric non-unital operads up to homotopy $(\mathcal{P}, \{\mu_t\})$ and $(\mathcal{Q}, \{\nu_t\})$, their* tensor product *$(\mathcal{P} \otimes \mathcal{Q}, \{\rho_t\})$ is given by the Hadamard tensor product of spaces $(\mathcal{P} \otimes \mathcal{Q})(n) := \mathcal{P}(n) \otimes \mathcal{Q}(n)$ and the operations*

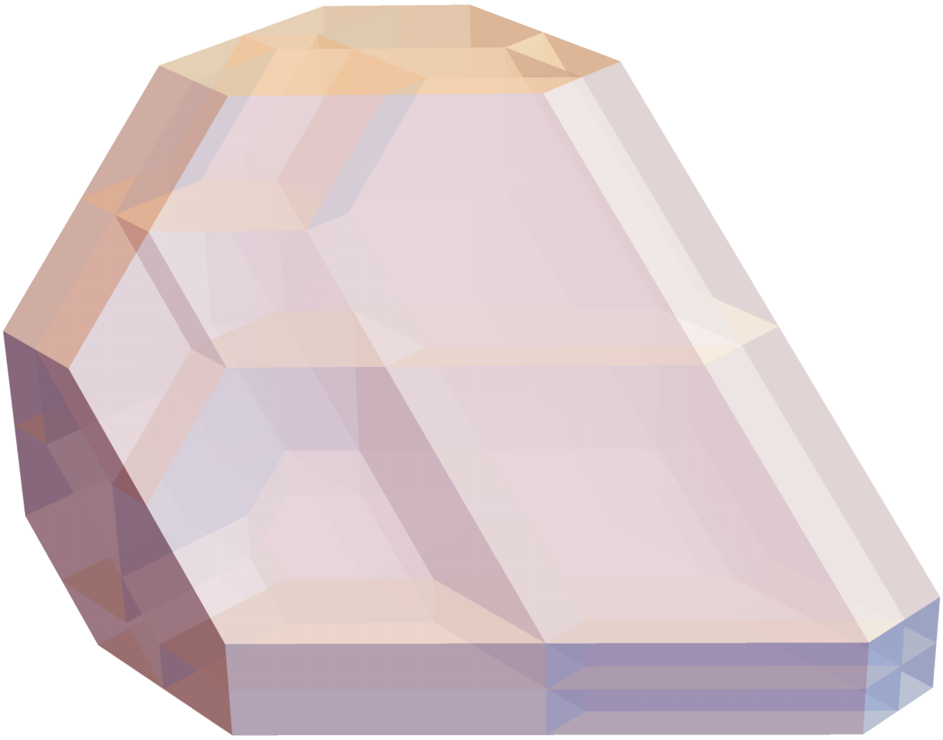
$$\rho_t := \sum_{\substack{\mathcal{N},\mathcal{N}' \in \mathcal{N}(t) \\ |\mathcal{N}|+|\mathcal{N}'|=|V(t)| \\ \forall (I,J) \in D(|E(t)|), \exists N \in \mathcal{N}, |N\cap I| > |N \cap J| \\ \text{or } \exists N' \in \mathcal{N}', |N' \cap I| < |N' \cap J|}} (-1)^{|Ad(\mathcal{N}) \cap Ad(\mathcal{N}')|} \mathrm{sgn}(\sigma_{\mathcal{N}\mathcal{N}'})\ \mathcal{N}(\mu_t) \otimes \mathcal{N}'(\nu_t)\ \sigma_t$$

where $\mathcal{N}(\mu_t)$ *and* $\mathcal{N}'(\nu_t)$ *denote the composition of the operations corresponding to the nests of* $\mathcal{N}$ *and* $\mathcal{N}'$ *in the increasing orders and where* σ_t *is the isomorphism* $\mathcal{P}(n_1)\otimes\mathcal{Q}(n_1)\otimes\cdots\otimes\mathcal{P}(n_k)\otimes\mathcal{Q}(n_k)\cong\mathcal{P}(n_1)\otimes\cdots\otimes\mathcal{P}(n_k)\otimes\mathcal{Q}(n_1)\otimes\cdots\otimes\mathcal{Q}(n_k)$.

Proof. This is just unravelling the definition of tensor product arising from Corollary 3.61. For a pair of faces $(F,G)\in\operatorname{Im}\triangle_{(t,\vec{v})}$, the sign comes from the comparison of our choice of orientation on $F\times G$, which is just the product of the orientations of F and G, with the orientation induced by the diagonal $\triangle_{(t,\vec{v})}$ when restricted to $(\mathring{F}+\mathring{G})/2$. Let (e_j) denote as before the positively oriented basis of the top cell of P_t. We need to compute the sign of the determinant of the vectors $\triangle_{(t,\vec{v})}(e_j)$ expressed in the basis $\{e_j^F\times 0\}\cup\{0\times e_j^G\}$ corresponding to the orientation of $F\times G$. By the very definition of $\triangle_{(t,\vec{v})}$ (Proposition 2.14), this is the same as computing the sign of the determinant of the e_j^F, e_j^G expressed in the basis (e_j), which gives the expression appearing above. □

Chapter 4

The diagonal of the multiplihedra



4.1 Introduction

The n-dimensional associahedron, a polytope whose faces are in bijection with planar trees with $n+1$ leaves, was first introduced as a topological cell complex by J. Stasheff to describe algebras whose product is associative only up to homotopy [Sta63]. The problem of giving polytopal realizations of these CW-complexes has a rich history [CZ12], and the algebras that they encode, called A_∞-algebras, are classical objects in algebraic topology. A_∞-algebras have found many applications, from iterated loop spaces [May72] to Fukaya categories [Sei08], through the interpretation of the associahedra as moduli spaces of metric trees [MW10]. Today, this notion is ubiquitous and appears in different branches of mathematics such as symplectic topology, mathematical physics, mirror symmetry, Galois cohomology or non-commutative probability.

The n-dimensional multiplihedron, a polytope whose faces are in bijection with 2-colored planar trees with n leaves (see Definition 4.2), was first introduced as a topological cell complex by J. Stasheff to describe morphisms between A_∞-algebras [Sta70]. The multiplihedron was only realized as a convex polytope recently, through the work of S. Forcey [For08a], and later S. Forcey and S. Devadoss [DF08], F. Ardila and J. Doker [AD13], and F. Chapoton and V. Pilaud [CP22]. The family of multiplihedra has been studied both in algebraic topology [BV73] and symplectic topology [MW10, Maz21], through their interpretation as moduli spaces of 2-colored metric trees.

A cellular approximation of the diagonal of the associahedra allows one to define the tensor product of two A_∞-algebras [SU04, MS06, MTTV21]. In this chapter, which is the result of a joint work with Thibaut Mazuir, our goal is to make this tensor product "functorial", by defining and studying a cellular approximation to the diagonal of the multiplihedra. Our results can be summarized as follows.

1. We define a cellular approximation of the diagonal of the Forcey–Loday realizations of the multiplihedra (Proposition 4.20);
2. We endow them with a compatible topological cellular operadic bimodule structure over the Loday realizations of the associahedra (Theorem 4.25);
3. We describe combinatorially the cellular image of the diagonal (Theorem 4.35);
4. We apply the cellular chains functor to obtain a "functorial" tensor product of A_∞-algebras and their categorification, known as A_∞-categories (Proposition 4.43).

To achieve these goals, we use the general theory developed in Chapter 2, based on the method introduced in [MTTV21]. For this purpose, we use the idea that the Forcey–Loday realization of the multiplihedron, as defined in [For08a], can be obtained from the Ardila–Doker realization of the multiplihedron [AD13] by projection. This last realization is a generalized permutahedron, in the sense of A. Postnikov [Pos09], which allows us to apply the results of Chapter 3 directly, both for the purpose of defining a cellular approximation of the diagonal, and to describe its cellular image combinatorially.

Our methods provide us with an universal tensor product, in the sense that its formula applies to any pair of A_∞-morphisms. We could call such a tensor product *operadic*. Unfortunately, our tensor product does not define a functor, since it is not strictly compatible with the composition of A_∞-morphisms. This is not a defect of our construction: in Theorem 4.46, we prove that there is no operadic tensor product satisfying this property. The proof is similar to a result of M. Markl and S. Schnider saying that there is no operadic tensor product of A_∞-algebras that satisfies associativity [MS06, Theorem 13]. However, our tensor product of A_∞-morphisms could still be a functor in some homotopical sense. We examine in Section 4.4.3 different perspectives regarding the possibility of endowing the category of A_∞-algebras with a (homotopy) symmetric monoidal structure.

Our present results can readily be applied to different fields, prominently to symplectic topology. The operadic bimodule structure in (2) above was used in the work of the second author, where the structures of A_∞-algebras and A_∞-morphisms are unraveled in the context of Morse theory [Maz21]. Moreover, our definition of a "functorial" tensor product of A_∞-categories by explicit formulas can be used to study the product of Fukaya algebras of Lagrangians [Amo17], and more generally the product of Fukaya categories of symplectic manifolds. It could also be used in the context of bordered Heegaard Floer homology [LOT20]. These applications are sketched in more details in Section 4.4.4. One could also think of applications in other areas where the multiplihedron appears, such as higher category theory [For08b].

Finally, the methods of this chapter can be straightfowardly extended to the "multiploperahedra", a family of polytopes which is to the operahedra of Chapter 3 what the multiplihedra are to the associahedra. They belong to both the families of graph-multiplihedra [DF08] and nestomultiplihedra [AD13]. Together with the results of Chapter 3, one would obtain a tensor product of ∞-morphisms between homotopy operads, defined by explicit formulas.

4.2 Realizations of the multiplihedra

Building on the work of S. Forcey in [For08a], we define Forcey–Loday realizations of the multiplihedra, and describe their general properties in Proposition 4.9. We show how they can be obtained from the Ardila–Doker realizations via projection.

4.2.1 Colored trees

We consider planar rooted trees, which we will abbreviate simply trees. The term "edge" refers to both internal and external edges. The external edges will sometimes be called leaves.

Definition 4.1 (Cut)**.** *A* cut *of a tree is a subset of edges or vertices which contains precisely one edge or vertex along the unique path from the root to any leaf.*

A cut divides a tree into two parts: an upper part, that we color in blue, and a lower part, that we color in red.

Definition 4.2 (2-colored tree)**.** *A* 2-colored tree *is a tree together with a cut. We call* 2-colored maximal tree *a 2-colored binary tree where the cut is made of edges only.*

We denote by CT_n (resp. CMT_n) the set of 2-colored trees (resp. 2-colored maximal trees) with n leaves, for $n \geq 1$.

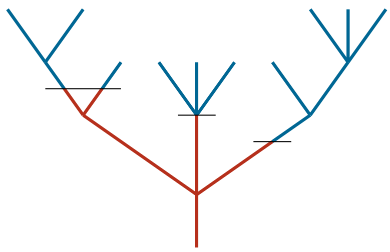
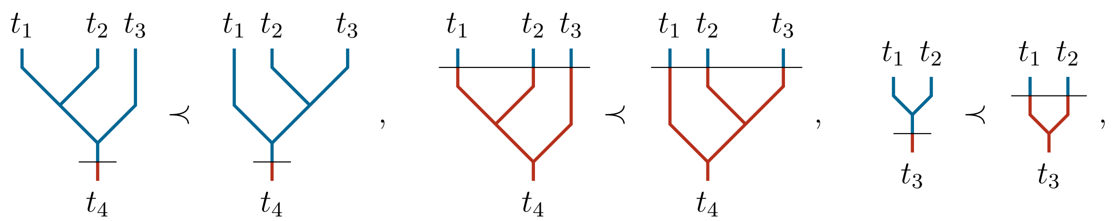


Figure 4.1: An example of a 2-colored tree.

