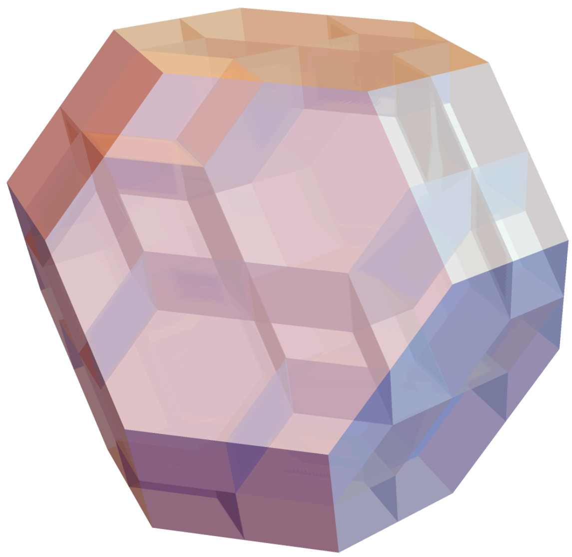
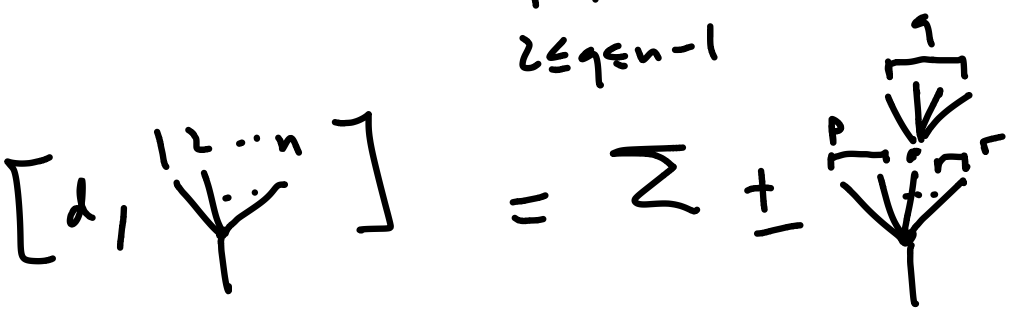
The diagonal of the operahedra

13.12.21 arXiv: 2110.14062



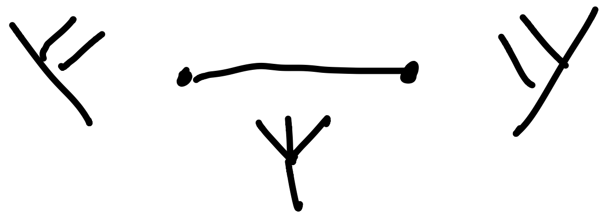
Def: An A_∞-algebra is a graded vector space A with operations $\mu_n : A^{\otimes n} \to A$, $n \geq 1$ of degree $n-2$, which satisfy

$$[\mu_1, \mu_n] = \sum_{\substack{p+q+r=n \\ 2 \leq q \leq n-1}} \pm \mu_{p+1+r}\left(\mathrm{id}^{\otimes p} \otimes \mu_q \otimes \mathrm{id}^{\otimes r}\right)$$



$\longrightarrow$ μ_1 is a differential
μ_2 is a product

μ_3 is a chain homotopy between $\mu_2(\mu_2 \otimes id)$ and $\mu_2(id \otimes \mu_2)$



Problem: If we have two A_∞-algebras (A, μ_n) and (B, ν_n). Endow their tensor product $A \otimes B$ with an A_∞-alg structure.

$$f_n : (A \otimes B)^{\otimes n} \longrightarrow A \otimes B$$

$$\begin{cases} f_1 = \mu_1 \otimes id + id \otimes \nu_1 \\ f_2 = \mu_2 \otimes \nu_2 \end{cases}$$

$f_3 := \cancel{\mu_3 \otimes \nu_3}$ $\quad f_3 := \mu_2(\mu_2 \otimes id) \otimes \nu_3 + \mu_3 \otimes \nu_2(id \otimes \nu_2)$

$f_4 := ???$

↪ this problem was solved by Saneblidze – Umble (2004)!

$\boxed{1}$ Cellular approximation of the diagonal of polytopes

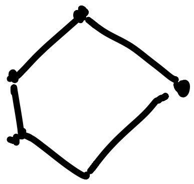
$$P \longrightarrow P \times P$$
$$z \longmapsto (z, z)$$

it is not cellular!

