Higher structures in Deformation Quantization

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A brief overview of deformation quantization

Definition

Let M be a smooth manifold. A **star product** *on M is an $\mathbb{R}[\![\hbar]\!]$-linear product $\star$ on $C^\infty(M)[\![\hbar]\!]$ such that:*

1. *$\star$ is associative: $f \star (g \star h) = (f \star g) \star h$,*
2. *for $f, g \in C^\infty(M) \subset C^\infty(M)[\![\hbar]\!]$, $f \star g = f \cdot g + \sum_{k=1}^{\infty} \hbar^k B_k(f, g)$ for some bidifferential operators B_k.*
3. *$1 \star f = f \star 1 = f, \forall C^\infty(M)[\![\hbar]\!]$.*

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Given a star product $\star$ one can define a bracket $\{f,g\}_\star=B_1(f,g)-B_1(g,f)$ for $f,g\in C^\infty(M)$ and it follows from the properties above that such bracket defines a Poisson structure on M.

Definition

Let $(M,\{-,-\})$ be a Poisson manifold. A (formal deformation) quantization of M is a star product $\star$ such that $\{-,-\}_\star=\{-,-\}$.

Deformation Quantization

Question

Can every Poisson manifold be quantized?
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The crucial object of study is the deformation complex where star products naturally live
This corresponds to the Lie algebra of formal multidifferential operators

$$\star \in D_{\mathrm{poly}}[\![\hbar]\!](M)$$

Multidifferential operators

The space of multidifferential operators $D^{\bullet}_{\mathrm{poly}}(M)$ is the chain complex of operators given by partial derivatives:

$$D^n_{\mathrm{poly}}(M) = \left\{ D\colon C^\infty(M)^{\otimes n} \to C^\infty(M) \middle| D \overset{\text{locally}}{=} \sum f \frac{\partial}{\partial x_{I_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{I_n}} \right\},$$

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$$D^{\bullet}_{\mathrm{poly}}(M) \simeq CH(C^\infty(M)) = \text{Hochschild complex of } C^\infty(M)$$

$$D \in D^2_{\mathrm{poly}}(M) \rightsquigarrow d(D)(f_1, f_2, f_3) =$$
$$= f_1 D(f_2, f_3) - D(f_1 f_2, f_3) + D(f_1, f_2 f_3) - D(f_1, f_2) f_3.$$

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D_{poly} is actually a differential graded Lie algebra:
$[D, D'] = D \circ D' - D' \circ D$.

$$\star \text{ associative} \Leftrightarrow d\star + \frac{1}{2}[\star, \star] = 0 \Leftrightarrow \star \in \mathrm{MC}(D_{\mathrm{poly}}[[\hbar]])$$

Multivector fields

The data of a Poisson structure on M is encoded by the Poisson bivector $\Pi \in \bigwedge^2 T_M$.

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The space of multivector fields on M is $T^d_{\text{poly}}(M) = \Gamma(M, \bigwedge^d T_M)$.

$\bullet$Lie bracket = Schouten-Nijenhuis bracket:
Extend Lie bracket on $T^1_{\text{poly}} = \Gamma(T_M)$ by
$[X, Y \wedge Z] = [X, Y] \wedge Z \pm Y \wedge [X, Z]$.

$$\Pi \text{ Poisson} \Leftrightarrow [\Pi, \Pi] = 0 \Leftrightarrow \Pi \in \mathrm{MC}(T_{\text{poly}})$$

Kontsevich Formality

Theorem (Kontsevich Formality, 1997)

There exists an L_∞ *(*Lie_∞/ Lie *up to homotopy) map*

$$\mathcal{U}\colon T_{\mathrm{poly}}(M) \to D_{\mathrm{poly}}(M)$$

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Corollary

There is a bijection

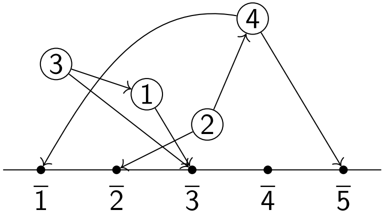
$$\mathrm{MC}(T_{\mathrm{poly}}[\![\hbar]\!])/\textit{gauge equiv.} \to \mathrm{MC}(D_{\mathrm{poly}}[\![\hbar]\!])/\textit{gauge equiv.}$$

$$\Pi \mapsto \sum_{n\geq 1} \frac{1}{n!}\mathcal{U}_{(n)}(\underbrace{\Pi,\ldots,\Pi}_{n \text{ times}})$$

How does the morphism look like?

