CEI wants to live in $H.(C,b)$

instead of $H.(L, b+uB)$

Goal:

~~cons~~ construct the L_∞ - quasi-iso $\widehat{k}: \widehat{h_A} \longrightarrow \widehat{h_A}^{\mathrm{triv.}}$

and represent the CEI $F^{A_s} \in MC(h_A^{\mathrm{Triv},+})$ as

Feyman graph sum.

Two main Theorems:

Theorem 8.3.

~~Let S~~ Given a homological symplectic chain level splitting $S: L \to L_+$ of the non-commutative Hodge filtration A. there exists a quasi-isomorphism $\widehat{k}: \widehat{h_A} \longrightarrow \widehat{h_A}^{\mathrm{triv}}$ of DGLAs. Moreover, we have the following ~~co~~ homotopy commutative diagram of DGLAs

$$\begin{tikzcd} h_A^+ \arrow[r, "\iota"] \arrow[d, "k"'] & \widehat{h_A} \arrow[d, "\widehat{k}"] \\ h_A^{+,\mathrm{triv}} \arrow[r, "\iota"'] & \widehat{h_A}^{\mathrm{triv}} \end{tikzcd}$$

ι are inclusions.

i.e. $L^{0,c} \hookrightarrow L^{0,c}$.

Thm 9.1

For any $g \geq 0$, $n \geq 1$, $2g-2+n > 0$, we have

$$\iota(F_{g,n}^{A_s}) = \sum_{m \geq 1} \sum_{G \in T(g,n-1)_m} (-1)^m \frac{wt(G)}{Aut(G)} \prod_v \mathrm{cont}(v) \prod_e \mathrm{cont}(e)$$

$\prod_C \mathrm{cont}(C)$ homology class in $H.(L_{1,n-1}, b)$.

~~Algebraic set up:~~

Recall. CEI.

The CEI ~~[illegible]~~ $F^{A\omega}$ is an element in $\widehat{\mathrm{Sym}}_{\hbar}\, H.(L)_{-}[[\hbar,\lambda]]$ where $L = (\cancel{CC.(A)}\ C.(A)[d],\ d)$

A = cyclic A_∞-alg. with $\mathbb{Z}/2\mathbb{Z}$-grading over $\mathbb{K}$, which is smooth, finite dimensional, unital and satisfies the Hodge de Rham degeneration property.

$L^{Tate} := L_{-} \oplus L_{+}$

Algebraic set up: $L = C.(A)[d]$, $L_{-} := L[u^{-1}]$ $L_{+} = L[[u]]$

DGLAs: $(h_A := \mathrm{sym}(L_{-})[[\hbar,\lambda]][1],\ b+uB+\hbar\Delta,\ \{-,-\})$

$\{-,-\} : h_A[1]^{\otimes 2} \longrightarrow h_A[1]$

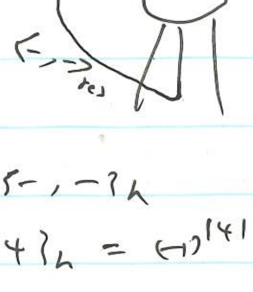
~~$\widehat{h}_A$~~ The Koszul resolution DGLA:

$$\widehat{h}_A := \bigoplus_{\substack{k\geq 1\\ l\geq 0}} \underbrace{\mathrm{Hom}^c(\mathrm{Sym}^k(L_{+}[1]),\ \mathrm{Sym}^l(L_{-}))[[\hbar,\lambda]]}_{L_{k,l}}$$

c : continuous homomorphism w.r.t. u-adic topology

$\partial + \hbar\Delta + \tau$. $\quad \partial := b+uB$.

$\tau : L_{k,l} \longrightarrow L_{k+1,l-1}$ $\quad \langle -,-\rangle_{res}$. contraction map



Lie-bracket $\{-,-\}_\hbar$

$$\{\Phi,\Psi\}_\hbar = (-1)^{|\Psi|}\sum_{r\geq 1}\left(\Psi\circ_r\Phi - (-1)^{|\Psi||\Phi|}\Phi\circ_r\Psi\right)\hbar^{r-1}$$

O_2 - operation.