Definition 4.3 (Face order and Tamari-type order)**.**

- ◇ *The* face order $s \subset t$ *on 2-colored trees is defined as follows: a 2-colored tree s is less than a 2-colored tree t if t can be obtained from s by a sequence of contractions of monochrome edges or moves of the color frontier from a family of edges to an adjacent vertex.*
- ◇ *The* Tamari-type order $s < t$ *on 2-colored maximal trees is generated by the following three covering relations:*



where t_i, for $1 \leq i \leq 4$, are 2-colored maximal trees, of respective colors each time.

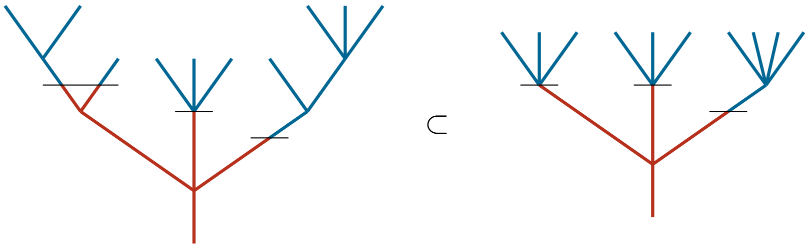


Figure 4.2: An example of the face order $s \subset t$.

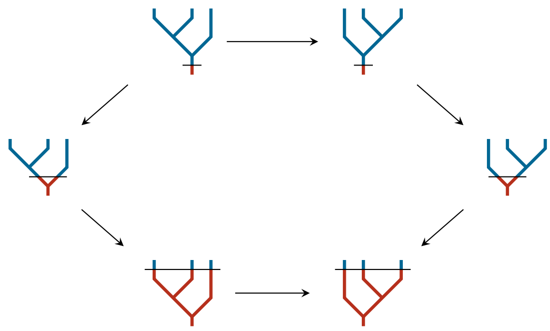


Figure 4.3: The Tamari-type poset $(\mathrm{CMT}_3, <)$ with minimum at the top.

We often add a minimum element $\emptyset_n$ to the poset of 2-colored trees.

Proposition 4.4. *The posets* $(\mathrm{CT}_n, \subset)$ *and* $(\mathrm{CMT}_n, <)$ *are lattices.*

Proof. The poset of 2-colored trees was proven in [For08a] to be isomorphic to the face lattice of a polytope, the multiplihedron; see Point (3) of Proposition 4.9. The Hasse diagram of the poset of 2-colored maximal trees was proven to be isomorphic to the oriented 1-skeleton of the multiplihedron, and also to be the Hasse diagram of a lattice in [CP22, Proposition 117]. □

Remark 4.5. F. Chapoton and V. Pilaud introduced in [CP22] the shuffle of two generalized permutahedra, which is again a generalized permutahedron (see Section 4.2.4 for definition and examples). The fact that the poset $(\mathrm{CMT}_n, <)$ is a lattice follows from the fact that the multiplihedron arises as the shuffle of the associahedron and the interval, which both have the lattice property, and that the shuffle operation preserves the lattice property in this case [CP22, Corollary 95]. However, the shuffle operation does not preserve the lattice property in general, see [CP22, Remark 140].

4.2.2 Multiplihedra

Definition 4.6 (Multiplihedra)**.** *For any $n \geq 1$, an* $(n-1)$-dimensional multiplihedron *is a polytope whose face lattice is isomorphic to the lattice* $(\mathrm{CT}_n, \subset)$ *of 2-colored trees with n leaves.*

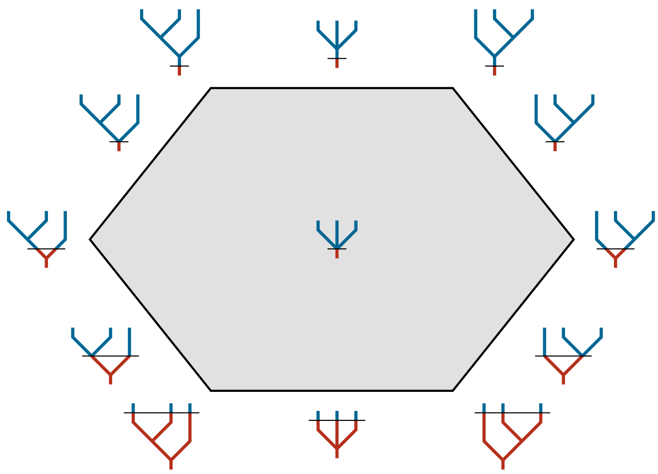
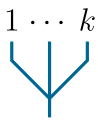
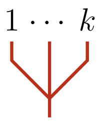
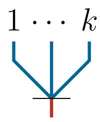


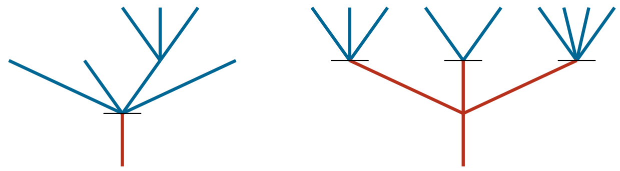
Figure 4.4: A 2-dimensional multiplihedron.

The dimension of a face labeled by a 2-colored tree is given by the sum of the degrees of its vertices defined by

|  | $= k-2$, |  | $= k-2$, |  | $= k-1$.

The codimension of a 2-colored tree is equal to the number of vertices of pure color. In the example of the 2-colored tree depicted on Figure 4.1, the dimension is equal to 4 and the codimension is equal to 5. As proven in [CP22, Proposition 117], the oriented 1-skeleton of a multiplihedron is the Hasse diagram of the Tamari-type poset.

Recall, for instance from [MTTV21], that one can also consider the set PT_n of planar trees and the set PBT_n of planar binary trees, with n leaves. They are equipped with similar orders and an $(n-2)$-dimensional polytope whose face lattice agrees with the lattice $(\mathrm{PT}_n, \subset)$ of planar trees is called an *associahedron*. The grafting of trees endows planar (binary) trees with a non-symmetric operad structure and 2-colored (maximal) trees with an operadic bimodule structure over it. Regarding the right action, we denote the operation of grafting a planar tree v at the i^{th}-leaf of a 2-colored tree u by $u \circ_i v$. Regarding the left action, we denote the grafting of a level of 2-colored trees $v_1, \ldots, v_k$ on the k leaves of a planar tree by $u(v_1, \ldots, v_k)$. We denote by c_n^{T} and by c_n^{B} the corollas with n leaves fully painted with the upper and the lower color respectively; we denote by c_n the corolla with n leaves with frontier color at the vertex. It is straightforward to see that these two grafting operations on corollas generate all the 2-colored trees of codimension 1: we call (B), for "bottom", the first type of 2-colored trees $c_{p+1+r} \circ_{p+1} c_q^{\mathrm{T}}$, with $p + q + r = n$ and $2 \leq q \leq n$, and we call (T), for "top", the second type of 2-colored trees $c_k^{\mathrm{B}}(c_1, \ldots, c_k)$, with $i_1 + \cdots + i_k = n$, $i_1, \ldots, i_k \geq 1$, and $k \geq 2$.



type (B) type (T)

Figure 4.5: Examples of 2-colored trees of type (B) and (T) respectively.

4.2.3 Forcey–Loday realizations of the multiplihedra

Jean-Louis Loday gave in [Lod04] realizations of the associahedra in the form of polytopes with integer coordinates. Stefan Forcey generalized this construction in [For08a] in order to give similar realizations for the multiplihedra.

Definition 4.7 (Weighted 2-colored maximal tree)**.** *A* weighted 2-colored maximal tree *is a pair (t, ω) made up of a 2-colored maximal tree $t \in \mathrm{CMT}_n$ with n leaves with some weight $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n_{>0}$. We call ω the* weight *and n the* arity *of the tree t or the* length *of the weight ω.*

Let (t, ω) be a weighted 2-colored maximal tree with n leaves. We order its $n-1$ vertices from left to right. At the i^{th} vertex, we consider the sum α_i of the weights of the leaves supported by its left input and the sum β_i of the weights of the leaves supported by its right input. If the i^{th} vertex is colored by the upper color, we consider the product $\alpha_i\beta_i$ and if the i^{th} vertex is colored by the lower color, we consider the product $2\alpha_i\beta_i$. The associated string produces a point with integer coordinates $M(t, \omega) \in \mathbb{R}^{n-1}_{>0}$. For example, if only the first and last vertices of t are blue, we obtain a point of the form

$$M(t, \omega) = \big(2\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_{n-2}\beta_{n-2}, 2\alpha_{n-1}\beta_{n-1}\big) \in \mathbb{R}^{n-1}_{>0} \ .$$

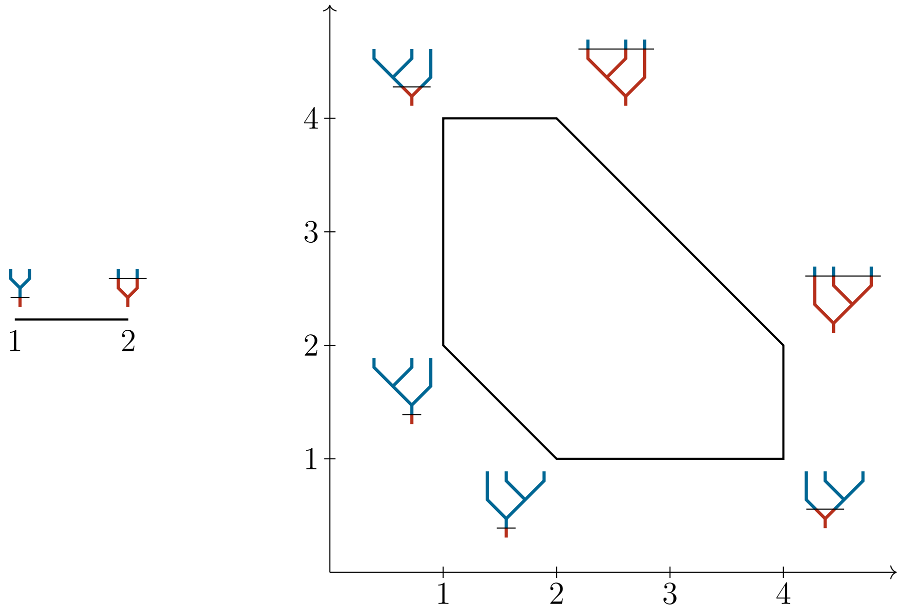


Figure 4.6: Examples of points associated to 2-colored maximal trees, with trivial weight.

Definition 4.8 (Forcey–Loday Realization)**.** *The* Forcey–Loday realization of weight *ω is the polytope*

$$\mathrm{J}_\omega := \mathrm{conv}\left\{ M(t,\omega) \mid t \in \mathrm{CMT}_n \right\} \subset \mathbb{R}^{n-1} \ .$$

The Forcey–Loday realization associated to the standard weight $(1,\ldots,1)$ is simply denoted by J_n. By convention, we define the polytope J_ω, with weight $\omega = (\omega_1)$ of length 1, to be made up of one point labeled by the 2-colored tree $\mathrm{i}_{\mathrm{B}}^{\mathrm{T}} :=$.

Proposition 4.9. *The Forcey–Loday realization J_ω satisfies the following properties.*

1. *Let $t \in \mathrm{CMT}_n$ be a 2-colored maximal tree.*

   For $p+q+r=n$, with $2 \leq q \leq n$, the point $M(t,\omega)$ is contained in the half-space defined by the inequality

$$x_{p+1} + \cdots + x_{p+q-1} \geq \sum_{p+1 \leq a < b \leq p+q} \omega_a \omega_b \ , \tag{B}$$

   with equality if and only if the 2-colored maximal tree t can be decomposed as $t = u \circ_{p+1} v$, where $u \in \mathrm{CMT}_{p+1+r}$ and $v \in \mathrm{PBT}_q$.