Remarkably, Kontsevich's morphism can be explicitly written in $\mathbb{R}^D$.

$$\mathcal{U}_{(n)} = \sum_{\Gamma \in \mathsf{Graphs}(n,\bullet)} \underbrace{W_\Gamma}_{\in \mathbb{R}} \mathcal{U}_\Gamma$$

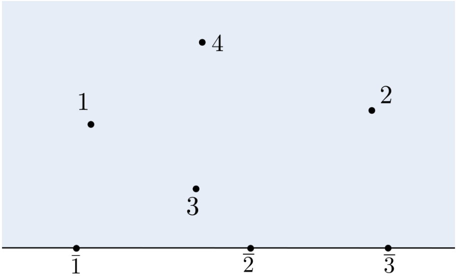


$\in \mathit{Graphs}(4,5).$

$$U_\Gamma \colon (T_{\rm poly}(\mathbb{R}^D))^{\wedge 4} \to D^5_{\rm poly}(\mathbb{R}^D)$$

Configuration spaces

Let $H = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$ be the upper half plane. We consider its configuration space $\mathbb{H}_{m,n} = Conf_{m,n}(H)$ of m non-overlapping points in the bulk and n non-overlapping points at the boundary.



$\in \mathbb{H}_{4,3}$

$$W_\Gamma = (\text{pre-factor}) \int_{\mathbb{H}_{m,n}} \bigwedge_{e \text{ edge}} \omega_e$$

Additional structure

A BV algebra A is a cochain complex with three operations:

- The product $- \wedge -$ of degree 0
- The Lie bracket $[-,-]$ of degree -1
- The BV operator $\Delta(-)$ of degree -1

satisfying relations such as

$$
\begin{aligned}
&[-,-] \text{ is a Lie bracket,}\\
&x_1 \wedge x_2 = x_2 \wedge x_1,\\
&(x_1 \wedge x_2) \wedge x_3 = x_1 \wedge (x_2 \wedge x_3),\\
&\Delta \circ \Delta = 0,\\
&[x_1, x_2 \wedge x_3] = [x_1, x_2] \wedge x_3 + x_2 \wedge [x_1, x_3],\\
&[x_1, x_2] = \Delta(x_1 \wedge x_2) - \Delta(x_1) \wedge x_2 - x_1 \wedge \Delta(x_2).
\end{aligned}
$$

for $\deg(x_1) = \deg(x_2) = 0$.

Additional structure - Multivector fields

Let (M, vol) be an oriented manifold. $T_{\mathrm{poly}}(M)$ is a BV algebra.
•BV operator:

$$\Delta\colon T^k_{\mathrm{poly}}(M) \stackrel{vol(\bullet)}{\to} \Omega^{D-k}(M) \stackrel{d_{dR}}{\to} \Omega^{D-k+1}(M) \to T^{k-1}_{\mathrm{poly}}(M)$$

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Can Kontsevich's map be made to preserve the BV *structure?*

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Candidates for BV-algebra structure on D_{poly}:
•$D \wedge D' = [f_1, \ldots, f_n \mapsto D(f_1, \ldots, f_k) \cdot D'(f_{k+1}, \ldots, f_n)]$,
•Δ Connes' B operator.

D_{poly} is **not** a BV-algebra, but these operations induce a BV-algebra structure on cohomology $H(D_{\mathrm{poly}})$.

The main result

Proposition

There is a BV_∞ *structure on* $D_{\mathrm{poly}}(M)$ *inducing this structure in cohomology.*

Theorem (C., 2016)

There exists a BV_∞ *quasi-isomorphism* $T_{\mathrm{poly}}(M) \to D_{\mathrm{poly}}(M)$ *extending Kontsevich's.*

Consequences/Applications

Star products

Corollary

The set of gauge equivalence classes of closed star products is isomorphic to the set of gauge equivalence classes of formal unimodular Poisson structures.

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String Topology

Object of interest: free loop space $LM = Map(S^1, M)$.

BV Structure on

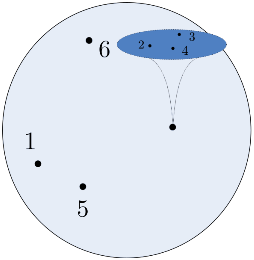
$$H(LM) = HH(\Omega(M)) = H(D_{\mathrm{poly}}(\Pi TM))$$

A closer look into Kontsevich's proof

$$W_\Gamma = (\text{pre-factor}) \int_{\mathbb{H}_{m,n}} \bigwedge_{e \text{ edge}} \omega_e$$

Consider the Fulton-MacPherson compactification of configuration spaces of points in $\mathbb{R}^2$

$$\mathsf{FM}_{\mathbb{R}^2}(n) = \overline{Conf_n(\mathbb{R}^2)}$$

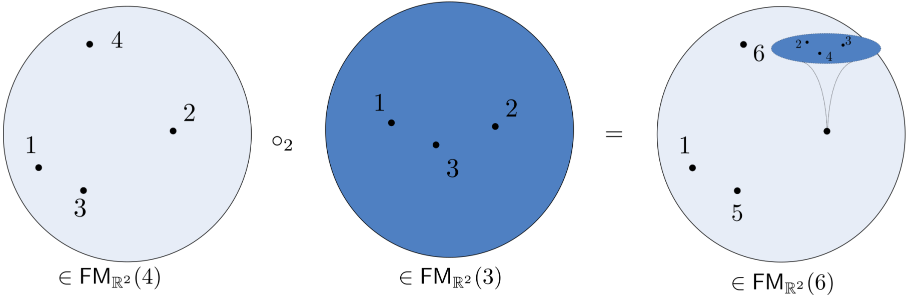


$\in \mathsf{FM}_{\mathbb{R}^2}(6)$

A topological operad

Consider the Fulton-MacPherson compactification of configuration spaces of points in $\mathbb{R}^2$ forms an operad, i.e. there are natural "insertion" operations

$$\circ_i \colon \mathrm{FM}_{\mathbb{R}^2}(m) \times \mathrm{FM}_{\mathbb{R}^2}(n) \to \mathrm{FM}_{\mathbb{R}^2}(m+n-1), i = 1, \dots, m$$



