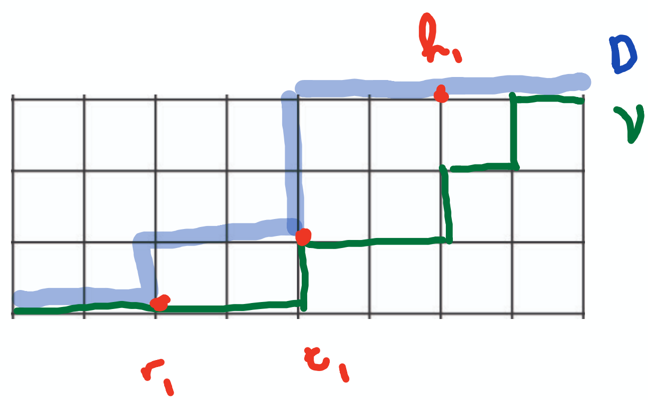
Maximal degree subposets of ν-Tamari. II

Given a ν-Dyck path D, define

r_i = base of each North step

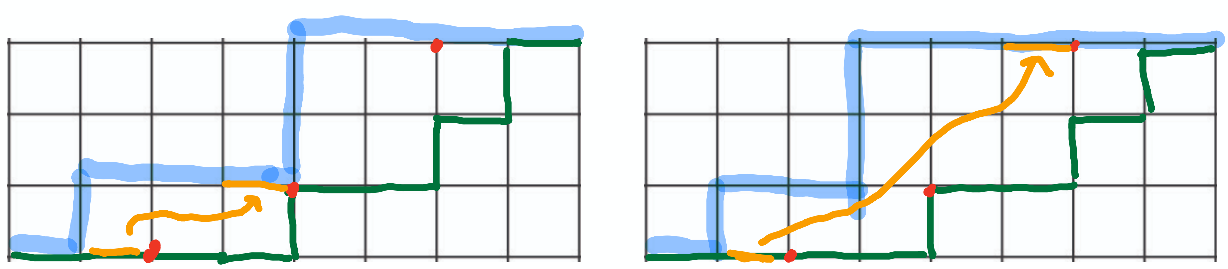
t_i = first point of same horizontal distance

h_i = first point of same horizontal distance followed by East step or final point.



ν-Tamari $\supset$ ν-Greedy

If r_i is preceded by E



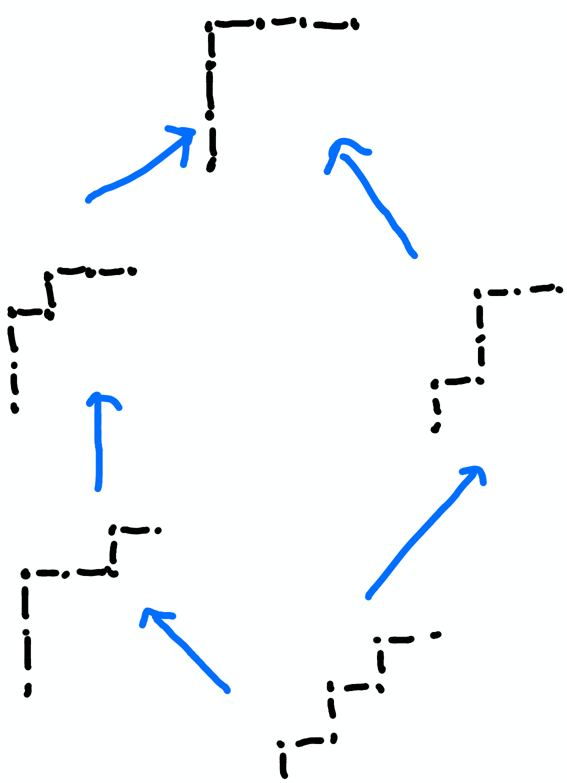
In the case of m-Tamari we have

Thm (Dermenjian '22)

