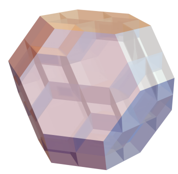
## 1. Introduction

Here are two seemingly unrelated problems:

**Problem 1**. *Compute explicitly some homotopy groups of spheres with coefficients in* $\mathbb{F}_p$.

**Problem 2**. *Define a tensor product of Fukaya categories in symplectic topology, by explicit formulas.*

Suprisingly, solving both problems involve finding a cellular approximation of the diagonal of a family of polytopes, and describing its cellular image combinatorially. This is what I have done in my thesis. My results apply to *any* family of polytopes, they can thus be used to solve both Problem 1 and Problem 2 (see Section 3.3 and Section 4.3), among others. They give rise to new geometric and combinatorial objects, which appear to be of independent interest.



## 2. Recent results

### 2.1. **The diagonal of the operahedra**.

My initial motivation in [13] came from the theory of operads. I wanted to solve the following problem.

**Problem 3**. *Define a tensor product of operads up to homotopy.*

This is the generalization of a smaller problem, that I will describe now, and which still conveys the essential ideas.

An $A_\infty$-algebra is an algebra where the associativity relation holds only up to homotopy. More precisely, it is a graded vector space $A$ together with operations $\mu_n : A^{\otimes n} \to A$ , $n \geq 1$ of degree $n-2$ which satisfy the $A_\infty$-*relations*

$$[\mu_1, \mu_n] = \sum_{\substack{p+q+r=n \\ 2\leq q\leq n-1}} \pm\mu_{p+q+r}(\mathrm{id}^{\otimes p} \otimes \mu_q \otimes \mathrm{id}^{\otimes r}) \ .$$

Thes relations say that $\mu_1$ is a differential, $\mu_2$ is a product, and $\mu_3$ is an homotopy between $\mu_2(\mu_2 \otimes \mathrm{id})$ and $\mu_2(\mathrm{id} \otimes \mu_2)$. The higher $\mu_n$'s are homotopies between homotopies, ensuring coherence.

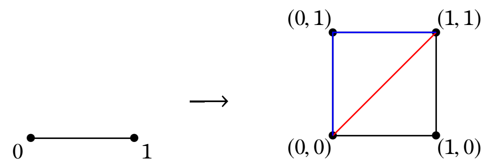
**Problem 4**. *Given two* $A_\infty$*-algebras* $(A, \{\mu_n\})$ *and* $(B, \{\nu_n\})$*, endow their tensor product* $A \otimes B$ *with an* $A_\infty$*-algebra structure.*

This problem is non-trivial. To solve it, one has to design, out of the $\mu_n$ and $\nu_n$, a family of operations $\rho_n : (A \otimes B)^{\otimes n} \to A \otimes B$ which satisfy the $A_\infty$-relations.

It turns out that the operad $A_\infty$, which encodes $A_\infty$-algebras, admits a description in terms of a family of polytopes called associahedra. First introduced by Stasheff [24], the

$(n-2)$-dimensional associahedron has its faces bijectively labeled by planar trees with $n$ leaves. Problem 4 can thus be tackled by more conceptual means as follows.

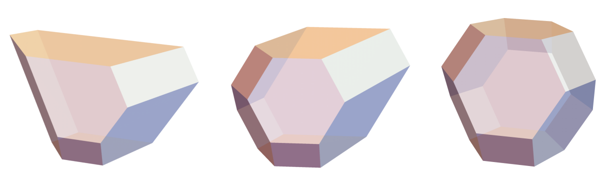
For a polytope $P$, the image of the set-theoretic diagonal $\triangle_P : P \to P \times P, x \mapsto (x, x)$ is not a union of faces of $P \times P$ (in red below). A *cellular approximation* is a cellular map $\triangle_P^{\text{cell}} : P \to P \times P$ which is homotopic to $\triangle_P$ (in blue).




**Proposition 1**. *Suppose that we are able to define a cellular approximation of the diagonal for the associahedra, and endow them with a compatible operad structure. Then, under the cellular chains functor, we get a functorial tensor product of $A_\infty$-algebras.*

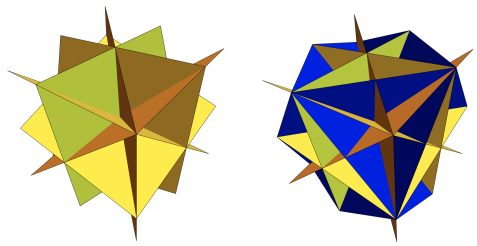
This proposition is not hard to prove, but finding the approximations and the operad structure is. To get explicit formulas for the tensor product, one needs an explicit description of the cellular image of the diagonal. Masuda–Tonks–Thomas–Vallette used in [16] this approach to solve Problem 4, recovering the formulas previously obtained by hand in [23, 15]. To do so, they introduced a new method coming from the theory of fiber polytopes of Billera-Sturmfels [3]. The operad structure that they define on the associahedra presents a fractal character, and is *uniquely* determined by the diagonal.