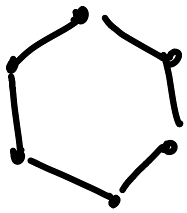
Problem: find a cellular approximations of the diagonal for families of polytopes

$\triangle$ simplices $\longrightarrow$ cup product!

$\square$ cubes $\longrightarrow$ Serre



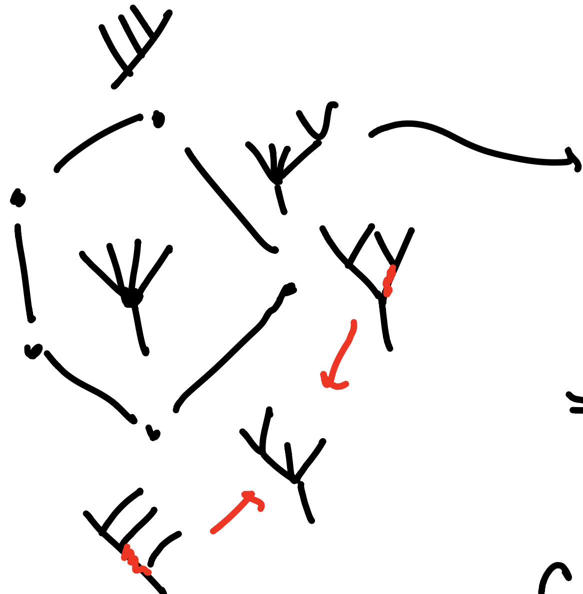
associahedra $\longrightarrow$ H-spaces (Stasheff)

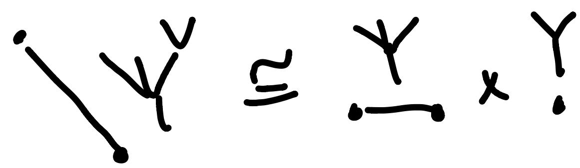


permutahedra $\longrightarrow$ iterated loop spaces (Milgram)

Associahedra

$$\{\text{faces of } K^n\} \cong \left\{\begin{array}{l}\text{planar trees}\\ \text{with } n \text{ leaves}\end{array}\right\}$$





$\Rightarrow$ operad structure

$$C^{cell}_\bullet(K^n) \cong \underline{A_\infty}(n)$$

encode A_∞-algebras

Prop: Suppose that we endow the associahedra with

- a topological cellular operad structure
- a family of compatible cellular approx. $K^n \xrightarrow[\Delta_n]{} K^n \times K^n$

morphisms of operads!

Then, 1) we get a topological model for A_∞-alg

2) we obtain a functorial tensor product of A_∞-alg.

Proof: $f : A_\infty \to End_A$

$g : A_\infty \to End_B$

$$
\begin{tikzcd}
A_\infty \arrow[r, "C_\bullet^{cell}(\Delta_n)"] & A_\infty \otimes A_\infty \arrow[r, "f \otimes g"] & End_A \otimes End_B \arrow[d] \\
& & End_{A \otimes B}.
\end{tikzcd}
$$

$\square$

↪ this was accomplished by

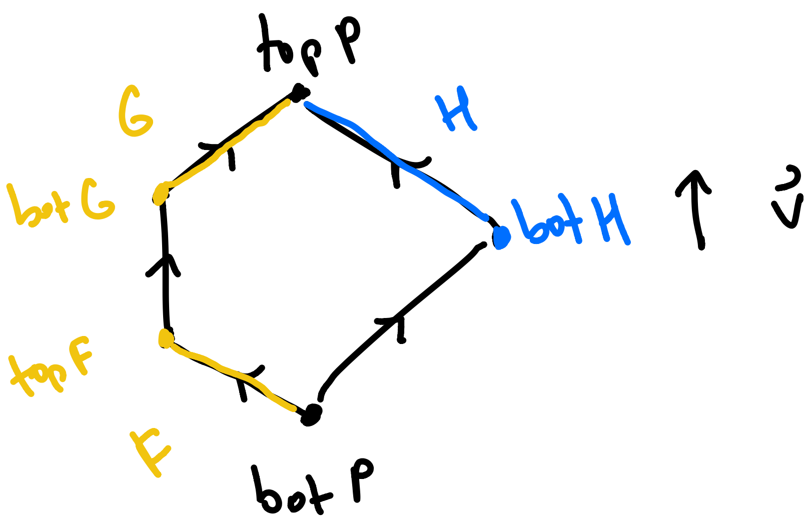
Masuda - Tonks - Thomas - Vallette (2019)

* Loday realizations
* introduce general method (fiber polytopes)
* recover the "magical formula" of

Saneblidze - Umble (2004)

Markl - Shnider (2006)

<u>Def</u>: A vector $\vec{v}$ orients P if it is not perpendicular to any edge of P.