$\theta : L_- \to L_+[1]$. $\theta(a) = B(a_0)$

$a = a_0 + a_{-1}u^{-1} + \cdots$

$h_A^+ = (\mathrm{Sym}^{\geq 1} L_-)[[\hbar, \lambda]][1]$ given by

$0 \to k[[\hbar,\lambda]] \to h_A \to h_A^+ \to 0$

$(\hat{h}_A^{triv}, b + uB + \mathcal{L})$

Operators needed to construct Γ:

used to trivialized the Lie bracket $[-,-]_\hbar$.

Def: Given a chain level splitting $s : L \to L_+$.

$s = id + s_1 u + s_2 u^2 + \cdots$ with inverse

$R : L_+ \to L$, define the odd map

$F : L_- \to L_+[1]$ by

$F(B) = -u^{-1} s_{\geq 1} R(B)$

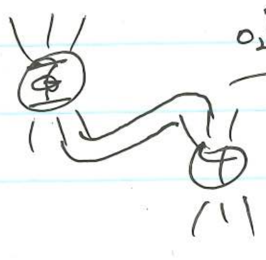
s.t. $i, j \geq 0 \sim F_{i,j} : L\cdot u^{-i} \to L\cdot u^{j}$

$F_{i,j}(x\cdot u^{-i}) = -\sum_{l=0}^{i} s_l R_{i+j+1-l}\, x\cdot u^{j}$

prop: $[\partial, F] = \theta$. $\partial = b + uB$.

Def: An even operator $\{-,-\}_\hbar^F : \mathrm{Sym}^2(\hat{h}_A[1]) \to \hat{h}_A[1]$ induced by F is defined as:

$\{\Psi, \Phi\}_\hbar^F = (-1)^{|\Psi|}(\Psi \circ_F \Phi - (-1)^{|\Psi||\Phi|}\Phi \circ_F \Psi)$



$\xrightarrow{O_2^F} \theta \otimes F \sim \sum_{j=1}^{r} \theta^{j-1} \otimes F \otimes \theta^{r-j}$

If S is homologically symplectic but not chain level symplectic

$\rightsquigarrow$ define another operator

$$\{-,-\}_h^{\delta} : Sym^2(\hat{h}_A[1]) \longrightarrow \hat{h}_A[1].$$

in the same way as $\{-,-\}_h^{F}$

$\delta : L_- \otimes L_- \to \mathbb{K}$. $[T_2, \delta] = H - H^{sym}$.

$H, H^{sym} : L_-^{\otimes 2} \to \mathbb{K}$.

take two outputs and pair them

$\{-,-\}_h^{\delta}$ and $\{-,-\}_h^{F}$ are used for homotopy contraction and

gluing. Later used for solving $\frac{d}{dt}\hat{k}_m(t)$

partially directed graphs.

A partially directed graph of type (g, k, l)

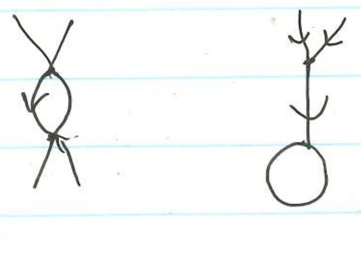
g = genus. k = input leaves, l = output leaves.

$G = (G, L_G^{in} \sqcup L_G^{out}, E^{dir}, T)$

~~take~~ It has labeling $f : \{1, \dots, m\} \to V(G)$.

E_G = internal edges of G, T = the spanning tree of G ($T \subseteq G$)

$E^{dir} \subseteq E_G$ $\rightsquigarrow$ directed edges.



$\star$: 1. No directed loop

2. Each vertex has at least one incoming edge

The weight of G : $wt(G)$

Def: the weight of a partially directed graph $wt(G) \in \mathbb{Q}$ is defined as:

~~1. If $T = \phi$ ⇝ $wt(G) := 1$~~

~~2. In general~~

$$wt(G) = \begin{cases} 1 & \text{if } T = \phi \\ \dfrac{1}{|T_G^{\text{non-loop}}|} \displaystyle\sum_{e \in T^{\text{contr}}} wt(G/e) & \end{cases}$$

G/e = partially directed graph obtained by contracting all directed edges connecting two end points of e.