As associahedra encode associative algebras up to homotopy, there is a larger family of polytopes which encodes operads up to homotopy. I called them operahedra. They include both the associahedra and the permutahedra, which are polytopes whose faces are in bijection with ordered partitions of $\{1, \ldots, n\}$. In fact, the operahedra are all generalized permutahedra, in the sense of Postnikov [21].



To define the tensor product of two operads up to homotopy, i.e. to solve Problem 3, I followed the same thread of thought. However, the simple and elegant formula of [15, 16] for the diagonal of the associahedra does not admit a straightforward generalization.

Refining the method of [16], I developed a general theory of cellular approximations of the diagonal for families of polytopes. My first contribution is a universal formula describing combinatorially the cellular image of such an approximation, for *any* family of polytopes. It is expressed in terms of a new conceptual object associated to a polytope: its *fundamental hyperplane arrangement* $\mathcal{H}_P$.



Given any vector $\vec{v}$ in a chamber of the fundamental hyperplane arrangement $\mathcal{H}_P$ of $P$, the theorem says that a pair of faces $(F,G)$ of $P$ is in the cellular image of the cellular approximation of the diagonal $\triangle_{(P,\vec{v})}$ if and only if their normal vectors $\vec{F}_i, \vec{G}_j$ satisfy a certain condition with respect to the normal vectors $\vec{d}_H$ of the hyperplanes $H \in \mathcal{H}_P$.

**Theorem 1** (Universal formula [13, Theorem 1.23]). *Let $(P,\vec{v})$ be a positively oriented polytope in $\mathbb{R}^n$. For each $H \in \mathcal{H}_P$, we choose a normal vector $\vec{d}_H$ such that $\langle \vec{d}_H, \vec{v}\rangle > 0$. We have*

$$(F,G) \in \mathrm{Im}\triangle_{(P,\vec{v})} \iff \forall H \in \mathcal{H}_P,\ \exists \vec{F}_i,\ \langle \vec{F}_i, \vec{d}_H\rangle < 0 \text{ or } \exists \vec{G}_j,\ \langle \vec{G}_j, \vec{d}_H\rangle > 0\ .$$

My second contribution is to apply this formula to the operahedra, and to define an operad structure on them. This case presents more degrees of freedom, and I had to make a coherent choice of approximations.

**Theorem 2** (Operad structure [13, Theorem 4.18]). *There exists a coherent choice of approximations of the diagonal which forces an operad structure on the operahedra.*

This allowed me to define a functorial tensor product of two homotopy operads. Theorem 1 then gave a very explicit, combinatorial formula for this tensor product [13, Proposition 4.27]. This formula, which is fundamentally expressed as pairs of ordered partitions of $\{1,\ldots,n\}$, appears to be of independent interest.

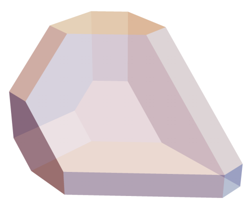
Moreover, a general geometric argument [13, Proposition 1.30] shows that this formula can in fact by applied *mutatis mutandis* to any generalized permutahedron. This prompts immediate applications to modular operads up to homotopy [29], multiplihedra (see Section 3.3) and many other algebraic structures up to homotopy [2].

## 3. Work in progress

3.1. **Möbius inversion and Koszul duality**. Nir Gadish introduced recently a far-reaching generalization of the Möbius inversion formula in the context of homotopy theory [9]. Applying his machinery to a presheaf on a category of trees, we are working on recovering Koszul duality as defined in [10]. The additional combinatorial information provided by his construction prompts a generalization of the operadic partition posets of Vallette [27] outside the set-theoretical world. This could lead to a generalization of the Cohen-Macaulay property, and a new combinatorial criterion for proving that an $(\infty\text{-})$operad is Koszul.

3.2. **Fiber polytopes and the Steenrod construction**. The fundamentally new idea of [16] is the following: for a polytope $P$, every vertex of the fiber polytope $\Sigma(P \times P, P)$ associated to the projection $P \times P \to P, (x, y) \mapsto (x + y)/2$ gives a cellular approximation of the diagonal of $P$. Such an approximation is not invariant under the transposition of the coordinates $(x, y) \mapsto (y, x)$. In the case of the simplices, Steenrod constructed in [26] a tower of homotopies controlling this lack of commutativity. In a joint work with Anibal M. Medina-Mardones, we construct such a tower for *any* polytope by means of iterated fiber polytopes, as in the works of Kapranov, Billera and Sturmfels [4, 5]. We are now working on defining explicit models in higher category theory, following Kapranov–Voevodsky [28] and Medina-Mardones [19].