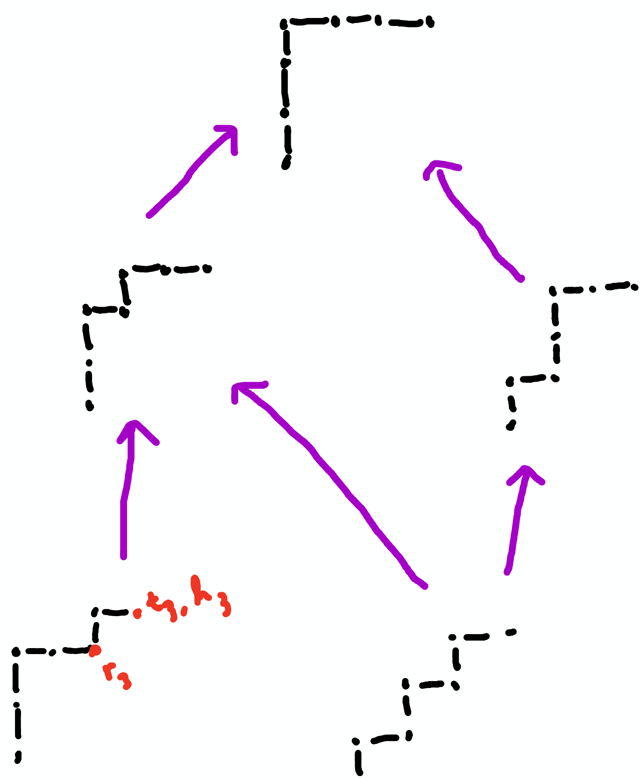
$$\boxed{G_m \cong \left(\left(T_{m+1}\right)_{in}\right)^{op} \cong \left(T_{m+1}^{op}\right)_{out}}$$