⇝ such $e \in T$ is called contractable edge of G.

T^{contr} = set of contractable edges of G.

Ex:  $wt E = 1$.

 $wt(G) = \frac{1}{2} = \frac{1}{|E^{\text{non-loop}}|}$

Construction of $\underbrace{\widehat{K}}_{L_\infty\text{-quasi-iso}} : \widehat{h_A} \longrightarrow \widehat{h_A}^{\text{triv.}}$

Define the "integrated" morphism $\widehat{K} : \widehat{h_A} \longrightarrow \widehat{h_A}^{\text{triv.}}$ and show it coincides with Fukaya's "path integral" on DGLA $\widehat{h_A} \otimes \Omega^\bullet_{[0,1]}$ ⇝ L_∞-morphism from $t=1$ to $t=0$.

Def:

For each $m \geq 1$, we construct a map
$$\hat{K}_m : Sym^m(\hat{h}_A[1]) \longrightarrow \hat{h}_A^{t\geq 0}[1]$$
as a graph sum ~~where~~
where for each $(\gamma_1 \cdot \hbar^{g_1}, \cdots, \gamma_m \hbar^{g_m}) \in Sym^m(\hat{h}_A[1])$

we have $\hat{K}_m(\gamma_1 \cdot \hbar^{g_1}, \cdots, \gamma_m \hbar^{g_m}) = \sum_{(G,f)} \frac{wt(G)}{|Aut(G)|} \hat{K}_{(G,f)}(\gamma_1 \cdot \hbar^{g_1}, \cdots, \gamma_m \hbar^{g_m})$

where $\hat{K}_{(G,f)} : \bigotimes_{i=1}^m L_{k_i, l_i}[1] \cdot \hbar^{g_i} \longrightarrow L_{k,l}[1] \hbar^g$ $\mathbb{K}$-linear map

(G,f) = partially directed graphs with labeling $f : \{1, \cdots, m\} \to V(G)$

- Each $\gamma_i \in L_{k_i, l_i}[1] \cdot \hbar^{g_i}$ is assigned to the vertex $f(i)$
- Homotopy operator T assigned to $E^{dir} \cap T$.
- Even operator ~~assigned~~ $\Theta \to$ other directed edges
- $\delta \to$ undirected edge in T

vertex $f(i)$	γ_i
$E^{dir} \cap T$	T
$E^{dir} \setminus T$	Θ
undirected edge $\cap T$	δ
undirected edge $\setminus T$	$H^{sym} : Sym^2 L \to \mathbb{K}$

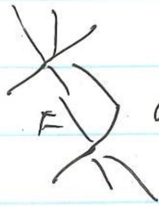
proof of Theorem 8.3.

Consider the DGLA $\widehat{h}_A \otimes \Omega^{\cdot}_{[0,1]}$, ~~define~~ extend the following operators as coderivations using bar construction of DG LAs

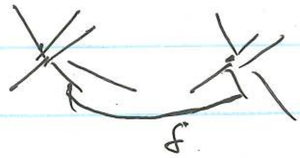
~~Δ^{Hsym}~~

$$\Delta^{Hsy} : \widehat{h}_A[1] \longrightarrow \widehat{h}_A[2]$$

$$\sum_{r\geq 1} t^{r+1} \{-,-\}_r^{F} \cdot \hbar^{r-1}$$



$$\sum_{r\geq 1} t^{r-1} \otimes \{-,-\}_r^{\delta} \hbar^{r-1}$$



$\rightsquigarrow$ coderivation

$$\widehat{V}_1(t),\ V_2^{F}(t),\ V_3^{\delta}(t) : \mathrm{Sym}(\widehat{h}_A[1]) \longrightarrow \mathrm{Sym}(\widehat{h}_A[1])$$

Define a family of morphism $\widehat{k}(t) : \widehat{h}_A(t) \longrightarrow \widehat{h}_A^{triv.}$

$$\widehat{k}_m(t)(\alpha_1 \hbar^{g_1}, \dots, \alpha_m \hbar^{g_m}) = \sum_{(G,f)} \frac{t^{|E_G|}\, wt(G)}{Aut(G)} \widehat{k}_{(G,f)}(\alpha_1 \hbar^{g_1} \cdots)$$

$$\widehat{k}_m(0) = id_{\widehat{h}_A^{triv.}} \qquad \widehat{k}_m(1) = \widehat{k}_m$$