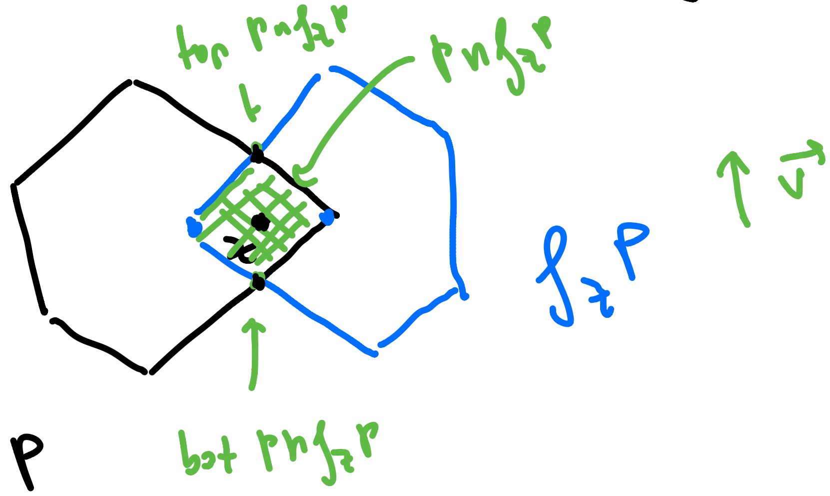
Magical formula $\qquad P \longrightarrow P \times P$

$$\boxed{\operatorname{Im} \Delta_n = \bigcup_{\operatorname{top} F \leq \operatorname{bot} G} F \times G}$$

$\dim F + \dim G = \dim P$

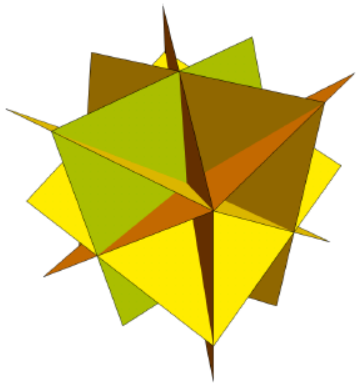
2 General theory

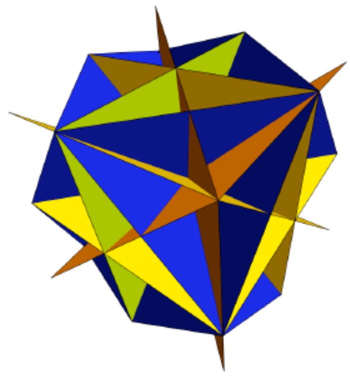
For P a polytope, $z \in P$, we write $\int_z P := 2z - P$



Def: P is positively oriented by $\vec{v}$ if $\forall z \in P$, $P \cap \int_z P$ is oriented by $\vec{v}$.

Def The fundamental hyperplane arrangement of P, $\mathcal{H}_P$, is the set of hyperplanes perpendicular to the edges of $P \cap f_z P$, $\forall z \in P$.


braid arrangement


fundamental hyp. arr. of 3D permutahedron.

Prop: Each chamber in $\mathcal{H}_P$ defines a diagonal of P, given by

$$\Delta_{(P,\vec{v})} : P \longrightarrow P \times P$$
$$z \longmapsto (\operatorname{bot} P \cap f_z P,\ \operatorname{top} P \cap f_z P)$$

THM [L.-A.]

