SPRING EXERCISES

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ABSTRACT. The goal of these exercises is to give a rigourous operadic interpretation of the constructions of Costello, Caldararu and Tu.

1. ABSTRACT SETTING

Let P be a wheeled prop.

Question 1. What structure does the totalisation of P possess?

We know from Harry's talk ([3]) that Tot(P) is an associative algebra, and in particular a Lie algebra. What if we take into account the contraction operations of P?

In [2] it is proven that the totalization of a modular operad is a BV algebra. Every wheeled prop has an underlying modular operad; does this structure carry to Tot(P)?

Question 2. Show that the totalisation construction is a functor: any morphism of wheeled props $P \to Q$ induces a morphism $\mathrm{Tot}(P) \to \mathrm{Tot}(Q)$ respecting this structure.

2. CONCRETE SETTING

Let A be a $\mathbb{Z}/2\mathbb{Z}$-graded cyclic A_∞-algebra over a field of characteristic zero, and we suppose further that A is smooth, finite dimensional, unital, and satisfies the Hodge-de Rham degeneration property.

Let d be its Calabi–Yau dimension, and let $L := CH_\bullet(A)[d]$ denote its shifted reduced Hochschild chain complex. Consider the dg Lie algebra

$$\hat{\mathfrak{h}}_A := \bigoplus_{k\geqslant 1, l\geqslant 0} \mathrm{Hom}^c\left(\mathrm{Sym}^k(L_+[1]), \mathrm{Sym}^l(L_-)\right)[[\hbar,\lambda]]$$

with differential $(\eth + \iota + \hbar\Delta)$ and bracket

$$\{\Psi,\Phi\}_\hbar := (-1)^{|\Psi|}\sum_{r\geq 1}\left(\Psi\circ_r\Phi - (-1)^{|\Psi||\Phi|}\Phi\circ_r\Psi\right)\hbar^{r-1}.$$

Question 3. Can we recover $\hat{\mathfrak{h}}_A$ as the totalisation Lie algebra associated to a wheeled prop? Does it possess more structure (e.g. is it a BV algebra)?

It seems like $\hat{\mathfrak{h}}_A$ is very closely related to the totalisation Lie algebra of the wheeled prop End_L.

Let $\mathscr{M}^{\mathrm{fr}}_{g,k,l}$ denote the moduli space of Riemann surfaces of genus g with $k+l$ framed marked points, k labeled as inputs and l labeled as outputs.

Question 4. Is the prop $C_*(\mathscr{M}^{\mathrm{fr}}_{g,k,l})$ endowed with self-sewing operations a wheeled prop?

The question might be easier if we consider its graphical model $M_{g,k,l}$.

Question 5. Is Costello's representation $\rho : C_*(\mathscr{M}^{\mathrm{fr}}_{g,k,l}) \to \mathrm{End}_L$ a morphism of wheeled props?

The existence of this morphism is stated as [1, Theorem 6.1].

Question 6. Can one give an explicit description of ρ, using the graphical model $M_{g,k,l}$?

REFERENCES

[1] Andrei Caldararu and Junwu Tu. Effective Categorical Enumerative Invariants, April 2024. arXiv:2404.01499 [math].

[2] Martin Doubek, Branislav Jurčo, Martin Markl, and Ivo Sachs. *Algebraic Structure of String Field Theory*, volume 973 of *Lecture Notes in Physics*. Springer, 2020.

[3] Sergei Merkulov and Bruno Vallette. Deformation theory of representations of prop(erad)s. I and II. *J. Reine Angew. Math.*, 634:51–106, 2009. Part II: **636** (2009), 123–174.