   For $i_1 + \cdots + i_k = n$, with $i_1, \ldots, i_k \geq 1$ and $k \geq 2$, the point $M(t,\omega)$ is contained in the half-space defined by the inequality

$$x_{i_1} + x_{i_1+i_2} + \cdots + x_{i_1+\cdots+i_{k-1}} \leq 2 \sum_{1 \leq j < l \leq k} \omega_{I_j} \omega_{I_l} \ , \tag{T}$$

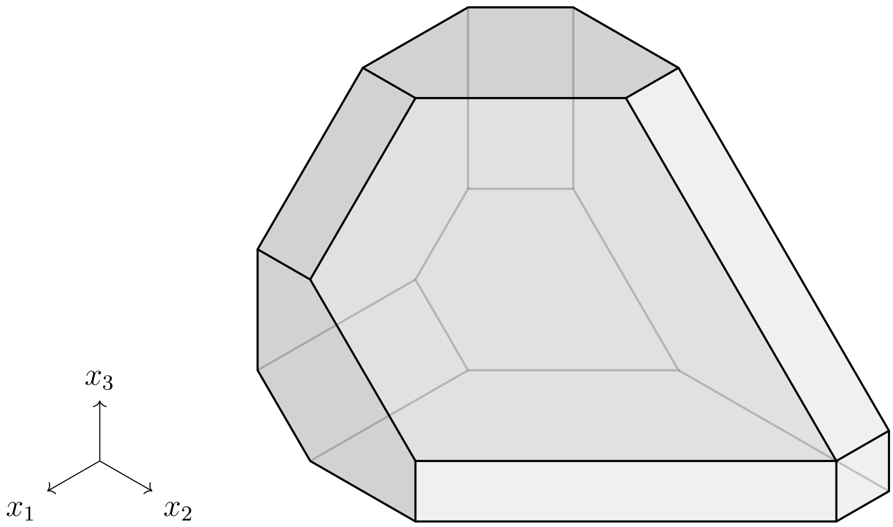


Figure 4.7: The Forcey–Loday realization of the multiplihedron J_4.

where $I_j = [i_1 + \cdots + i_{j-1} + 1, \ldots, i_1 + \cdots + i_j]$ *and* $\omega_{I_j} \coloneqq \sum_{a \in I_j} \omega_a$*, with equality if and only if the 2-colored maximal tree* t *can be decomposed as* $t = u(v_1, \ldots, v_k)$*, where* $u \in \mathrm{PBT}_k$ *and* $v_j \in \mathrm{CMT}_{i_j}$*, for* $1 \leq j \leq k$*.*

2. *The polytope* J_ω *is the intersection of the half-spaces defined in* (1)*.*

3. *The face lattice* $(\mathcal{L}(\mathrm{J}_\omega), \subset)$ *is isomorphic to the lattice* $(\mathrm{CT}_n, \subset)$ *of 2-colored trees with* n *leaves.*

4. *Any face of a Forcey–Loday realization of a multiplihedron is isomorphic to a product of a Loday realization of an associahedron with possibly many Forcey–Loday realizations of multiplihedra, via a permutation of coordinates.*

Proof. Points (1)–(3) were proved in [For08a]. We prove Point (4) by induction on n. It clearly holds true for $n = 1$. Let us suppose that it holds true up to $n - 1$ and let us prove it for the polytopes J_ω, for any weight ω of length n. We examine first facets. In the case of a facet of type (B) associated to $p + q + r = n$ with $2 \leq q \leq n - 1$, we consider the following two weights

$$\overline{\omega} \coloneqq (\omega_1, \ldots, \omega_p, \omega_{p+1} + \cdots + \omega_{p+q}, \omega_{p+q+1}, \ldots, \omega_n) \quad \text{and} \quad \widetilde{\omega} \coloneqq (\omega_{p+1}, \ldots, \omega_{p+q})$$

and the isomorphism

$$\begin{array}{rccc} \Theta_{p,q,r} : & \mathbb{R}^{p+r} \times \mathbb{R}^{q-1} & \xrightarrow{\cong} & \mathbb{R}^{n-1} \\ & (x_1, \ldots, x_{p+r}) \times (y_1, \ldots, y_{q-1}) & \mapsto & (x_1, \ldots, x_p, y_1, \ldots, y_{q-1}, x_{p+1}, \ldots, x_{p+r}) \,. \end{array}$$

The image of the vertices of $\mathrm{J}_{\overline{\omega}} \times \mathrm{K}_{\widetilde{\omega}}$ are sent to the vertices of the facet of J_ω labelled by the 2-colored tree $c_{p+1+r} \circ_{p+1} c_q^{\mathrm{T}}$. In other words, the permutation of coordinates Θ sends bijectively $\mathrm{J}_{\overline{\omega}} \times \mathrm{K}_{\widetilde{\omega}}$ to J_ω. Similarly, in the case of a facet of type (T) associated to $i_1 + \cdots + i_k = n$ with $i_1, \ldots, i_k \geq 1$ and $k \geq 2$, we consider the following weights

$$\overline{\omega} := \left(\sqrt{2}\omega_{I_1}, \ldots, \sqrt{2}\omega_{I_k}\right) \quad \text{and} \quad \widetilde{\omega}_j := (\omega_{i_1+\cdots+i_{j-1}+1}, \ldots, \omega_{i_1+\cdots+i_{j-1}+i_j}), \text{ for } 1 \leq j \leq k,$$

and the isomorphism

$$\Theta^{i_1,\ldots,i_k} \; : \; \mathbb{R}^{k-1} \times \mathbb{R}^{i_1-1} \times \cdots \times \mathbb{R}^{i_k-1} \;\xrightarrow{\cong}\; \mathbb{R}^{n-1}$$

which sends

$$(x_1, \ldots, x_{k-1}) \times (y_1^1, \ldots, y_{i_1-1}^1) \times \cdots \times (y_1^k, \ldots, y_{i_k-1}^k)$$

to

$$(y_1^1, \ldots, y_{i_1-1}^1, x_1, y_1^2, \ldots, y_{i_2-1}^2, x_2, y_1^3, \ldots, x_{k-1}, y_1^k, \ldots, y_{i_k-1}^k) \ .$$

The image of the vertices of $\mathrm{K}_{\overline{\omega}} \times \mathrm{J}_{\widetilde{\omega}_1} \times \cdots \times \mathrm{J}_{\widetilde{\omega}_k}$ are sent to the vertices of the facet of J_ω labelled by the 2-colored tree $c_k^{\mathrm{B}}(c_1, \ldots, c_k)$. In other words, the permutation of coordinates Θ sends bijectively $\mathrm{K}_{\overline{\omega}} \times \mathrm{J}_{\widetilde{\omega}_1} \times \cdots \times \mathrm{J}_{\widetilde{\omega}_k}$ to J_ω.

We can finally conclude the proof with these decompositions of facets of J_ω, the induction hypothesis, and [MTTV21, Proposition 1, Point (5)]. □

4.2.4 The multiplihedra as generalized permutahedra

Definition 4.10 (Permutahedron). *The* $(n-1)$-dimensional permutahedron *is the polytope in* $\mathbb{R}^n$ *equivalently defined as:*

- *the convex hull of the points* $\sum_{i=1}^{n} i e_{\sigma(i)}$ *for all permutations* $\sigma \in \mathbb{S}_n$, *or*
- *the intersection of the hyperplane* $\left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i = \binom{n+1}{2} \right\}$ *with the affine half-spaces* $\left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i \geq \binom{|I|+1}{2} \right\}$ *for all* $\emptyset \neq I \subseteq [n]$.

Definition 4.11 (Generalized permutahedron). *A* generalized permutahedron *is a polytope equivalently defined as:*

- *a polytope whose normal fan coarsens the one of the permutahedron, or*

- *the convex set*

$$\left\{ x \in \mathbb{R}^n \ : \ \sum_{i=1}^{n} x_i = z_{[n]} \ , \sum_{i \in I} x_i \geq z_I \text{ for all } I \subseteq [n] \right\} \ ,$$

where $\{z_I\}_{I \subseteq [n]}$ *are real numbers which satisfy the inequalities* $z_I + z_J \leq z_{I \cup J} + z_{I \cap J}$ *for all* $I, J \subseteq [n]$*, and where* $z_\emptyset = 0$*.*

Generalized permutahedra were introduced by A. Postnikov [Pos09], and are subject of a vast literature. They present many interesting combinatorial, geometric and algebraic properties. For instance, they are the universal family of polytopes possessing a certain Hopf algebraic structure [AA]. They are also a class of polytopes to which the results of Chapter 3 apply directly.

Loday realizations of the associahedra are all generalized permutahedra, while Forcey–Loday realizations of the multiplihedra are not. However, F. Ardila and J. Doker introduced in [AD13] realizations of the multiplihedra that are generalized permutahedra. They are obtained from the Loday realizations of the associahedra via the operation of "lifting". We consider here the special case $q = 1/2$ in their construction.

Definition 4.12 (Lifting of a generalized permutahedron [AD13, Definition 2.3])**.** *For a generalized permutahedron* $P \subset \mathbb{R}^n$*, its* $\frac{1}{2}$-lifting $P\left(\frac{1}{2}\right) \subset \mathbb{R}^{n+1}$ *is defined by*

$$P\left(\tfrac{1}{2}\right) := \left\{ x \in \mathbb{R}^{n+1} \ : \ \sum_{i=1}^{n+1} x_i = z_{[n]} \ , \sum_{i \in I} x_i \geq \tfrac{1}{2} z_I \ , \sum_{i \in I \cup \{n+1\}} x_i \geq z_I \text{ for all } I \subseteq [n] \right\} .$$

Proposition 4.13 ([AD13, Proposition 2.4])**.** *The* $\frac{1}{2}$*-lifting* $P\left(\frac{1}{2}\right)$ *of a generalized permutahedron is again a generalized permutahedron.*

Proposition 4.14. *The* $\frac{1}{2}$*-lifting* $\mathrm{K}_\omega\left(\frac{1}{2}\right)$ *of the Loday realization of the associahedron is a realization of the multiplihedron.*

Proof. This is a particular case of [AD13, Corollary 4.10]. □

We call the lifting of the Loday associahedron $\mathrm{K}_\omega\left(\frac{1}{2}\right)$ the Ardila–Doker realization of the multiplihedron. It is related to the Forcey–Loday realization via the projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ which forgets the last coordinate.

Proposition 4.15. *The Forcey-Loday realization of the multiplihedron is the image under the projection* π *of the* $\frac{1}{2}$*-lifting of the Loday realization of the associahedron, scaled by* 2*. That is, we have*

$$\mathrm{J}_\omega = \pi\left(2\mathrm{K}_\omega\left(\tfrac{1}{2}\right)\right) \ .$$

Proof. This follows from the vertex description of $\frac{1}{2}$-lifting given in [Dok11, Definition 3.5.3], together with the description of the projection from the permutahedron to the multiplihedron given in the proof of [Dok11, Theorem 3.3.6]. The coordinates of a vertex in $2\mathrm{K}_\omega$ are of the form $(2\alpha_1\beta_1, \ldots, 2\alpha_n\beta_n)$. A coordinate $2\alpha_i\beta_i$ is then multiplied by $1/2$ in the lifting if and only if its associated vertex in the 2-colored maximal tree is of the upper color. We thus recover the description of Definition 4.8. □

In summary, we have the following diagram:

Loday associahedron		Ardila–Doker multiplihedron		Forcey–Loday multiplihedron
$2\mathrm{K}_\omega$	$\hookrightarrow$	$2\mathrm{K}_\omega\left(\frac{1}{2}\right)$	$\overset{\pi}{\twoheadrightarrow}$	J_ω
$\mathbb{R}^n$	$\hookrightarrow$	$\mathbb{R}^{n+1}$	$\twoheadrightarrow$	$\mathbb{R}^n$
Gen. permutahedron		Gen. permutahedron		*Not* a gen. permutahedron

4.3 The diagonal of the multiplihedra

In this section, we define a cellular approximation of the diagonal of the Forcey–Loday realizations of the multiplihedra, and we endow them with an operadic bimodule structure over the Loday realizations of the associahedra. We use the method of [MTTV21] and the general theory developed in Chapter 2. Results of Chapter 3 can be applied directly to the Ardila–Doker multiplihedron, from which one obtains the Forcey–Loday multiplihedron by projection. One can transfer the results to these realizations, using the crucial fact that the projection preserves orthogonality for a certain class of vectors with coordinates equal to 0, 1 or -1. Another salient feature is that, in order to obtain the operadic structure, we need to make a choice of orientation vectors that is coherent with operadic composition, see Proposition 4.24. Finally, choosing an orientation and applying the cellular chains functor, we recover the operadic bimodule M_∞ with its usual sign conventions [Maz21].