When $m=1$

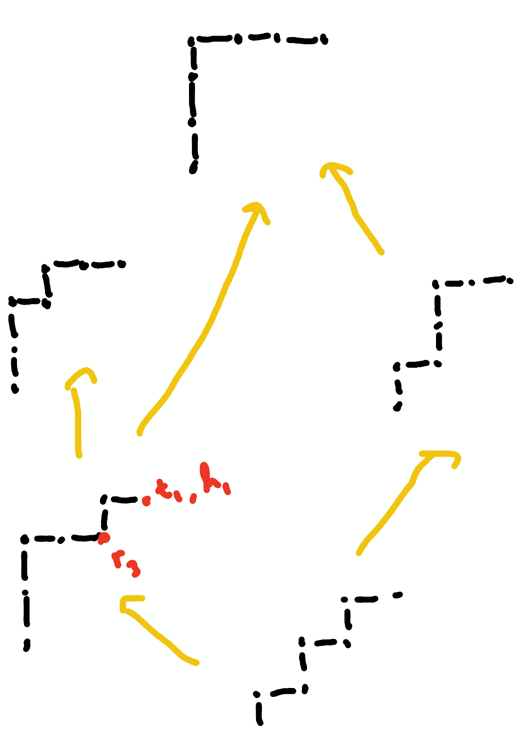
Tamari



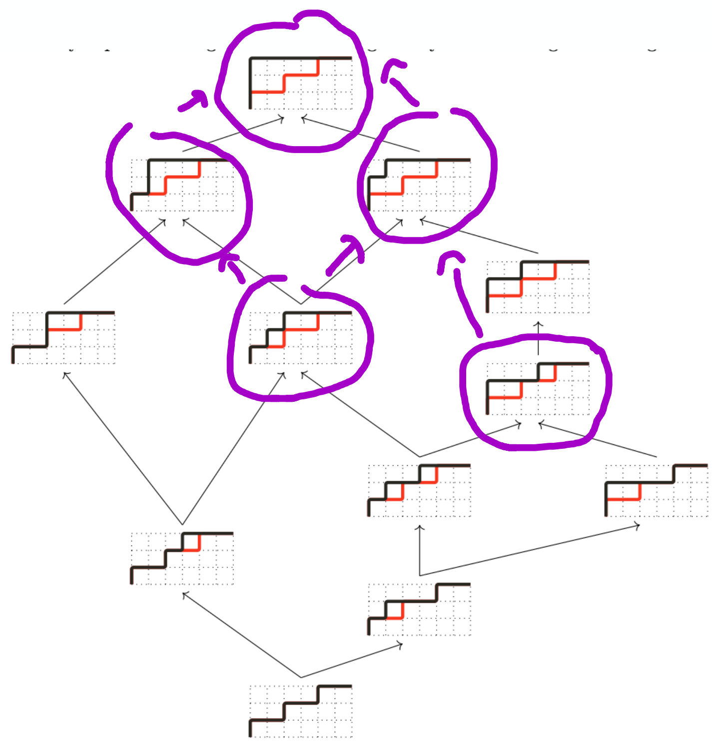
Greedy



Dexter ($\cong$ Greedy^{op})



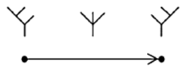
2-Tamari



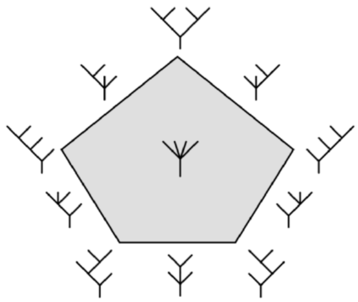
A geometrical interpretation

<u>Obs</u>: The Hasse diagram of the Tamari lattice is the oriented 1-skeleton of the associahedron

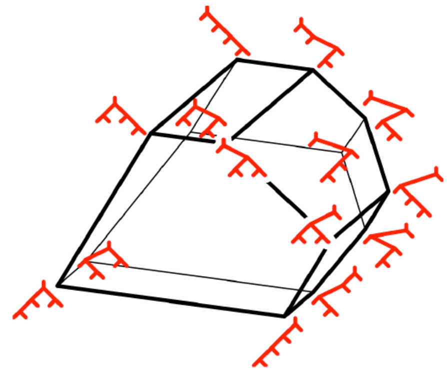
K_0 



K_1



K_2



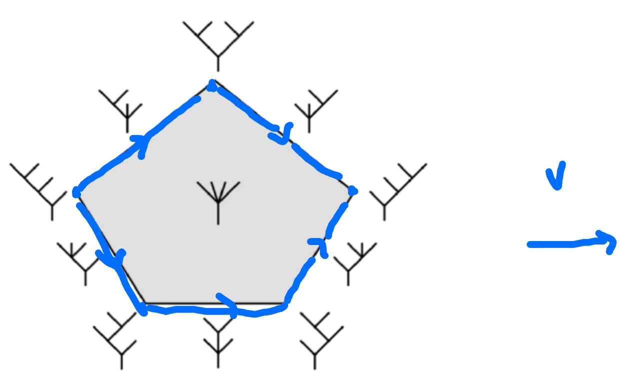
K_3

$$\left\{\begin{matrix}\text{faces of}\\ K_n\end{matrix}\right\} \cong \left\{\begin{matrix}\text{planar trees with}\\ n+2 \text{ leaves}\end{matrix}\right\}$$

$$\begin{matrix}\cup & \qquad\qquad & \cup\end{matrix}$$

$$\left\{\begin{matrix}\text{vertices of}\\ K_n\end{matrix}\right\} \cong \left\{\begin{matrix}\text{planar binary trees}\\ \text{with } n+2 \text{ leaves}\end{matrix}\right\}$$