There is a universal formula describing

the cellular image of $\Delta_{(P,Q)}$ | for any polytope P, in terms of $\mathcal{H}_P$

3 The operahedra

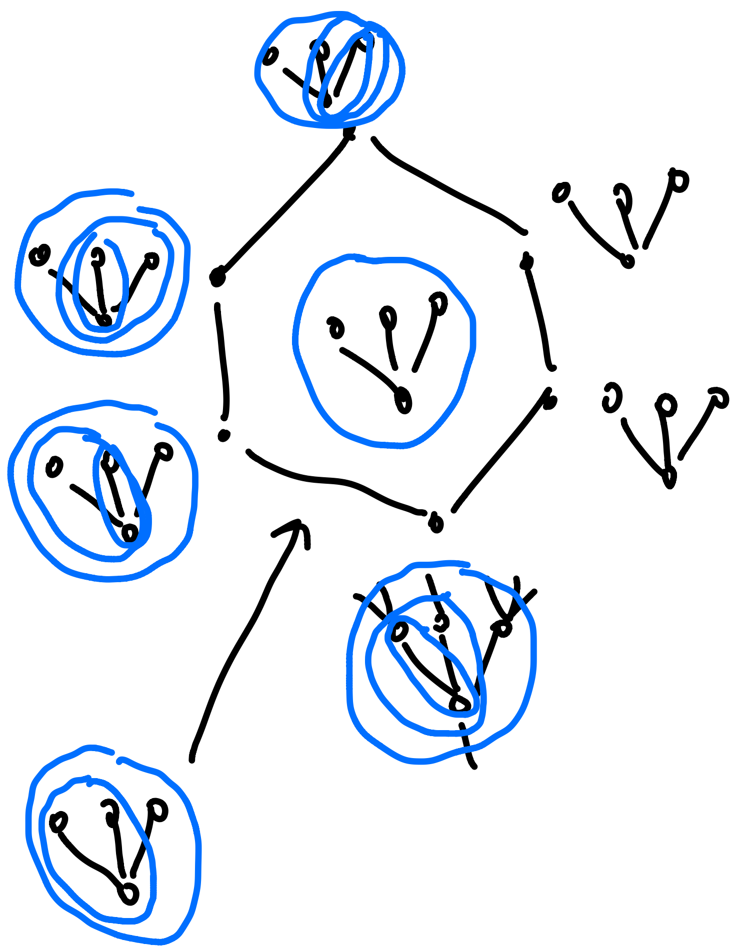
associahedra

$$
\begin{tikzcd}
 & \mathrm{Ass} \arrow[dl, hook] \arrow[dd] & & A_\infty \arrow[ll, "\sim"'] \arrow[dl, hook] \arrow[dd] & \cong C^{cell}_\bullet(K^n) \\
\Theta \arrow[dd] & & \Theta_\infty \arrow[ll, "\sim"'] \arrow[dd] & & \cong C^{cell}_\bullet(P_t) \\
 & \text{ass. alg.} \arrow[dl, hook] & & A_\infty\text{-alg} \arrow[ll, "\sim"'] \arrow[dl, hook] & \\
\text{operads} & & \text{operad up to htpy} \arrow[ll, "\sim"'] & &
\end{tikzcd}
$$

operahedra

$$\mathrm{Im}\,\Delta_n = \bigcup_{\mathrm{top}\,F \leq \mathrm{bot}\,G} F \times G$$

Def: An operahedron is a polytope whose face lattice is isomorphic to the lattice of nestings of a planar tree.



Def: $D(n) := \{ I, J \subset \{1, \dots, n\} \mid |I| = |J| \neq 0,$
$I \cap J = \emptyset,$
$\min(I \cup J) \in I \}.$

THM [Universal formula for operahedra]

For t a planar tree with $n+1$ vertices, we have

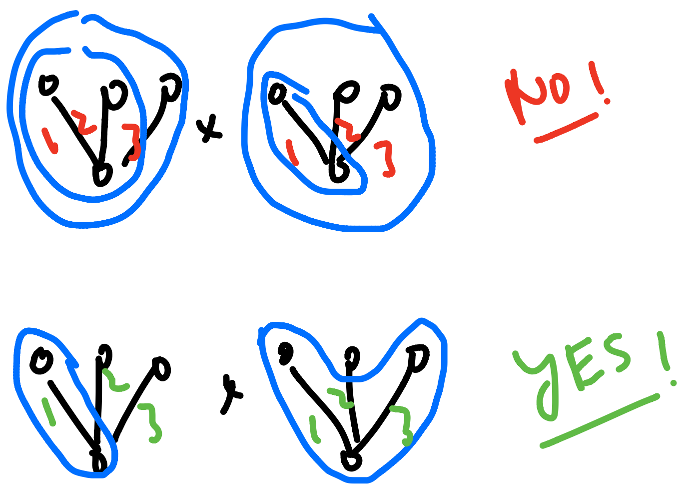
$$(\mathcal{N}, \mathcal{N}') \in \mathrm{Im}\, \Delta_{(1,2)}$$