4.3.1 Diagonal of the Forcey–Loday realizations of the multiplihedra

The projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ forgetting the last coordinate defines an affine isomorphism between any hyperplane H of equation $\sum_{i=1}^{n+1} x_i = c \in \mathbb{R}$, and $\mathbb{R}^n$. The inverse map $(\pi_{|H})^{-1}$ is given by the assignment

$$(x_1, \ldots, x_n) \mapsto \left(x_1, \ldots, x_n, c - \sum_{i=1}^{n} x_i \right) .$$

If a polytope P is contained in the hyperplane H, then the polytope $\pi(P)$ is affinely isomorphic to P, and the projection π defines a bijection between the faces of P and the faces of $\pi(P)$. Moreover, for every face F of P, we have $\dim F = \dim \pi(F)$.

However, the projection π does not preserve orthogonality in general, so if P is positively oriented by $\vec{v}$, the projection $\pi(P)$ might not be positively oriented by $\pi(\vec{v})$. We restrict our attention to a certain class of orientation vectors for which this property holds, in the case where P is a generalized permutahedron.

Definition 4.16. *A* good orientation vector *is a vector* $\vec{v} = (v_1, \ldots, v_{n+1}) \in \mathbb{R}^{n+1}$ *satisfying*

$$v_i \geqslant 2v_{i+1} \ , \ \textit{for any } 1 \leqslant i \leqslant n \ , \quad \textit{and} \quad v_{n+1} > 0 \ .$$

Observe that the family of good orientation vectors is stable under the projection forgetting the last coordinate: if $\vec{v}$ is a good orientation vector, then so is $\pi(\vec{v})$.

Being a good orientation vector is a more restrictive condition than being a principal orientation vector, in the sense of Definition 3.31. Thus, a good orientation vector orients positively any generalized permutahedron.

Proposition 4.17. *Let $P \subset \mathbb{R}^{n+1}$ be a generalized permutahedron, and let $\vec{v} \in \mathbb{R}^{n+1}$ be a good orientation vector. Then, the polytope $\pi(P)$ is positively oriented by $\pi(\vec{v})$. Moreover, the projection π commutes with the diagonal maps of P and $\pi(P)$, that is $\triangle_{(\pi(P),\pi(\vec{v}))} = (\pi \times \pi)\triangle_{(P,\vec{v})}$.*

Proof. Since P is a generalized permutahedron, the direction of the edges of the intersection $P \cap \rho_z P$, for any $z \in P$, are vectors with coordinates equal to $0, 1$ or -1, and the same number of 1 and -1 (combine Proposition 2.29 and Proposition 3.20). The direction $\vec{d}$ of such an edge satisfies $\langle \vec{d}, \vec{v} \rangle \neq 0$, since the first non-zero coordinate of $\vec{d}$ will contribute a greater amount than the sum of the remaining coordinates in the scalar product. For the same reason, we have $\langle \pi(\vec{d}), \pi(\vec{v}) \rangle \neq 0$. Indeed, we have that $\pi(P \cap \rho_z P) = \pi(P) \cap \rho_{\pi(z)} \pi(P)$, and in particular that the image of the edges of $P \cap \rho_z P$ under π are the edges of $\pi(P) \cap \rho_{\pi(z)} \pi(P)$. Thus, $\pi(P)$ is positively oriented by $\pi(\vec{v})$. For the last part of the statement, observe that π preserves the orientation of the edges: if we have $\langle \vec{d}, \vec{v} \rangle > 0$, then we have $\langle \pi(\vec{d}), \pi(\vec{v}) \rangle > 0$. Hence, the image of the vertex $\mathrm{top}_{\vec{v}}(P \cap \rho_z P)$, which maximizes $\langle -, \vec{v} \rangle$ over $P \cap \rho_z P$, under π is equal to the vertex $\mathrm{top}_{\pi(\vec{v})}(\pi(P) \cap \rho_{\pi(z)} \pi(P))$ which maximizes $\langle -, \pi(\vec{v}) \rangle$ over $\pi(P) \cap \rho_{\pi(z)} \pi(P)$. The argument for the minimum $\mathrm{bot}(P \cap \rho_z P)$ is the same. □

Proposition 4.18. *Let $P \subset \mathbb{R}^{n+1}$ be a generalized permutahedron. Any two good orientation vectors $\vec{v}, \vec{w}$ define the same diagonal maps on P and $\pi(P)$, that is, we have $\triangle_{(P,\vec{v})} = \triangle_{(P,\vec{w})}$ and $\triangle_{(\pi(P),\pi(\vec{v}))} = \triangle_{(\pi(P),\pi(\vec{w}))}$.*

Proof. Good orientation vectors are principal orientation vectors as in Definition 3.31. Since all principal orientation vectors live in the same chamber of the fundamental hyperplane arrangement of the permutahedron, they all define the same diagonal on the permutahedron (Proposition 2.22), and thus the same diagonal on any generalized permutahedron (Corollary 2.30). So, we have $\triangle_{(P,\vec{v})} = \triangle_{(P,\vec{w})}$. Finally, using Proposition 4.17, we have $\triangle_{(\pi(P),\pi(\vec{v}))} = (\pi \times \pi)\triangle_{(P,\vec{v})} = (\pi \times \pi)\triangle_{(P,\vec{w})} = \triangle_{(\pi(P),\pi(\vec{w}))}$. □

Definition 4.19. *A* well-oriented realization of the multiplihedron *is a positively oriented polytope which realizes the multiplihedron and such that the orientation vector induces the Tamari-type lattice on the set of vertices.*

Proposition 4.20. *Any good orientation vector induces a well-oriented realization* $(\mathrm{J}_\omega, \vec{v})$ *of the Forcey–Loday multiplihedron, for any weight* ω*.*

Proof. The proof of Proposition 4.17 shows that any edge of the realization of the muliplihedron J_ω is directed, according to the Tamari type order, by either $\vec{e}_i$ or $\vec{e}_i - \vec{e}_j$, for $i < j$. Since $\vec{v}$ has strictly decreasing coordinates, in each case the scalar product is positive. It remains to show that $P \cap \rho_z P$ is oriented by $\vec{v}$, for any $z \in P$. This follows directly from Proposition 4.17, and the fact that J_ω arises as the projection under π of a generalized permutahedron, see Proposition 4.15. □

Any good orientation vector defines a diagonal map $\triangle_\omega : \mathrm{J}_\omega \to \mathrm{J}_\omega \times \mathrm{J}_\omega$, for any weight ω. These maps are all equivalent, up to isomorphism in the category Poly.

Proposition 4.21. *For any pair of weights* ω *and* θ *of length* n*, there exists a unique isomorphism* $\mathrm{tr} = \mathrm{tr}_\omega^\theta : \mathrm{J}_\omega \to \mathrm{J}_\theta$ *in the category* Poly*, which preserves homeomorphically the faces of the same type and which commutes with the respective diagonals.*

Proof. The arguments of [MTTV21, Sections 3.1-3.2] hold in the present case using Proposition 4.9. Requiring commutation with the respective diagonals is really what makes the map tr unique. □

Definition 4.22. *We denote by* $\triangle_n : \mathrm{J}_n \to \mathrm{J}_n \times \mathrm{J}_n$ *the diagonal induced by any good orientation vector for the Forcey–Loday realization of standard weight* $\omega = (1, \ldots, 1)$*.*

4.3.2 Operadic bimodule structure

We will use the transition map tr of Proposition 4.21 above to endow the family of standard weight Forcey–Loday multiplihedra with an operadic bimodule structure over the standard weight Loday associahedra. We will use the uniqueness property of the map tr in a crucial way.

Definition 4.23 (Action-composition maps)**.** *For any $n, m \geq 1$ and any $1 \leq i \leq m$, for any $k \geq 2$ and any $i_1, \ldots, i_k \geq 1$, we define the* action-composition maps *by*

$$\circ_{p+1} \ : \ \mathrm{J}_{p+1+r} \times \mathrm{K}_q \xrightarrow{\mathrm{tr}\times\mathrm{id}} \mathrm{J}_{(1,\ldots,q,\ldots,1)} \times \mathrm{K}_q \xrightarrow{\Theta_{p,q,r}} \mathrm{J}_n \quad \textit{and}$$

$$\gamma_{i_1,\ldots,i_k} \ : \ \mathrm{K}_k \times \mathrm{J}_{i_1} \times \cdots \times \mathrm{J}_{i_k} \xrightarrow{\mathrm{tr}\times\mathrm{id}} \mathrm{K}_{(i_1,\ldots,i_k)} \times \mathrm{J}_{i_1} \times \cdots \times \mathrm{J}_{i_k} \xrightarrow{\Theta^{i_1,\ldots,i_k}} \mathrm{J}_{i_1+\cdots+i_k} \ ,$$

where the last inclusions are given by the block permutations of the coordinates introduced in the proof of Proposition 4.9.

Now, we show that the choice of diagonal maps $\triangle_n : \mathrm{J}_n \to \mathrm{J}_n \times \mathrm{J}_n$ is coherent with operadic composition.

Proposition 4.24. *The diagonal maps $\triangle_n$ commute with the maps Θ.*

Proof. First observe that a good orientation vector has decreasing coordinates, so it induces the diagonal maps $\triangle_n : \mathrm{K}_n \to \mathrm{K}_n \times \mathrm{K}_n$ and the non-symmetric operad structure on $\{\mathrm{K}_n\}$ defined in [MTTV21]. As shown in Proposition 3.51, to prove the claim it suffices to show that the preimage under Θ^{-1} of a good orientation vector is still a good orientation vector for each associahedron and multiplihedron. This is easily seen to be the case from the definition of Θ, in the proof of Point (4) of Proposition 4.9. □

Theorem 4.25.

1. *The collection $\{\mathrm{J}_n\}_{n\geq 1}$ together with the action-composition maps $\circ_i$ and $\gamma_{i_1,\ldots,i_k}$ form an operadic bimodule over the non-symmetric operad $\{\mathrm{K}_n\}$ in the category* Poly*.*
2. *The maps $\{\triangle_n : \mathrm{J}_n \to \mathrm{J}_n \times \mathrm{J}_n\}_{n\geq 1}$ form a morphism of $(\{\mathrm{K}_n\}, \{\mathrm{K}_n\})$-operadic bimodules in the category* Poly*.*

Proof. Once we have in hand Proposition 4.24 asserting that the diagonal maps commute with the maps Θ, we can apply the proof of [MTTV21, Theorem 1] *mutatis mutandis*. The uniqueness of the transition map tr is the essential ingredient, as it forces the operadic axioms to hold. □

This theorem was mentioned in [Maz21], where associahedra and multiplihedra were interpreted as compactifications of moduli spaces of metric trees, and used to unravel A_∞ structures on the Morse cochains of a smooth compact manifold.

From the general theory of operads, we know that the data of a $(\mathcal{P}, \mathcal{Q})$-operadic bimodule $\mathcal{M}$ is equivalent to the data of a 2-colored operad. Under the cellular chains functor, Theorem 4.25 gives a quasi-free 2-colored operad, spanned by blue corollas in degree $|c_n^B| = n - 2$, red corollas in degree $|c_n^T| = n - 2$ and bicolored corollas in degree $|c_n| = n - 1$. An algebra over this differential graded 2-colored operad is a pair of A_∞-algebras related by an A_∞-morphism.