WTS:

$$\frac{d}{dt}\widehat{k}_m(t) = \hbar\, \widehat{k}_m(t)\, \widehat{V}_1 + \widehat{k}_{m-1}(t)\, V_2^{F}(t) + \widehat{k}_{m-1}(t)\, V_2^{\delta}(t)$$

$$\therefore\ \frac{d}{dt}\widehat{k}_m(t) = \sum_{g,m} \sum_{(G,f)} \frac{|E_G|\, t^{|E_G|-1}}{|Aut(G,f)|} \cdot wt(G) \cdot \widehat{k}_m$$

$$= \int_{D_m} |E_G|\, t^{|E_G|-1} \cdot wt(G) \cdot \widehat{k}_m \, d\mu$$

D_m = groupoid of marked partially directed graphs with m vertices. ([Fio06], groupoid integrals)

Right hand side is $\cancel{E}$.

$$h \widehat{K_m}(t) \widehat{V_1} = \int_{D_m} |E_G^{loop}| \cdot t^{|E_G|-1} \cdot wt \cdot \widehat{K_m}\, d\mu$$

$$\widehat{K_{m-1}}(t)\, \widehat{V_2}^F(t) + \widehat{K_{m-1}}(t)\, \widehat{V_2}^\delta(t)$$
$$= \widehat{K_{m-1}}(t) \underbrace{\left(\widehat{V_2}^F(t) + \widehat{V_2}^\delta(t) \right)}_{\widehat{V_2}(t)}$$

Consider the follong maps of groupoids

$$D_m \xleftarrow{F} D_m' \xrightarrow{c} D_{m-1}$$
$$(G, f) \longleftarrow (G, f, e) \longmapsto (G/e, f')$$
$$e \in T$$

$$=$$

$$\widehat{K_{m-1}}(t)(\widehat{V_2}(t)) = \int_{D_{m-1}} wt \cdot \widehat{K_{m-1}}(t) V_2(t)\, d\mu$$

$$= \int_{D_m'} c^*(wt \cdot \widehat{K_{m-1}}(t) V_2(t))\, d\mu$$

$$= \int_{D_m} F_* c^*(wt\, \widehat{K_{m-1}}(t) V_2(t))\, d\mu \quad ①$$

For $(G, f) \in D_m$, $\rightsquigarrow$ $F_* c^*(wt \cdot \widehat{K_{m-1}}(t) V_2(t))(G, f)$

$$= \sum_{e \in T^{contr.}} c^*(wt\, \widehat{K_{m-1}}(t) V_2(t))(G, f, e)$$
$$= \sum_{e \in T^{contr.}} wt(G/e)\, \widehat{K_m}(G, f) \cdot t^{|E_G|-1}$$
$$= |E_G^{non-loop}| \cdot wt(G) \cdot t^{|E_G|-1} \widehat{K_m}(G, f)$$

$$\Rightarrow ① = \int_{D_m} |E_G^{non-loop}|\, wt(G) \cdot t^{|E_G|-1} \widehat{K_m}(G, f).$$

show homotopy diagram

we have

~~$\xi_A : h_A^+$~~

ξ lifts to $\xi_\Omega : h_A^+ \otimes \Omega_{[0,1]} \longrightarrow \hat{h}_A \otimes \Omega_{[0,1]}$

$$\begin{tikzcd}
h_A^+ \otimes \Omega_{[0,1]} \arrow[r] \arrow[d, "\frac{d}{dt}\xi_\Omega(t)"'] & \hat{h}_A \otimes \Omega_{[0,1]} \arrow[d, "\frac{d}{dt}\hat{\xi}_\Omega(t)"] \\
h_A^{+\,triv} \otimes \Omega_{[0,1]} \arrow[r, "\xi"'] & \hat{h}_A^{triv} \otimes \Omega_{[0,1]}
\end{tikzcd}$$

$a(t) + B(t)\,dt \in h_A \otimes \Omega_{[0,1]}$

Pseudo-isotopies:

$\exists\ L_\infty$-morphism $H : h_A^+ \longrightarrow \hat{h}_A \otimes \underbrace{\Omega_{[0,1]}}$ encode homotopy betw L_∞-morphism

$ev_0(H) = \xi \circ k$

$ev_1(H) = \hat{\xi} \circ L$.