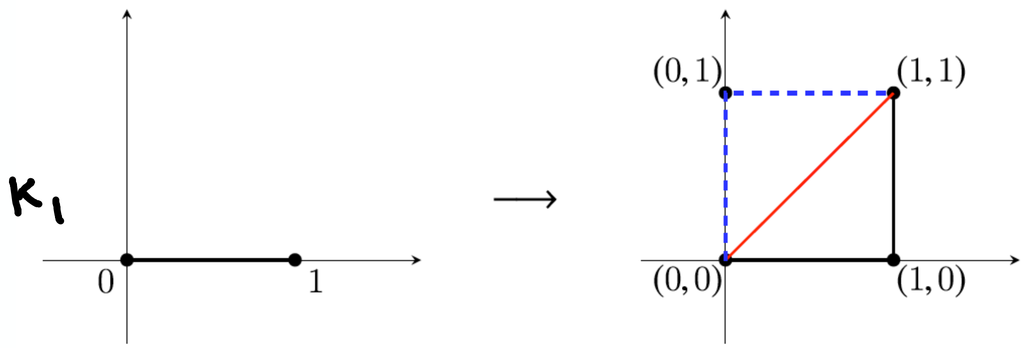
Def: A vector v orients K_n if it is not perpendicular to any edge of K_n.



The diagonal of K_n $\quad \Delta : K_n \longrightarrow K_n \times K_n$

$x \longmapsto (x, x)$

is <u>not</u> cellular

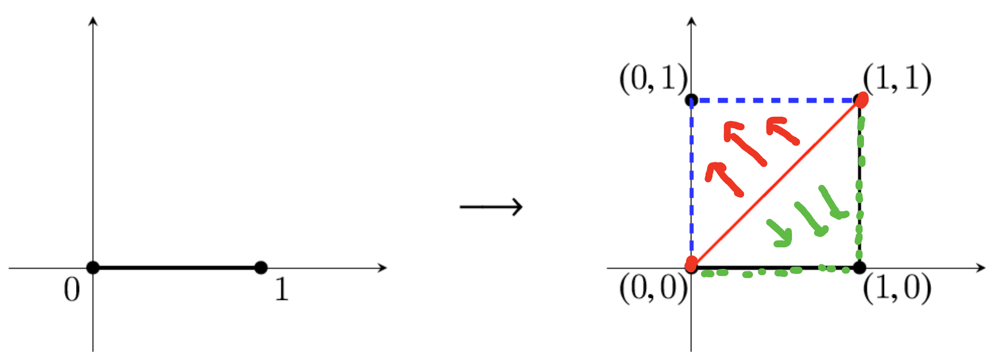


Cellular approximation:

a) agrees on vertices with Δ

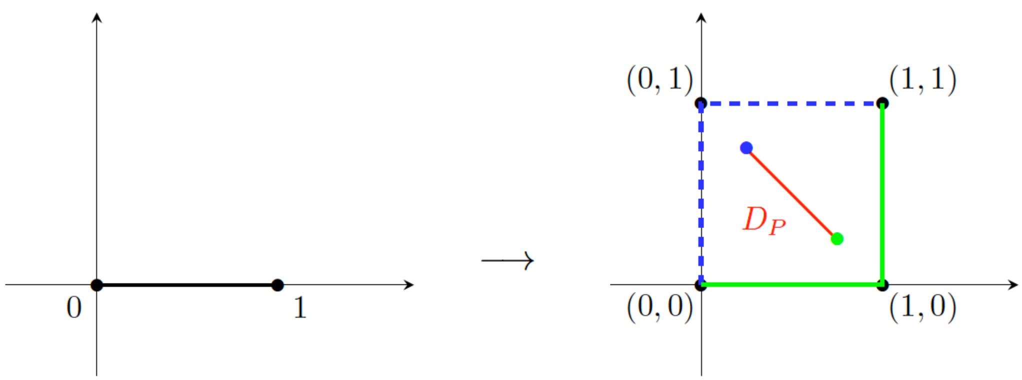
b) homotopic to Δ

c) image is a union of faces of $K_n \times K_n$

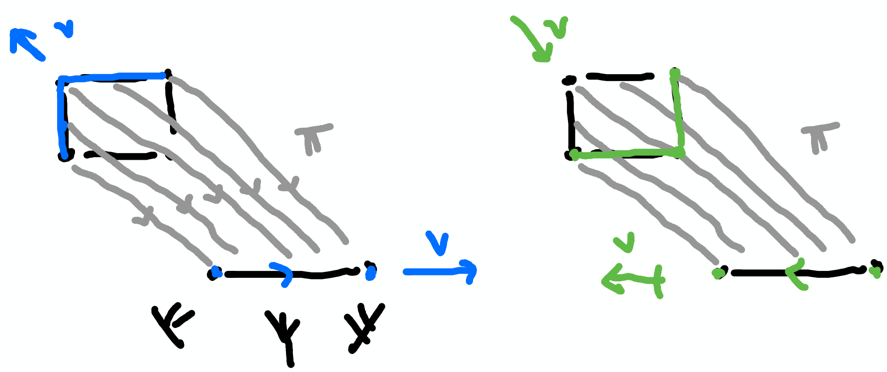


Cellular approximations are given by vertices of the fiber polytope

$$\Sigma\left(\begin{array}{cc} K_n \times K_n & (x,y) \\ \Big\downarrow \pi & \Big\downarrow \\ K_n & \frac{x+y}{2} \end{array}\right) =: D(K_n)$$



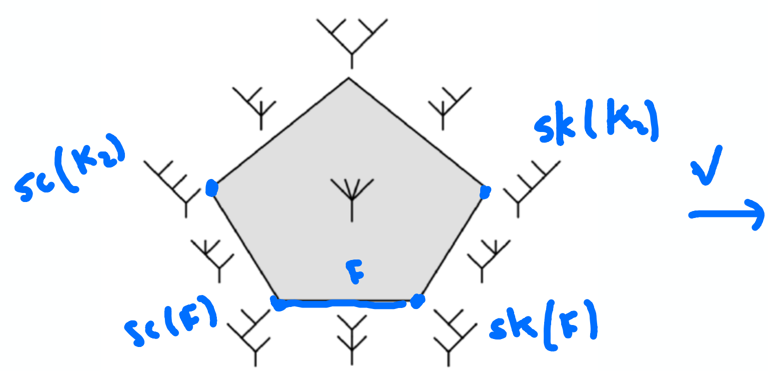
They are selected by generic vectors, which are in particular orientation vectors



Thm (Masuda-Tonks-Thomas-Vallette)

There is a unique vertex Δ_n of $D(K_n)$ which induces the Tamari lattice on K_n.

Obs: Oriented polytope $\Rightarrow$ unique source and sink on each face



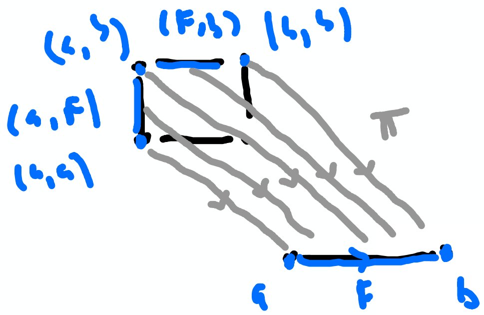
$\Delta_n : K_n \to K_n \times K_n$

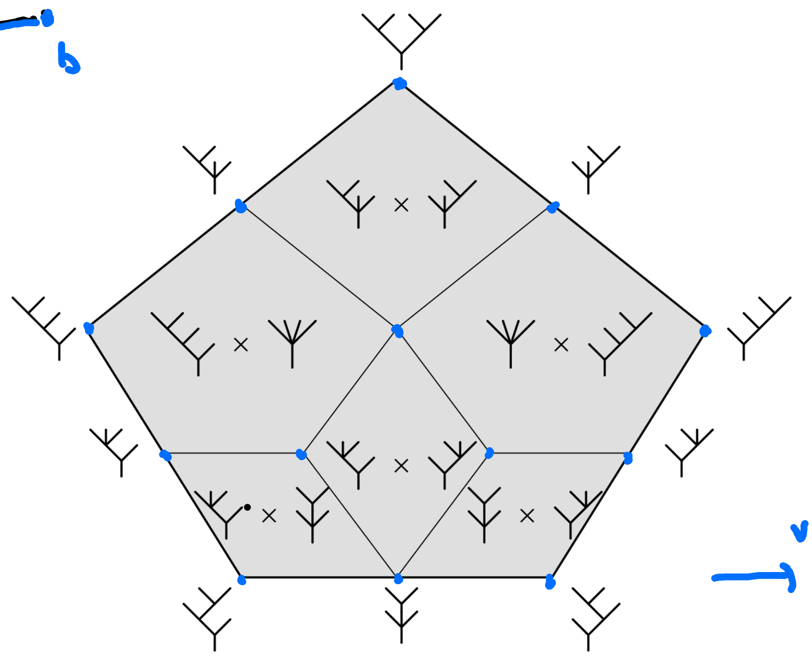
Thm (MTTV, Markl-Schnider, Saneblidze-Umble)