4.3.3 Differential graded structures

Let us quickly recall the definitions of A_∞-algebra and A_∞-morphism, and at the same time establish our sign conventions. For more details, we refer to [LV12, Chapter 9].

Definition 4.26 (A_∞-algebra). *An* A_∞-algebra *is a graded vector space A together with operations*

$$m_n : A^{\otimes n} \to A \ , \ n \geq 1$$

of degree $|m_n| = n - 2$, satisfying the equations

$$\sum_{p+q+r=n} (-1)^{p+qr} m_{p+1+r}(\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r}) = 0 \ , \ n \geq 1 \ .$$

An A_∞-algebra is an algebra over the differential graded non-symmetric operad A_∞. This quasi-free operad is generated by the operations m_n and its differential encodes the relations that they satisfy.

Definition 4.27 (A_∞-morphism). *An* A_∞-morphism *$A \rightsquigarrow B$ between two A_∞-algebras $(A, \{m_n\})$ and $(B, \{m'_n\})$ is a family of linear maps*

$$f_n : A^{\otimes n} \to B \ , n \geq 1$$

of degree $|f_n| = n - 1$, satisfying the equations

$$\sum_{i_1+\cdots+i_k=n} (-1)^{\varepsilon} m'_k(f_{i_1} \otimes \cdots \otimes f_{i_k}) = \sum_{p+q+r=n} (-1)^{p+qr} f_{p+1+r}(\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r}) \ , \ n \geq 1 \ ,$$

where $\varepsilon = \sum_{u=1}^{k} (k-u)(1-i_u)$.

An A_∞-morphism is an algebra over the differential graded operadic bimodule M_∞. This quasi-free operadic bimodule is generated by a family of elements ν_n of degree $|\nu_n| = n-1$ and its differential encodes the relations satisfied by the f_n above.

Now, we promote the operadic bimodule $\{J_n\}_{n\geq 1}$ to an operadic bimodule in the category of CW complexes by making a choice of orientation on the multiplihedra. We aim at recovering, via the cellular chains functor, the previous sign conventions for A_∞-algebras and A_∞-morphisms.

We make the same choices of cellular orientations as in [Maz21, I, Section 4]. For a (2-colored) tree t, we order its vertices from bottom to top and from left to right, proceeding one level at a time. We call this the *left-levelwise order* on t. There is a unique decomposition of $t = (\cdots((c_{n_1} \circ_{i_1} c_{n_2}) \circ_{i_2} c_{n_3}) \cdots \circ_{i_k} c_{n_{k+1}})$ where the (2-colored) corollas are grafted according to this total order.

- For the associahedra $\mathrm{K}_n \subset \mathbb{R}^{n-1}$, we choose as positively oriented basis of the top dimensional cell the basis $\{e_1 - e_{j+1}\}_{1\leq j\leq n-2}$. Then, we choose the orientation of any other cell t of K_n to be the image of the positively oriented basis of the top cells of the polytopes K_{n_i} under the sequence of compositions following the left-recursive order on t.
- For the multiplihedra $\mathrm{J}_n \subset \mathbb{R}^{n-1}$, we choose as positively oriented basis of the top dimensional cell the basis $\{-e_j\}_{1\leq j\leq n-1}$. Then, we choose the orientation of any other cell t of J_n to be the image of the positively oriented basis of the top cells of the polytopes K_{n_i} and J_{n_j} under the sequence of action-compositions following the left-recursive order on t.

We then proceed as in Proposition 3.59 to endow the K_n and J_n with a CW structure.

Proposition 4.28. *The above cellular orientations on the associahedra and multiplihedra yield an isomorphism of differential graded non-symmetric operads $C_\bullet^{cell}(\{\mathrm{K}_n\}) \cong \mathrm{A}_\infty$ and an isomorphism of operadic bimodules $C_\bullet^{cell}(\{\mathrm{J}_n\}) \cong \mathrm{M}_\infty$.*

Proof. Computing the signs in the (action-)composition maps amounts to comparing bases where the vectors have been permuted. Since we have choosen the left-levelwise order on trees, we recover precisely the usual sign conventions for the operad A_∞ and of the operadic bimodule M_∞. The signs for the differentials, arising from the boundary maps, were computed in [Maz21, I, Section 4]. □

The image of the diagonal maps $\triangle_n : \mathrm{J}_n \to \mathrm{J}_n \times \mathrm{J}_n$ under this functor gives a morphism of differential graded operadic bimodules $\mathrm{M}_\infty \to \mathrm{M}_\infty \otimes \mathrm{M}_\infty$, and thus defines the tensor product of two A_∞-morphisms.

4.4 Tensor product of A-infinity morphisms

We provide a combinatorial description of the cellular image of the diagonal of the Forcey–Loday multiplihedra, defined in the preceding section. We make use of the universal formula of Theorem 2.25, as well as the computation of the fundamental hyperplane arrangement of the permutahedra from Chapter 3. The key fact that the Ardila–Doker multiplihedron is a generalized permutahedron allows us to avoid the computation of the fundamental hyperplane arrangement of the multiplihedron itself. Applying the cellular chains functor, we obtain an explicit and universal formula for the tensor product of A_∞-morphisms between A_∞-algebras, and at the same time for A_∞-functors between A_∞-categories. However, this formula is not strictly compatible with the composition of A_∞-morphisms. We prove in Section 4.4.3 that this is in fact the case for *any* operadic tensor product, and we discuss some perspectives regarding the possibility of endowing the category of A_∞-algebras with a (homotopy) symmetric monoidal structure. We conclude with an overview of some possible applications of our results in symplectic topology.

4.4.1 Cellular formula

We introduce an equivalent description of 2-colored trees as 2-colored nested linear graphs, a language that is more suitable for the description of the cellular image of the diagonal maps $\triangle_n$.

Let ℓ be a linear graph with n vertices. We number its edges from 1 to $n-1$ from bottom to top. We write $V(\ell)$ and $E(\ell)$ for its sets of vertices and edges, respectively. Any subset of edges $N \subset E(\ell)$ defines a subgraph of g whose edges are N and whose vertices are all the vertices adjacent to an edge in N. We call this graph the *closure* of N.

Definition 4.29 (Nest and nesting)**.**

- *A* nest *of a linear graph ℓ with n vertices is a non-empty set of edges $N \subset E(\ell)$ whose closure is a connected subgraph of ℓ.*
- *A* nesting *of a graph ℓ is a set $\mathcal{N} = \{N_i\}_{i \in I}$ of nests such that*
  1. *the* trivial nest *$E(\ell)$ is in $\mathcal{N}$,*
  2. *for every pair of nests $N_i \neq N_j$, we have either $N_i \subsetneq N_j$, $N_j \subsetneq N_i$ or $N_i \cap N_j = \emptyset$, and*
  3. *if $N_i \cap N_j = \emptyset$ then no edge of N_i is adjacent to an edge of N_j.*

Two nests that satisfy Conditions (2) and (3) are said to be *compatible*. We denote the set of nestings of ℓ by $\mathcal{N}(\ell)$. We naturally represent a nesting by circling the closure of each nest as in Figure 4.8. A nesting is *maximal* if it has maximal cardinality $|\mathcal{N}| = |E(\ell)|$.

Definition 4.30 (2-colored nesting)**.** *A* 2-colored nesting *is a nesting where each nest is either colored in blue, red or purple, and which satisfy the following properties:*

1. *if a nest N is blue or purple, then all nests contained in N are blue, and*
2. *if a nest N is red or purple, then all nests that contain N are red.*

One should think of purple nests as both colored by red and blue. We call *monochrome* the nests that are either blue or red, and *bicolored* the purple nests. We denote by $\mathrm{mono}(\mathcal{N})$ the set of monochrome nests of a 2-colored nesting $\mathcal{N}$. We denote by $\mathcal{N}_2(\ell)$ the set of 2-colored nesting of ℓ. A 2-colored nesting is *maximal* if it has maximal cardinality, and it is made of blue and red nests only.

Remark 4.31. The data of a 2-colored nesting on a graph is equivalent to the data of a marked tubing on its line graph, as defined in [DF08]. See also Remark 3.4.

Lemma 4.32. *There is a bijection between 2-colored trees with n leaves and 2-colored nested linear graphs with n vertices.*

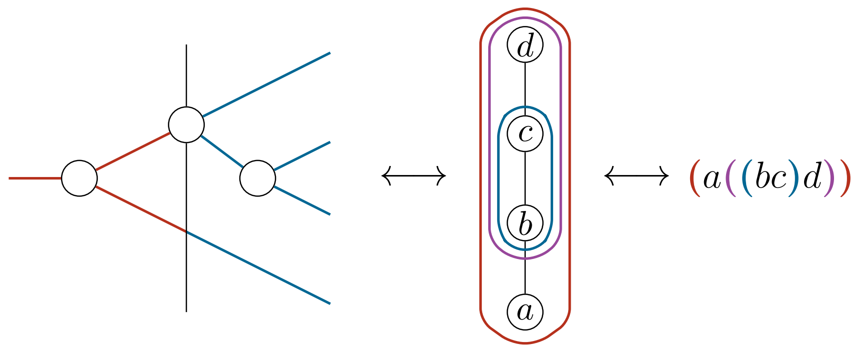


Figure 4.8: Bijections between 2-colored trees, 2-colored nested linear graphs, and 2-colored parenthesizations.

Proof. This is a simple generalization of the bijection between planar trees and nested linear graphs, see Figure 4.8. Under this bijection, vertices of 2-colored trees correspond to nests, and their colors agree under the previous conventions. Also, 2-colored maximal trees are in bijection with maximal 2-colored nested linear graphs. □

Definition 4.33. *Let $\mathcal{N}$ be a 2-colored nesting of a linear graph with n vertices. We respectively denote by $B(\mathcal{N})$, $P(\mathcal{N})$ and $R(\mathcal{N})$ the set of blue, purple and red nests of $\mathcal{N}$, and we define*

$$Q(\mathcal{N}) := \bigcup_{R_1,\ldots,R_k \in R(\mathcal{N})} \left\{ \bigcup_{i=1}^{k} R_i \cup \bigcup_{B \in B(\mathcal{N})} B \cup \bigcup_{P \in P(\mathcal{N})} P \right\},$$

where the braces are the set braces, and where we allow $\cup R_i = \emptyset$.

We number the edges of the linear graph with n vertices from bottom to top as represented in Figure 4.8, starting at 1 and ending at $n-1$. To each blue nest $B \in B(\mathcal{N})$ in a 2-colored nesting $\mathcal{N}$ of a linear graph with n vertices, we associate the *characteristic vector* $\vec{B} \in \mathbb{R}^n$ which has a 1 in position i if $i \in B$, 0 in position i if $i \notin B$ and 0 in position n. To each union of nests $Q \in Q(\mathcal{N})$, we associate the characteristic vector $\vec{Q} \in \mathbb{R}^n$ which has a 1 in position i if $i \in Q$, 0 in position i if $i \notin Q$ and 1 in position n. We denote moreover by $\vec{n}$ the vector $(1, \ldots, 1) \in \mathbb{R}^n$.