$k(1)$

$k(t) + \underline{v(t)\,dt} \in h_A^+ \otimes \Omega_{[0,1]}$

$t=0$. $k(0) \otimes = Id_{h_A^{triv}}$

$t=1$ $k(1) = k$ ~~full~~ $: h_A \to h_A^{triv}$

$C = \mathbb{I}$.

$L^{\mathrm{Triv}} = (L, b)$.

A chain level splitting $S : (L, b) \longrightarrow (L[[u]], b[[u]])$

induces an isomorphism $L_-^{\mathrm{Triv}} \longrightarrow L_-$

and an isomorphism of DGLAs

$$S : h_A^{\mathrm{Triv}} \longrightarrow h_A^{\mathrm{Triv}}.$$

$\mathbb{I}$ The inverse of S, $R : L[[u]] \longrightarrow L$

induceds an iso of DGLAs

$$R : h_A^{\mathrm{Triv}} \longrightarrow h_A^{\mathrm{Triv}}.$$

$$\rightsquigarrow \begin{tikzcd} h_A^{+} \arrow[r, "K"] \arrow[d, "\iota"'] & h_A^{\mathrm{Triv},+} \arrow[r, "R"] \arrow[d, "\iota"'] & h_A^{\mathrm{Triv},+} \arrow[d, "\iota"'] \\ \hat{h}_A \arrow[r, "\hat{K}"] & \hat{h}_A^{\mathrm{Triv}} \arrow[r, "\hat{R}"] & \hat{h}_A^{\mathrm{Triv}} \end{tikzcd}$$

quasi-iso of DGLA $\Rightarrow$ Isomorphism of MC ~~mod~~ moduli spaces,

$\hat{K}$, $\hat{R}$ are ~~~~ corresponding Koszul resolution

$$\begin{tikzcd} MC(h_A^{+}) \arrow[r, "K."] \arrow[d, "\iota."'] & MC(h_A \cdots) \arrow[r, "R."] \arrow[d, "\iota."] & MC(h_A^{\mathrm{Triv}+}) \ni F^{A,S} \arrow[d, "\iota."] \\ MC(\;) \arrow[r, "\hat{K}"] & (\qquad) \arrow[r, "\hat{R}\,\Theta"] & MC(\hat{h}_A^{\mathrm{Triv}}) \end{tikzcd}$$

$\cup$

$\hat{B}^A$

$\Downarrow$

$\hat{B}^A$

$$\iota.(F^{A,S}) = \hat{R}.\hat{K}.\hat{B}^A$$

rules for graph sum

$\hat{B}^A_{g(v),k(v)}(v)$	$\rightsquigarrow$	vertex.
S		incoming edges
R		out going edges
F		directed eges in T
Θ		other directed eges

$$\frac{dk(t)}{dt} = k(t)v(t). \quad k(0) = id$$

$\rightsquigarrow k(1) = P\exp\int_0^1 v(t)dt.$

$\rightsquigarrow k(1) =$ Loo –

δ undirected edges in T

H^{sym} other undirected edges.

$$(-1)^{m-1} \frac{wt(G)}{Aut(G)}$$

$\tau(F^{A,S}_{0,3}) = \frac{1}{2}$  $wt(G) = 1$

$Aut(G) = 2$

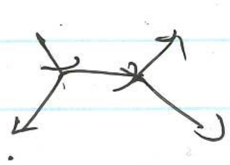
$\tau(F^{A,S}_{1,1})$

$=$  $+ \frac{1}{2}$  $wt(G) = 1$

$g=1$ $Aut(G) = 2$

$wt(G) = 1$ loop $= \langle -, - \rangle$.

two output legs.

$\tau(F^{A,S}_{0,4}) = \frac{1}{3!}$  $- \frac{1}{2!}$ 

$Aut(G) = 3.$ $Aut(G) = 2.$