cellular approx

$$\mathrm{Im}\,\Delta_n = \bigcup_{sk(F) \leq sc(G)} F \times G$$

We can "see" $\mathrm{Im}\,\Delta_n$ by drawing $\pi(\mathrm{Im}\,\Delta_n)$





Vertices are thus the intervals in the Tamari lattice A000260 1, 3, 13, 68, 399 ...

Faces of maximal dimension are in bijection with canopy/synchronized intervals (via choice of a specific vertex on each one) A000139

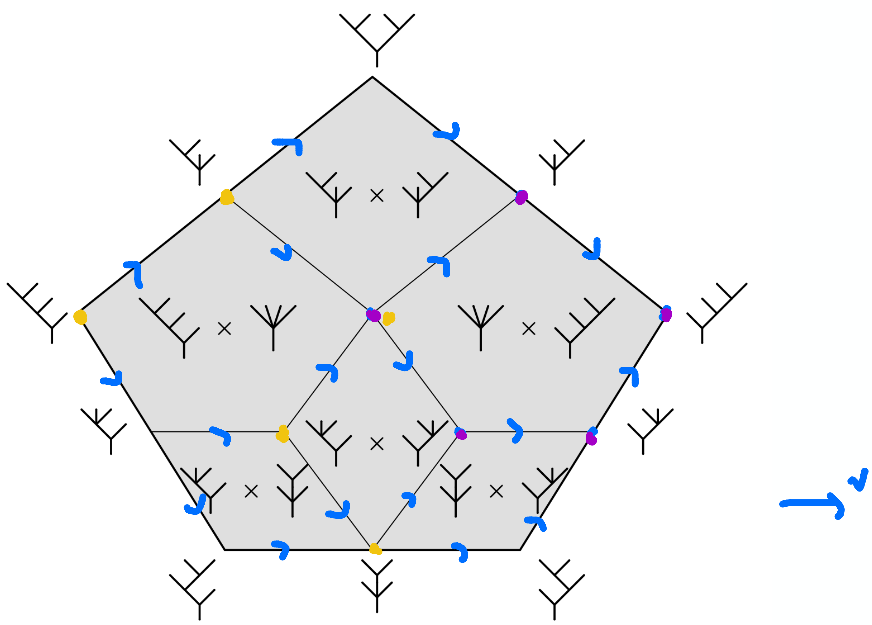
1, 2, 6, 22, 91, 408, ...

Construction:

1) Consider the cell $C := K_n \times sk(K_n) \in \mathrm{Im}\,\Delta_n$

2) Each vertex of C is $sk(F \times G)$ for a unique pair $F \times G$, $\dim F + \dim G = n$

3) Consider the subposet L_n of $\mathcal{T}_n \times \Pi_n$ spanned by $sc(F \times G)$ for these pairs



Conjecture (Chapoton):

$$\boxed{\begin{aligned} L_n &\cong \mathrm{Dexter}(\mathcal{T}_n) \\ &\cong \mathrm{Greedy}(\Pi_n)^{op} \end{aligned}}$$

Remarks:

a) Possible strategy: use Camille Combe's coordinates to characterize L_n

b) Dermenjian's thm is distinct

2-Tamari $\ncong$ 1-skeleton of $\mathrm{Im}\,\Delta_n \subset \Pi_n \times \Pi_n$

But it has a geometrical interpretation as subdivision of the associahedron!