Lemma 4.34. *The normal cone of the face of the Ardila–Doker realization of the multiplihedron labeled by the 2-colored nesting $\mathcal{N}$ is given by*

$$\text{Cone}\left(\{-\vec{B}\}_{B\in B(\mathcal{N})} \cup \{-\vec{Q}\}_{Q\in Q(\mathcal{N})} \cup \{\vec{n}, -\vec{n}\}\right) .$$

Proof. This follows from the description of the Ardila–Doker multiplihedron as a generalized permutahedron. The normal cone of a face of the former is a union of normal cones of faces of the permutahedron. These can be easily determined from the projection from the permutahedron to the multiplihedron, written down explicitly in the proof of [Dok11, Theorem 3.3.6]. □

We are now ready to apply Theorem 2.25 to the multiplihedra. We define

$$D(n) := \{(I, J) \mid I, J \subset \{1, \dots, n\}, |I| = |J|, I \cap J = \emptyset, \min(I \cup J) \in I\}.$$

Theorem 4.35. *Let J_n be the Forcey–Loday realization of standard weight of the multiplihedron, and let $\vec{v} \in \mathbb{R}^n$ be a good orientation vector. The cellular image of the diagonal map $\triangle_n : \mathrm{J}_n \to \mathrm{J}_n \times \mathrm{J}_n$ admits the following description. For $\mathcal{N}$ and $\mathcal{N}'$ two 2-colored nestings of the linear graph with n vertices, we have*

$$\begin{aligned}
(\mathcal{N}, \mathcal{N}') \in \operatorname{Im} \triangle_n \quad \Longleftrightarrow \quad & \forall (I, J) \in D(n), \\
& \exists B \in B(\mathcal{N}), |B \cap I| > |B \cap J| \ \textit{or} \\
& \exists Q \in Q(\mathcal{N}), |(Q \cup \{n\}) \cap I| > |(Q \cup \{n\}) \cap J| \ \textit{or} \\
& \exists B' \in B(\mathcal{N}'), |B' \cap I| < |B' \cap J| \ \textit{or} \\
& \exists Q' \in Q(\mathcal{N}'), |(Q' \cup \{n\}) \cap I| < |(Q' \cup \{n\}) \cap J| \ .
\end{aligned}$$

Proof. The essential ingredient is the computation of the fundamental hyperplane arrangement of the permutahedron, which was already done in Section 3.3.1. The result follows in four steps:

1. Since a good orientation vector $\vec{v}$ is also a principal orientation vector (Definition 3.31), it orients positively the permutahedron.

2. Theorem 2.25 then gives a combinatorial description of the cellular image of the diagonal of the permutahedron induced by $\vec{v}$, which is precisely Theorem 3.32.

3. Using Proposition 2.32 and the description of the normal cones of the faces of the multiplihedron in Lemma 4.34, we get the above formula for the Ardila–Doker realizations of the multiplihedra.

4. Proposition 4.17 garantees that this formula holds for the Forcey–Loday realizations, which completes the proof.

□

The cellular image of $\triangle_n$ can be represented on J_n as follows: for each pair of faces $(F, G) \in \operatorname{Im} \triangle_n$, draw the polytope $(F+G)/2$. This defines a polytopal subdivision of J_n. The polytopal subdivision of J_4 is illustrated on the first page of this Chapter.

Let us make this formula explicit in low dimensions. We write 2-colored nestings of a linear graph with n vertices as 2-colored parenthesizations of a word with n letters, which are easier to read and shorter to type, see Figure 4.8. We show only pairs of faces (F, G) such that $\dim F + \dim G = \dim P$; the other pairs can be deduced by taking faces.

$$\triangle_2((ab)) = (ab) \times (ab) \cup (ab) \times (ab)$$

$$\begin{aligned}\triangle_3((abc)) = {} & ((ab)c) \times (abc) \cup (abc) \times (a(bc)) \cup (abc) \times (a(bc)) \\ \cup {} & (abc) \times (a(bc)) \cup (a(bc)) \times (a(bc)) \cup ((ab)c) \times ((ab)c) \\ \cup {} & ((ab)c) \times (abc) \cup ((ab)c) \times (abc)\end{aligned}$$

$$\triangle_4((abcd)) =$$

$$\begin{aligned}
& (((ab)c)d) \times (abcd) \cup (abcd) \times (a(b(cd))) \cup ((abc)d) \times (a(bc)d) \\
\cup\ & ((ab)(cd)) \times (ab(cd)) \cup ((abc)d) \times (a(bcd)) \cup ((ab)cd) \times (ab(cd)) \\
\cup\ & (a(bc)d) \times (a(bcd)) \cup ((abc)d) \times (a(bc)d) \cup ((ab)cd) \times (ab(cd)) \\
\cup\ & ((abc)d) \times (a(bcd)) \cup (((ab)c)d) \times ((abc)d) \cup (ab(cd)) \times (a(b(cd))) \\
\cup\ & ((ab)(cd)) \times ((ab)(cd)) \cup (a(bc)d) \times (a((bc)d)) \cup ((ab)cd) \times ((ab)(cd)) \\
\cup\ & (a(bc)d) \times (a(bcd)) \cup (a((bc)d)) \times (a(bcd)) \cup ((ab)cd) \times ((ab)(cd)) \\
\cup\ & (a(bc)d) \times (a(bcd)) \cup ((abc)d) \times (a(bcd)) \cup ((ab)cd) \times (ab(cd)) \\
\cup\ & (((ab)c)d) \times ((ab)cd) \cup (a(bcd)) \times (a(b(cd))) \cup (((ab)c)d) \times ((ab)cd) \\
\cup\ & (a(bcd)) \times (a(b(cd))) \cup (a(bc)d) \times (a(bcd)) \cup (((ab)c)d) \times (abcd) \\
\cup\ & (abcd) \times (a(b(cd))) \cup (((ab)c)d) \times (abcd) \cup (abcd) \times (a(b(cd))) \\
\cup\ & (((ab)c)d) \times (abcd) \cup (abcd) \times (a(b(cd))) \cup ((ab)(cd)) \times (ab(cd)) \\
\cup\ & ((abc)d) \times ((a(bc))d) \cup ((ab)(cd)) \times (ab(cd)) \cup ((abc)d) \times (a((bc)d)) \\
\cup\ & ((abc)d) \times (a(bc)d) \cup ((ab)cd) \times (ab(cd)) \cup ((abc)d) \times (a(bcd)) \\
\cup\ & ((a(bc))d) \times (a(bc)d) \cup ((abc)d) \times (a(bc)d) \cup (((ab)c)d) \times (a(bc)d)
\end{aligned}$$

The number of pairs of faces of complementary dimensions in the image of $\triangle_n$ are given, for $n = 0$ to 6, by $1, 2, 8, 42, 254, 1678$ and 11790, respectively. This sequence of integers does not (yet) appear on the Online Encyclopedia of Integer Sequences (OEIS).

For every face F of the multiplihedron J_n, a good orientation vector $\vec{v}$ defines a unique vertex $\operatorname{top} F$ (resp. $\operatorname{bot} F$) which maximizes (resp. minimizes) the scalar product $\langle -, \vec{v} \rangle$. By Proposition 2.16, we have that any pair of faces $(F, G) \in \operatorname{Im} \triangle_n$ satisfies $\operatorname{top} F \leq \operatorname{bot} G$. In the cases of the simplices, the cubes and the associahedra, the converse also holds: we can characterize the diagonal with a formula of the form $(F, G) \in \operatorname{Im} \triangle_n \iff \operatorname{top} F \leq \operatorname{bot} G$. However, it was shown in Chapter 3 that this does not hold anymore for the operahedra, and in particular for the permutahedra. This property does not hold either in the case of the multiplihedra. For instance, in dimension 3 the condition $\operatorname{top} F \leq \operatorname{bot} G$ determines completely all the pairs in $\operatorname{Im} \triangle_4$ except four of them: $((abc)d) \times (a((bc)d))$, $((abc)d) \times (a(bc)d)$, $((abc)d) \times (a(bc)d)$ and $(((ab)c)d) \times (a(bc)d)$. It seems likely that the condition $\operatorname{top} F \leq \operatorname{bot} G$ is equivalent to the conditions in Theorem 4.35 for the pairs (I, J) such that $|I| = |J| = 1$, as it is for the permutahedron Proposition 3.33.

Remark 4.36. Observe that our choice of diagonal differs from the one of [SU04]. For instance, the four pairs of faces above are not part of the image of the Saneblidze–Umble diagonal in dimension 3. A way to relate the two constructions would be to find a choice of chambers in the fundamental hyperplane arrangement of the permutahedra (or the multiplihedra) recovering the latter diagonal, see also Remark 3.35.

The formula above, even though easily implemented in the computer, is not optimal. For instance, one could use Equation (2.1) in Theorem 2.25 to reduce the number of pairs (I, J) on which the conditions have to be tested. One could also compute directly the fundamental hyperplane arrangement of the Ardila–Doker or Forcey–Loday multiplihedron. It would be desirable to obtain equivalent combinatorial descriptions of $\operatorname{Im} \triangle_n$, with a view towards possible applications. We sketch some of them in the next sections.

4.4.2 Application to A-infinity algebras and categories

The diagonal of the associahedra gives a tensor product of A_∞-algebras, and the diagonal of the multiplihedra gives a tensor product of A_∞-morphisms. They also define tensor products on the categorical enrichment of these two notions.

Definition 4.37. *An A_∞-category $\mathcal{A}$ consists of*

- *a set of objects* $\mathrm{Ob}(\mathcal{A})$,
- *for each pair of objects* $X, Y \in \mathrm{Ob}(\mathcal{A})$, *a graded vector space* $\mathcal{A}(X, Y)$, *and*

- *for each* $n \geq 1$ *and each family of objects* $X_0, \ldots, X_n \in \mathrm{Ob}(\mathcal{A})$*, a linear map*

$$m_n : \mathcal{A}(X_0, X_1) \otimes \cdots \otimes \mathcal{A}(X_{n-1}, X_n) \to \mathcal{A}(X_0, X_n)$$

of degree $|m_n| = n - 2$*, satisfying the equations*

$$\sum_{p+q+r=n} (-1)^{p+qr} m_{p+1+r}(\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r}) = 0 \ , \ n \geq 1 \ .$$

Observe that an A_∞-algebra is just an A_∞-category with one object. Thus, the diagonal of the associahedra provides us with a canonical A_∞-category structure on the tensor product $\mathcal{A} \otimes \mathcal{B}$ of two A_∞-categories, which we describe now.

Definition 4.38. *Let* $(\ell, \mathcal{N})$ *be a nested linear graph. The* left-levelwise order *on* $\mathcal{N}$ *is defined as follows: we order the nests by decreasing cardinality, and we order two nests of same cardinality by comparing their minimal elements.*

The left-levelwise order on a nesting is induced by the left-levelwise order on trees, under the bijection of Lemma 4.32.

Definition 4.39. *Let* ℓ *be a linear graph.*

1. *For a nesting* $\mathcal{N}$ *of* ℓ*, we denote by* N_i *the unique minimal nest of* $\mathcal{N}$ *containing the edge* i*, with respect to nest inclusion.*

2. *An edge* i *of* ℓ *is* admissible *with respect to* $\mathcal{N}$ *if* $i \neq \min N_i$*. We denote the set of admissible edges of* $\mathcal{N}$ *by* $\mathrm{Ad}(\mathcal{N})$*.*

3. *Given a pair of nestings* $\mathcal{N}, \mathcal{N}'$*, we give the set* $\mathrm{Ad}(\mathcal{N}) \sqcup \mathrm{Ad}(\mathcal{N}')$ *the total order by using the left-levelwise order on the nestings and within a nest by following the numbering of the edges in increasing order.*

4. *The function* $\sigma_{\mathcal{N}\mathcal{N}'} : \mathrm{Ad}(\mathcal{N}) \sqcup \mathrm{Ad}(\mathcal{N}') \to (1, 2, \ldots, |\mathrm{Ad}(\mathcal{N}) \sqcup \mathrm{Ad}(\mathcal{N}')|)$ *defined on* $i \in \mathrm{Ad}(\mathcal{N})$ *by*

$$\sigma_{\mathcal{N}\mathcal{N}'}(i) = \begin{cases} \min N_i - 1 & \text{if } i \in \mathrm{Ad}(\mathcal{N}) \cap \mathrm{Ad}(\mathcal{N}') \text{ and } 1 \neq \min N_i < \min N_i' \\ i - 1 & \text{otherwise ,} \end{cases}$$

and similarly on $i \in \mathrm{Ad}(\mathcal{N}')$ *by inversing the roles of* $\mathcal{N}$ *and* $\mathcal{N}'$*, induces a permutation of the set* $\{1, 2, \ldots, |\mathrm{Ad}(\mathcal{N}) \sqcup \mathrm{Ad}(\mathcal{N}')|\}$ *that we still denote by* $\sigma_{\mathcal{N}\mathcal{N}'}$*.*

We are now ready to define the tensor product of two A_∞-categories. We denote by σ_n the isomorphism $(A_1 \otimes B_1) \otimes (A_2 \otimes B_2) \otimes \cdots \otimes (A_n \otimes B_n) \cong (A_1 \otimes \cdots \otimes A_n) \otimes (B_1 \otimes \cdots \otimes B_n)$.