$$\Leftrightarrow \quad \forall\, (I, J) \in D(n),$$

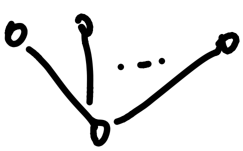
$$\exists\, N \in \mathcal{N},\ |N \cap I| > |N \cap J|, \text{ or}$$

$$\exists\, N' \in \mathcal{N}',\ |N' \cap I| < |N' \cap J|.$$

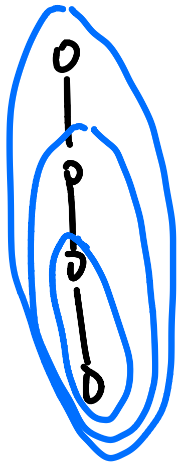
ex



I	J	
1	2	✓
1	3	✓
2	3	✓

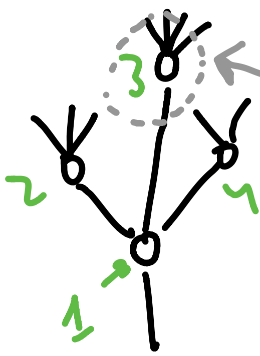


⟶ permutahedra

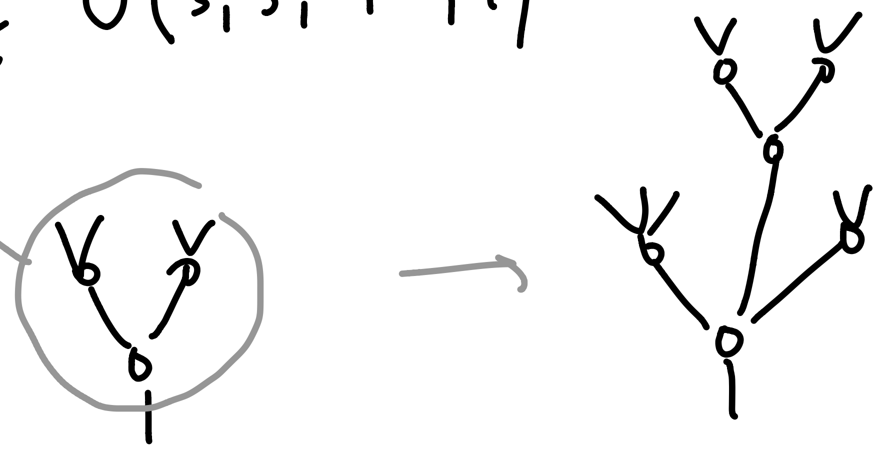


$\rightsquigarrow$ associahedra

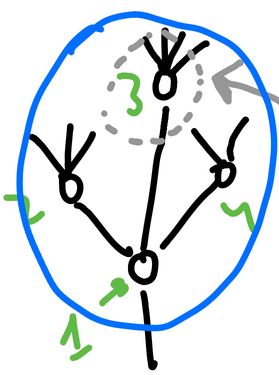
$\mathcal{O}$ is $\mathbb{N}$-colored operad



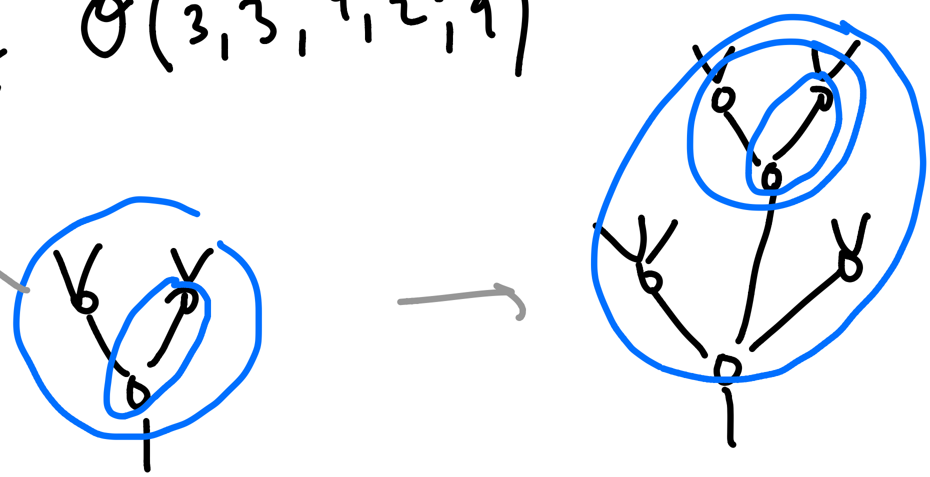
$\in \mathcal{O}(3,3,4,2;9)$



$\mathcal{O}_\infty$ = remember the nests!