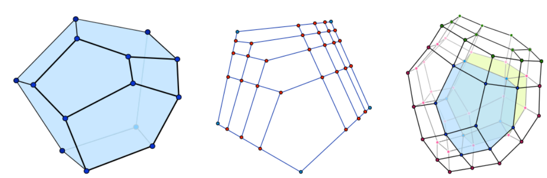
GEOMETRY OF ν-TAMARI LATTICES IN TYPES A AND B

CESAR CEBALLOS, ARNAU PADROL, AND CAMILO SARMIENTO

ABSTRACT. In this paper, we exploit the combinatorics and geometry of triangulations of products of simplices to derive new results in the context of Catalan combinatorics of ν-Tamari lattices. In our framework, the main role of "Catalan objects" is played by $(I,\overline{J})$-trees: bipartite trees associated to a pair $(I,\overline{J})$ of finite index sets that stand in simple bijection with lattice paths weakly above a lattice path $\nu = \nu(I,\overline{J})$. Such trees label the maximal simplices of a triangulation whose dual polyhedral complex gives a geometric realization of the ν-Tamari lattice introduced by Préville-Ratelle and Viennot. In particular, we obtain geometric realizations of
polyhedral subdivisions of associahedra induced by an ar
hyperplanes, giving a positive answer to an open questio

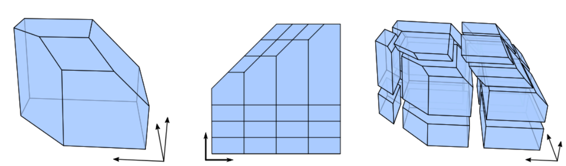
The simplicial complex underlying our triangulation
lattice with a full simplicial complex structure. It is a nat
the classical simplicial associahedron, alternative to the r
of Armstrong, Rhoades and Williams, whose h-vector e
suitable generalization of the Narayana numbers.

Our methods are amenable to cyclic symmetry, whi
type B analogues of our constructions. Notably, we defin
generalizes the type B Tamari lattice, introduced indep
and Reading, along with corresponding geometric realiza



1-Tamari $n=4$ 4-Tamari $n=3$ 2-Tamari $n=4$

Figure 1. Bergeron's pictures "by hand" of m-Tamari lattices reproduced with permission from [5, Figures 4, 5 and 6].



1-Tamari $n=4$ 4-Tamari $n=3$ 2-Tamari $n=4$

Figure 2. Geometric realizations of m-Tamari lattices by cutting classical associahedra with tropical hyperplanes. Compare with Bergeron's pictures in Figure 1.

c) Similar definition for permutaheders?

BHZ order could be a candidate

⚠ Many diagonals on permutahedra that induce Bruhat order on permutations

Chains in shard lattices and BHZ posets

Pierre Baumann,[*] Frédéric Chapoton,[†]
Christophe Hohlweg[‡] & Hugh Thomas[§]

September 13, 2016

Abstract

For every finite Coxeter group W, we prove that the number of chains in the shard intersection lattice introduced by Reading on the one hand and in the BHZ poset introduced by Bergeron, Zabrocki and the third author on the other hand, are the same. We also show that these two partial orders are related by an equality between generating series for their Möbius numbers, and provide a dimension-preserving bijection between the order complex on the BHZ poset and the pulling triangulation of the permutahedron arising from the right weak order, analogous to the
bijection defined by Reading bet
and the same triangulation of t

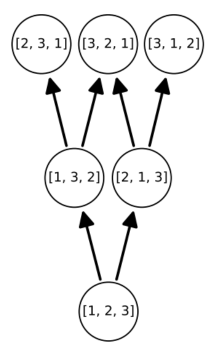


Figure 2: The BHZ order on the symmetric group S_3

Thank you for your attention!