From topology to algebra

$H_\bullet(X \times Y) = H_\bullet(X) \otimes H_\bullet(Y) \Rightarrow H_\bullet(\text{Top. operad}) = \text{Alg. operad}.$

$H(\mathrm{FM}_{\mathbb{R}^2}) = \mathrm{Ger}$, the operad governing Gerstenhaber algebra structures.

$$\begin{array}{ccccc} \mathrm{Lie} & \subset & \mathrm{Ger} & \subset & \mathrm{BV} \\ [-,-] & & - \wedge - & & \Delta \end{array}$$

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Kontsevich's Formality Theorem can be expressed in terms of the natural map of operads

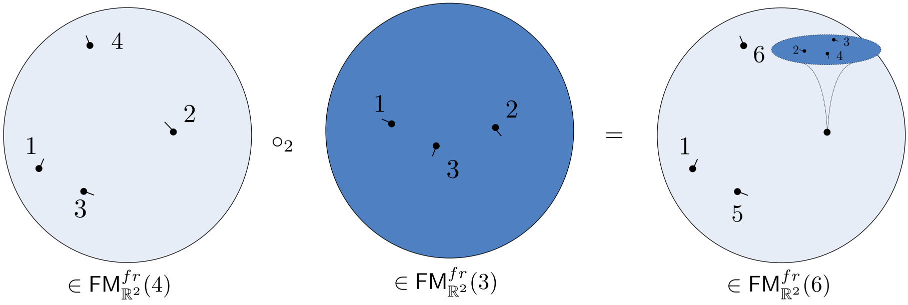
$$\mathrm{Lie}_\infty \to \mathit{Chains}(\mathrm{FM}_{\mathbb{R}^2})$$

sending the k-th bracket to the fundamental chain of $\mathrm{FM}_{\mathbb{R}^2}(k)$.

Configuration spaces - The Fulton-MacPherson operad

The framed Fulton-MacPherson operad $\mathsf{FM}^{fr}_{\mathbb{R}^2}$ is given by

$$\mathsf{FM}^{fr}_{\mathbb{R}^2}(n) = (S^1)^{\times n} \ltimes \overline{Conf_n(\mathbb{R}^2)}$$



$$H_\bullet(\mathrm{FM}^{fr}_{\mathbb{R}^2}) = \mathrm{BV}.$$

Proposition

There is a quasi-isomorphism of operads $\mathrm{BV}_\infty \to Chains(\mathrm{FM}^{fr}_{\mathbb{R}^2})$

From Topology to Algebra

$$H_\bullet(\mathrm{FM}^{fr}_{\mathbb{R}^2}) = \mathrm{BV}.$$

Proposition

There is a quasi-isomorphism of operads $\mathrm{BV}_\infty \to \mathit{Chains}(\mathrm{FM}^{fr}_{\mathbb{R}^2})$

This + expressing the spaces of graphs in terms of operads + relating all objects by appropriate maps yield

Theorem

Let M be an orientable manifolds.
There exists a BV_∞ *quasi-isomorphism* $T_{\mathrm{poly}}(M) \to D_{\mathrm{poly}}(M)$ *extending Kontsevich's.*

Thank you for your attention

References:

BV Formality - R. Campos- Advances in Mathematics 306, (2016)

Operadic Torsors - R. Campos & T. Willwacher- Journal of Algebra 458, (2016)

In case someone asked me the difficult questions

$$
\begin{array}{ccccc}
\mathsf{CBr}_\infty & \circlearrowright & \mathsf{CBr}_\infty^{\mathrm{bimod}} & \circlearrowleft & \mathsf{CBr}_\infty \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{CBr} & \circlearrowright & Chains(\mathsf{F}\mathbb{H}_{\bullet,0}) & \circlearrowleft & Chains(\mathsf{FFM}_2) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{CBr} & \circlearrowright & \prod_n \mathsf{BVKGra}^\mu(\cdot, n)[-n] & \circlearrowleft & \mathsf{BVGra} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{End}\, D_{\mathrm{poly}} & \circlearrowright & \mathrm{Hom}(T_{\mathrm{poly}}^{\otimes \bullet}, D_{\mathrm{poly}}) & \circlearrowleft & \mathrm{End}\, T_{\mathrm{poly}}.
\end{array}
$$