Proposition 4.40. *The* tensor product $\mathcal{A} \otimes \mathcal{B}$ *of two* A_∞*-categories* $\mathcal{A}$ *and* $\mathcal{B}$ *is given by*

- *the set of objects* $\mathrm{Ob}(\mathcal{A} \otimes \mathcal{B}) \coloneqq \mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{B})$,
- *for each pair of objects* $X_1 \times Y_1, X_2 \times Y_2 \in \mathrm{Ob}(\mathcal{A} \otimes \mathcal{B})$, *the space of morphisms* $\mathcal{A} \otimes \mathcal{B}(X_1 \times Y_1, X_2 \times Y_2) \coloneqq \mathcal{A}(X_1, X_2) \otimes \mathcal{B}(Y_1, Y_2)$,
- *for each* $n \geq 1$ *and each family of objects* $X_0 \times Y_0, \ldots, X_n \times Y_n \in \mathrm{Ob}(\mathcal{A} \otimes \mathcal{B})$, *the linear map*

$$\rho_n : \mathcal{A} \otimes \mathcal{B}(X_0 \times Y_0, X_1 \times Y_1) \otimes \cdots \otimes \mathcal{A} \otimes \mathcal{B}(X_{n-1} \times Y_{n-1}, X_n \times Y_n) \to \mathcal{A} \otimes \mathcal{B}(X_0 \times Y_0, X_n \times Y_n)$$

defined by

$$\rho_n \coloneqq \sum_{\substack{\mathcal{N},\mathcal{N}' \in \mathcal{N}(\ell) \\ |\mathcal{N}|+|\mathcal{N}'|=|V(\ell)| \\ \mathrm{top}(\mathcal{N}) \leq \mathrm{bot}(\mathcal{N})}} (-1)^{|\mathrm{Ad}(\mathcal{N}) \cap \mathrm{Ad}(\mathcal{N}')|} \mathrm{sgn}(\sigma_{\mathcal{N}\mathcal{N}'}) \mathcal{N}(m_n) \otimes \mathcal{N}'(m_n')\ \sigma_n\ ,$$

where ℓ *is a linear graph, and where* $\mathcal{N}(m_n)$ *and* $\mathcal{N}'(m_n')$ *denote the composition of the structural maps* $\{m_n\}$ *and* $\{m_n'\}$ *of* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively, corresponding to the nests of* $\mathcal{N}$ *and* $\mathcal{N}'$ *in the left-levelwise order.*

Proof. The formula for the ρ_n's stems from our choice of diagonal on the associahedra; the fact that they satisfy the A_∞ relations follows from the construction and the functoriality in Proposition 4.28. The signs depend on our choice of cellular orientations on the associahedra, and they are obtained via the computation of a determinant. We refer to the proof of Proposition 3.64 for more details. □

Now we treat the case of A_∞-functors between A_∞-categories.

Definition 4.41. *An* A_∞*-functor* $f : \mathcal{A} \rightsquigarrow \mathcal{B}$ *between two* A_∞*-categories consists of*

- *a function* $\mathrm{Ob}(f) : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$,
- *for each* $n \geq 1$ *and each family of objects* $X_0, \ldots, X_n \in \mathrm{Ob}(\mathcal{A})$, *a linear map*

$$f_n : \mathcal{A}(X_0, X_1) \otimes \cdots \otimes \mathcal{A}(X_{n-1}, X_n) \to \mathcal{B}(f(X_0), f(X_n))$$

of degree $|f_n| = n - 1$, *satisfying the equations*

$$\sum_{i_1+\cdots+i_k=n} (-1)^{\varepsilon} m_k'(f_{i_1} \otimes \cdots \otimes f_{i_k}) = \sum_{p+q+r=n} (-1)^{p+qr} f_{p+1+r}(\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r})\ ,$$

for $n \geq 1$, *where* $\varepsilon = \sum_{u=1}^{k} (k-u)(1-i_u)$.

Observe that an A_∞-functor between two A_∞-categories with one object is just an A_∞-morphism between two A_∞-algebras. Thus, the diagonal of the multiplihedra provides us with a canonical A_∞-functor structure from $\mathcal{A}\otimes\mathcal{B}$ to $\mathcal{A}'\otimes\mathcal{B}'$ associated to two A_∞-functors $\mathcal{A} \rightsquigarrow \mathcal{A}'$ and $\mathcal{B} \rightsquigarrow \mathcal{B}'$.

Definition 4.42. *For 2-colored nestings $\mathcal{N}$ and $\mathcal{N}'$, we use the same definitions as in Definition 4.39, but with the following two modifications:*

(2) We say that an edge i of g is admissible *with respect to $\mathcal{N}$ when N_i is bicolored, or if $i \neq \min N_i$ when N_i is monochrome.*

(4) The function $\sigma_{\mathcal{N}\mathcal{N}'} : \mathrm{Ad}(\mathcal{N}) \sqcup \mathrm{Ad}(\mathcal{N}') \to (1, 2, \ldots, |\mathrm{Ad}(\mathcal{N}) \sqcup \mathrm{Ad}(\mathcal{N}')|)$ is defined on $i \in \mathrm{Ad}(\mathcal{N})$ by

$$\sigma_{\mathcal{N}\mathcal{N}'}(i) = \begin{cases} \min N_i & \text{if } i \in \mathrm{Ad}(\mathcal{N}) \cap \mathrm{Ad}(\mathcal{N}'), N_i \text{ is monochrome and } N_i' \text{ is not} \\ \min N_i & \text{if } i \in \mathrm{Ad}(\mathcal{N}) \cap \mathrm{Ad}(\mathcal{N}'), N_i \text{ and } N_i' \text{ are monochrome} \\ & \text{and } \min N_i < \min N_i' \\ i & \text{otherwise ,} \end{cases}$$

and similarly on $i \in \mathrm{Ad}(\mathcal{N}')$ by inversing the roles of $\mathcal{N}$ and $\mathcal{N}'$.

For convenience, let us recall that

$$D(n) \coloneqq \{(I, J) \mid I, J \subset \{1, \ldots, n\}, |I| = |J|, I \cap J = \emptyset, \min(I \cup J) \in I\}.$$

Let us denote by $\mathrm{mono}(\mathcal{N})$ the set of monochrome nests of a nesting $\mathcal{N}$.

Proposition 4.43. *The* tensor product $f \otimes g$ *of two A_∞-functors $f : \mathcal{A} \rightsquigarrow \mathcal{A}'$ and $g : \mathcal{B} \rightsquigarrow \mathcal{B}'$ is given by*

- *the function $\mathrm{Ob}(f \otimes g) \coloneqq \mathrm{Ob}(f) \times \mathrm{Ob}(g) : \mathrm{Ob}(\mathcal{A} \otimes \mathcal{B}) \to \mathrm{Ob}(\mathcal{A}' \otimes \mathcal{B}')$,*
- *for each $n \geq 1$ and each family of objects $X_0 \times Y_0, \ldots, X_n \times Y_n \in \mathrm{Ob}(\mathcal{A} \otimes \mathcal{B})$, the linear map*

$$h_n : (\mathcal{A} \otimes \mathcal{B})(X_0 \times Y_0, X_1 \times Y_1) \otimes \cdots \otimes (\mathcal{A} \otimes \mathcal{B})(X_{n-1} \times Y_{n-1}, X_n \times Y_n) \to (\mathcal{A}' \otimes \mathcal{B}')(f(X_0) \times g(Y_0), f(X_n) \times g(Y_n))$$

defined by

$$h_n \coloneqq \sum_{\mathcal{N},\mathcal{N}'} (-1)^{|\mathrm{Ad}(\mathcal{N}) \cap \mathrm{Ad}(\mathcal{N}')|} \mathrm{sgn}(\sigma_{\mathcal{N}\mathcal{N}'}) \mathcal{N}(f) \otimes \mathcal{N}'(g)\ \sigma_n\ ,$$

where the sum runs over the pairs $\mathcal{N}, \mathcal{N}' \in \mathcal{N}_2(\ell)$ such that $|\mathrm{mono}(\mathcal{N})| + |\mathrm{mono}(\mathcal{N}')| = |E(\ell)|$ which satisfy the conditions in Theorem 4.35. Here, ℓ is a linear graph, and $\mathcal{N}(f)$ and $\mathcal{N}'(g)$ denote the composition of the structural maps $\{f_n\}$ and $\{g_n\}$ corresponding to the nests of $\mathcal{N}$ and $\mathcal{N}'$ in the left-levelwise order.

Proof. The formula for the h_n's stems from our choice of diagonal on the multiplihedra; the fact that they satisfy the A_∞ relations follows from the construction and the functoriality in Proposition 4.28. The signs depend on our choice of cellular orientations on the multiplihedra, and they are obtained via the computation of a determinant. We refer to Section 3.4.3 for more details. □

4.4.3 Monoidal structure on the category of A-infinity algebras

The introduction of the tensor product of A_∞-morphisms raises questions about the properties that it satisfies. For simplicity we focus on A_∞-algebras, however all statements have an A_∞-categorical counterpart. We are interested in tensor products of A_∞-algebras and A_∞-morphisms that are universal, in the sense that they provide a formula that applies to any pair of A_∞-algebras and A_∞-morphisms, respectively. These tensor products are obtained via a diagonal of the operad A_∞ or a diagonal of the operadic bimodule M_∞, we therefore call them *operadic*. We would like to know if the category of A_∞-algebras is made into a symmetric monoidal category by the operadic tensor product defined above.

1. A first idea is to look at the category $\mathrm{A}_\infty\text{-}\mathsf{alg}$ of A_∞-algebras with *strict* morphisms, that is linear maps $f : A \to B$ which commute with the structure operations of A and B (equivalently, strict morphisms are A_∞-morphisms where all the $f_n : A^{\otimes n} \to B$, $n \geq 2$ are equal to zero). In this category, one checks directly that the previous tensor product defines a bifunctor $- \otimes - : \mathrm{A}_\infty\text{-}\mathsf{alg} \times \mathrm{A}_\infty\text{-}\mathsf{alg} \to \mathrm{A}_\infty\text{-}\mathsf{alg}$. However, this tensor product is not, and in fact *cannot* be monoidal, for the following reason:

   Proposition 4.44 ([MS06, Theorem 13])**.** *There is no operadic tensor product that satisfies associativity. That is, there is no coassociative diagonal for the operad A_∞. Therefore, the operad A_∞ does not admit the structure of a Hopf operad.*

   The proof is a direct attempt to construct inductively a coassociative diagonal, a process which leads to a contradiction in arity 4. One can prove with similar methods that there is no cocommutative diagonal.

   Remark 4.45. It would be interesting to know if the bifunctor $- \otimes -$ admits a right adjoint.