$\in \mathcal{O}(3,3,4,2;9)$



Recall that a face F of a polytope P is equal to the intersection of a family of facets $\{F_i\}_{i\in I}$. If we choose an outward pointing normal vector $\vec{F}_i$ for each facet F_i, then the normal cone of F is spanned by these normal vectors, i.e. we have $\mathcal{N}_P(F) = \mathrm{Cone}(\{\vec{F}_i\}_{i\in I})$.

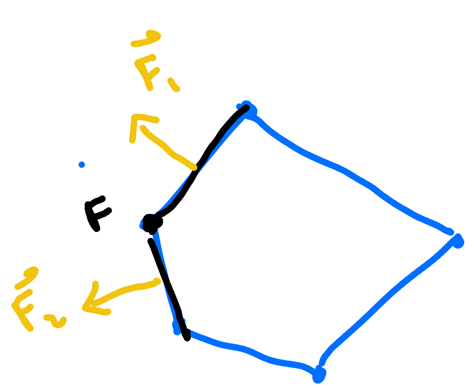
For a pair of faces F, G of P, let us set the notation

$$\mathcal{H}_P(F,G) := \{H \in \mathcal{H}_P \mid H \text{ intersects a codimension 1 face of } \mathrm{Cone}(-\mathcal{N}_P(F) \cup \mathcal{N}_P(G))\} \ .$$

Theorem 1.23 (Universal formula for the bot-top diagonal). *Let $(P, \vec{v})$ be a positively oriented polytope in $\mathbb{R}^n$. For each $H \in \mathcal{H}_P$, we choose a normal vector $\vec{d}_H$ such that $\langle \vec{d}_H, \vec{v}\rangle > 0$. We have*

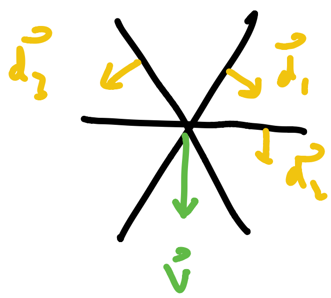
$$(1) \qquad (F,G) \in \mathrm{Im}\, \triangle_{(P,\vec{v})} \quad \cancel{\iff \forall H \in \mathcal{H}_P(F,G),\ \exists \vec{F}_i,\ \langle \vec{F}_i, \vec{d}_H\rangle < 0 \text{ or } \exists \vec{G}_j,\ \langle \vec{G}_j, \vec{d}_H\rangle > 0}$$

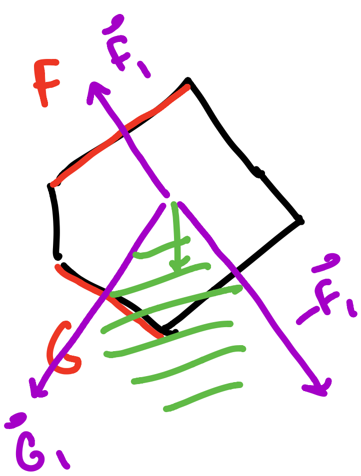
$$(2) \qquad \iff \forall H \in \mathcal{H}_P,\ \exists \vec{F}_i,\ \langle \vec{F}_i, \vec{d}_H\rangle < 0 \text{ or } \exists \vec{G}_j,\ \langle \vec{G}_j, \vec{d}_H\rangle > 0\ .$$



$\mathcal{N}_P(F) = \mathrm{Cone}(\vec{F}_1, \vec{F}_2)$

$P \longrightarrow \mathcal{H}_P$





$(F,G) \in \mathrm{Im}\, \Delta_{(P,\vec{v})}$

$\iff \vec{v} \in \mathrm{Cone}(-\mathcal{N}_P(F) \cup \mathcal{N}_P(G))$