2. A second idea is to consider the category $\infty\text{-}\mathrm{A}_\infty\text{-}\mathsf{alg}$ of A_∞-algebras and A_∞-morphisms. In this case our operadic tensor product does not even define a bifunctor $- \otimes - : \infty\text{-}\mathrm{A}_\infty\text{-}\mathsf{alg} \times \infty\text{-}\mathrm{A}_\infty\text{-}\mathsf{alg} \to \infty\text{-}\mathrm{A}_\infty\text{-}\mathsf{alg}$, as a corollary of the following

Theorem 4.46. *There is no operadic diagonal of* M_∞ *which is strictly compatible with the composition of* A_∞*-morphisms.*

Proof. The proof is a straightforward computation, similar to the one of Proposition 4.44. We try to construct inductively a diagonal $\triangle$ which respects the following condition: given two A_∞-morphisms $f^i : A_i \rightsquigarrow B_i$ and $g^i : B_i \rightsquigarrow C_i$ for $i = 1, 2$, the following identity is satisfied

$$(g^1 \circ f^1) \otimes (g^2 \circ f^2) = (g^1 \otimes g^2) \circ (f^1 \otimes f^2) \ , \qquad (\star)$$

where $\circ$ denotes the composition of A_∞-morphisms. In arity 1, there is only one possible diagonal $\triangle((x)) = (x) \otimes (x)$ and the equation $(\star)$ is directly seen to be satisfied. In arity 2, the most general diagonal has the form

$$\triangle((xy)) = (xy) \otimes [\alpha(xy) + \beta(xy)] + [\gamma(xy) + \delta(xy)] \otimes (xy) \ .$$

Compatibility with the differential imposes $\beta+\delta = 1$, $\alpha = \delta$, $\beta = \gamma$ and $\alpha+\gamma = 1$. Thus, we have

$$\triangle((xy)) = \alpha[(xy) \otimes (xy) + (xy) \otimes (xy)] + (1-\alpha)[(xy) \otimes (xy) + (xy) \otimes (xy)] \ .$$

Applying this formula to both sides of $(\star)$ and comparing the terms, we obtain that in order for the equality to hold, we must have both $\alpha = 0$ and $1 - \alpha = 0$, a contradiction. □

It seems likely that one could show with a similar method that there are no coassociative nor cocommutative diagonals on M_∞. Nevertheless, in the category ∞-A_∞-alg we have the following

Proposition 4.47 ([MSS02, LOT20])**.** *For any operadic tensor product of* A_∞*-algebras, there exist* A_∞*-isomorphisms*

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C) \ \text{and} \ A \otimes B \cong B \otimes A \ ,$$

for any A_∞*-algebras* A, B *and* C*.*

Proof. Using the fact that the operad A_∞ is a cofibrant resolution of the associative operad, it is possible to show that there exists an homotopy between the two morphisms of operads $(\Delta \otimes \mathrm{id})\Delta$ and $(\mathrm{id} \otimes \Delta)\Delta$, see for instance [MSS02, Proposition 3.136]. This means that there exists an A_∞-isotopy between the A_∞-algebras $A \otimes (B \otimes C)$ and $(A \otimes B) \otimes C$, which is in particular an A_∞-isomorphism. Alternatively, one can use the fact that the operad A_∞ and the operadic bimodule M_∞ are contractible to deduce the existence a morphism of operadic bimodules (a "trigonal") $\mathrm{M}_\infty \to \mathrm{M}_\infty \otimes \mathrm{M}_\infty \otimes \mathrm{M}_\infty$ compatible with the morphisms of operads $(\Delta \otimes \mathrm{id})\Delta$ on the left and $(\mathrm{id} \otimes \Delta)\Delta$ on the right, see the proof of [LOT20, Theorem 1.1]. Similar methods can be employed to show that there exists an A_∞-isomorphism $A \otimes B \cong B \otimes A$. □

However, this kind of arguments cannot be used to determine if a certain choice of A_∞-isomorphisms makes the pentagon or the hexagon diagrams commute, and there are a priori no reasons for them to do so.

In summary, the category ∞-A_∞-alg might be the right place to work in, but to do so one would have to come up with the right homotopical notions of bifunctor $- \otimes -$ and monoidal category. A definition of "monoidal A_∞-category" was proposed recently in the context of ∞-categories [Pas18, Definition A.5.3]. It would be interesting to know if the image of our tensor product under the Faonte–Lurie nerve [Fao17] makes the category of A_∞-algebras into a monoidal A_∞-category in this sense.

3. A third idea is to see A_∞-algebras as particular instances of a more general structure, where the preceding tensor product extends to a symmetric monoidal one. One can see Proposition 4.44 as saying that the operad A_∞ cannot be an operad in coassociative coalgebras. It is then tempting to ask if A_∞ could be an operad in A_∞-coalgebras; however, this cannot be since (by Proposition 4.44) there is no symmetric monoidal structure on the category of A_∞-coalgebras. One could then consider the category of AA_∞-algebras, that is algebras over the bar-cobar resolution (also called W-construction, or Boardman–Vogt resolution) of the associative operad, which is symmetric monoidal with respect to the tensor product defined by J.-P. Serre's cubical diagonal. Alternatively, S. Arkhipov and D. Poliakova propose in [AP21] the category of integrated A_∞-coalgebras, which is also symmetric monoidal, as an appropriate setting to study the operad A_∞.

4.4.4 Tensor products in symplectic topology

Tensor products of Fukaya algebras and Fukaya categories

Let M be a closed symplectic manifold and $L \subset M$ a closed spin Lagrangian submanifold. Using Lagrangian Floer theory and pseudo-holomorphic disks curves with Lagrangian boundary conditions, K. Fukaya constructs in [Fuk10] a filtered A_∞-algebra $\mathcal{F}(L)$ associated to the Lagrangian L, called the *Fukaya algebra of L*. In [Amo17], L. Amorim shows that given two symplectic manifolds M_1 and M_2 together with Lagrangians $L_i \subset M_i$, the Fukaya algebra $\mathcal{F}(L_1 \times L_2)$ of the product Lagrangian $L_1 \times L_2$ is A_∞-quasi-isomorphic to the tensor product of their Fukaya algebras $\mathcal{F}(L_1) \otimes \mathcal{F}(L_2)$. This tensor product of filtered A_∞-algebras was previously defined in [Amo16] as follows.

The idea is first to consider, for any filtered A_∞-algebras A and B, two differential graded associative algebras $\mathrm{End}(A)$ and $\mathrm{End}(B)$, which are A_∞-quasi-isomorphic to A and B, respectively. Then, it is possible to write the tensor product $\mathrm{End}(A) \otimes \mathrm{End}(B)$ as a deformation retract of the ordinary tensor product of A and B

$$h \circlearrowright \mathrm{End}(A) \otimes \mathrm{End}(B) \underset{i}{\overset{p}{\rightleftarrows}} A \otimes B \ ,$$

and to apply the homotopy transfer theorem to transfer the dg-algebra structure on $\mathrm{End}(A) \otimes \mathrm{End}(B)$ to an A_∞-algebra structure on $A \otimes B$. It is even possible to choose i, p and h as to recover, when specializing to non-filtered A_∞-algebras, the "magical formula" defining the tensor product of A_∞-algebras in [MS06, MTTV21], see [Amo16, Corollary 4.2].

If the rectification process $A \mapsto \mathrm{End}(A)$ could be applied to A_∞-morphisms as well, we could use an extended retract diagram

$$\begin{array}{ccc} h \circlearrowright \mathrm{End}(A_1) \otimes \mathrm{End}(B_1) & \underset{i}{\overset{p}{\rightleftarrows}} & A_1 \otimes B_1 \\ \downarrow {\scriptstyle \mathrm{End}(f) \otimes \mathrm{End}(g)} & & \downarrow {\scriptstyle f \otimes g} \\ h \circlearrowright \mathrm{End}(A_2) \otimes \mathrm{End}(B_2) & \underset{i}{\overset{p}{\rightleftarrows}} & A_2 \otimes B_2 \end{array}$$

to get a tensor product of A_∞-morphisms for filtered A_∞-algebras. It would then be interesting to know if the choices of i, p and h made in [Amo16] would recover the formula obtained in the present work for the tensor product of non-filtered A_∞-morphisms. However, the rectification process $A \mapsto \mathrm{End}(A)$ used in [Amo16] is not functorial. One could think of applying the same idea with another type of rectification, for instance the one of [LV12, Theorem 11.4.4], even though in this case it is less obvious what the retract diagram should be.

A tensor product of A_∞-morphisms between filtered A_∞-algebras could prove useful to study the A_∞-morphisms between the Fukaya algebras of products of two Lagrangian submanifolds, and more generally the A_∞-functors between Fukaya categories of products of two symplectic manifolds. In [Fuk17], K. Fukaya shows that for two closed symplectic manifolds M_0 and M_1 there exists a unital A_∞-functor

$$\mathrm{Fuk}(M_0) \otimes \mathrm{Fuk}(M_1) \rightsquigarrow \mathrm{Fuk}(M_0^- \times M_1)$$

which is a homotopy equivalence into its image. This A_∞-functor can be composed with the categorification A_∞-functor of [MWW18, Theorem 1.1] to give an A_∞-functor

$$\mathrm{Fuk}(M_0) \otimes \mathrm{Fuk}(M_1) \rightsquigarrow \mathrm{Fuk}(M_0^- \times M_1) \rightsquigarrow \mathrm{Func}(\mathrm{Fuk}(M_0), \mathrm{Fuk}(M_1)) \ .$$

It would be interesting to know when this composition becomes an homotopy equivalence. Given two A_∞-categories $\mathcal{A}$ and $\mathcal{B}$, one could also ask whether there exists a purely algebraic A_∞-functor

$$\mathcal{A} \otimes \mathcal{B} \rightsquigarrow \mathrm{Func}(\mathcal{A}, \mathcal{B}) \ ,$$

such that the previous composition is homotopy equivalent to this A_∞-functor when $\mathcal{A} := \mathrm{Fuk}(M_0)$ and $\mathcal{B} := \mathrm{Fuk}(M_1)$. A third question of interest could finally be to understand how the tensor product of A_∞-functors could be realized in symplectic topology, using Lagrangian correspondences.

Tensor products in Bordered Heegaard Floer homology

Heegaard Floer homology gives invariants of 3-manifolds, 4-dimensional cobordisms, and closed 4-manifolds. The invariant of closed 4-manifolds is called the "Heegaard Floer mixed invariant" [OS06]. The 3-manifold invariants are known to be isomorphic to Seiberg-Witten Floer homology [KLT20, CGH11]. It is expected that the invariants of cobordisms and closed manifolds are also equal, but that is still an open question. In particular, the Heegaard Floer mixed invariant is expected to be the same as the Seiberg-Witten invariant. It is known that they have many of the same properties and agree in many examples; in particular, the Heegaard Floer mixed invariant distinguishes exotic smooth structures in many cases.

The variant of Heegaard Floer homology (or Seiberg-Witten Floer homology) used in the previous paragraph is a kind of S^1-equivariant theory, at least philosophically. It is denoted HF^+ or HF^-. The simpler, non-equivariant theory is denoted $\widehat{HF}$. One can compute $\widehat{HF}$ using bordered Heegaard Floer homology [LOT14].

We learned through private communication with Robert Lipshitz that a long term goal of his work with P. Osváth and D. P. Thurston is to develop algorithms to compute HF^+ and HF^- explicitly. This involves extending bordered Heegaard Floer homology from the non-equivariant to the equivariant setting, which requires explicit diagonals of the associahedra and multiplihedra as follows. Given a 3-manifold $Y = Y_1 \cup_F Y_2$ with two boundary components Y_1, Y_2, the aforementioned authors build in [LOT20] a "bimodule twisted complex" $CFDD^-(Y)$, also called a "type DD-bimodule". The operation of gluing the two boundary components Y_1 and Y_2 along a surface F to obtain Y requires, at the algebraic level, to tensor together A_∞-algebras, and thus a diagonal of the associahedra. When re-associating gluings in this theory, one then needs a way to relate different tensor products, which can be done through a diagonal of the multiplihedra. In particular, such a diagonal allows one to describe associativity of tensor products. It is worth noting that the notion of A_∞-algebra needed here (among other homotopy algebraic structures) is a certain kind of curved A_∞-algebra, which is called weighted A_∞-algebra in [LOT20], and would require a non-trivial extension of the results of the present chapter.

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