3.3. **The diagonal of the multiplihedra and the tensor product of $A_\infty$-categories**. In order to define the tensor product of Fukaya $A_\infty$-categories in symplectic topology, for example the category of products of symplectic manifolds, one needs both a tensor product of $A_\infty$-algebras and a compatible tensor product of $A_\infty$-morphisms [14]. This construction is given by a cellular approximation of another family of polytopes, first introduced by J. Stasheff: the multiplihedra [25, 8, 18].



The multiplihedra are, like the operahedra, part of the family of generalized permutahedra [1]. As mentioned at the end of Section 2.1, one can apply *mutatis mutandis* the formula obtained in [13]. In a current work with Thibaut Mazuir and Naruki Masuda, we are working on the following theorem. Taking cellular chains, this would be the first instance of a tensor product of $A_\infty$-categories, defined by explicit formulas.

**Expected result 1**. *The preceding choices of diagonals endow the multiplihedra with a compatible bimodule structure over the associahedra.*

## 4. Future projects

4.1. **Quillen homology of configuration spaces**. In a recent paper [30], Adela Y. Zhang introduces a general method to compute the mod 2 André-Quillen homology of spectral Lie algebras, and applies it to compute the mod 2 homology of configuration of points in a parallelizable manifold. In the case where this manifold is $\mathbb{R}^n$, she makes a precise conjecture about higher differentials in a certain spectral sequence [30, Conjecture 5.7].

This conjecture could be solved by conceptual means, as follows. As suggested in the accompanying remark [30, Remark 5.8], the spectral sequence arises from the bar-cobar resolution of the spectral Lie operad. This is the analogue of a spectral sequence of May [17] for associative algebras, where higher differentials correspond to Massey products, i.e.

to an $A_\infty$-algebra structure. In the case of the spectral Lie operad, they should correspond to operadic Massey products, i.e. to an $\mathbb{O}_\infty$-algebra structure, where the operad $\mathbb{O}_\infty$ is encoded by the operahedra of Section 2.1.

In fact, one would need a version of $\mathbb{O}_\infty$ taking both symmetric group actions (this was done in [7]) and power operations (since we are working over $\mathbb{F}_2$) into account. First steps in this direction are made in [11]. Understanding the $\mathbb{O}_\infty$ structure on the singular chains of the spectral Lie operad would then allow for an explicit computation, via the homotopy transfer theorem, of the desired operadic Massey products.

4.2. **Associativity in $A_\infty$-categories**. What is the "monoidal" structure formed by $A_\infty$-algebras and their $A_\infty$-morphisms? A first step to answer this question is to understand the "up to homotopy" coassociativity of the diagonal of the associahedra. Indeed, by a result of Markl and Schnider [15, Section 6], a strictly coassociative diagonal does not exist. The idea would be to construct a family of homotopy coherent "higher diagonals" $\triangle_n : C_\bullet(P) \to C_\bullet(P)^{\otimes n}$ resolving the lack of coassociativity of $\triangle_2$. I see three different approaches to perform this construction.

(1) Use iterated fiber polytopes, as for the Steenrod diagonal in Section 3.2.
(2) Define geometrically the retraction of Markl-Schnider [15] and apply the homotopy transfer theorem between the minimal and bar-cobar resolutions of the assocative operad.
(3) Restrict the operadic construction of [20, Chapter 4] to a certain class of coherent sequences of faces, as in [5].

4.3. **Homotopy groups of spheres: the unfinished symphony of Alain Prouté**. In his Ph. D. thesis [22], Alain Prouté pursues the goal of computing explicitly the mod $p$ homology of fibered spaces by means of $A_\infty$-structures, a technique initiated by Kadeishvili in [12], whose ideas go back to Brown [6]. Given a principal fibration $G \to E \to X$, the idea is to go around Serre's spectral sequence and compute directly the homology of $E$ in terms of the homology of the twisted tensor product $H_\bullet(X) \otimes_t H_\bullet(G)$, where $H_\bullet(X)$ is endowed with a $A_\infty$-coalgebra structure [22, Theorem 5.14].

Prouté treats the case where the fiber is an Eilenberg-MacLane space $K(\mathbb{Z}/p, n)$. He is able to do part of the computation, but he has to stop because he needs an explicit formula for the tensor product of $A_\infty$-coalgebras. Using Hurewicz's theorem, this explicit computation would give a way to determine iteratively homotopy groups of spheres.

As we have seen in Section 2.1, the problem of defining the tensor product of two $A_\infty$-coalgebras (or equivalently two $A_\infty$-algebras) has been solved since the early 2000's [23, 15]. However, to my knowledge, nobody has continued Prouté's computations. This is probably due to the fact that the formula for the tensor product of two $A_\infty$-algebras, in its actual form, is not very amenable to computations. The new combinatorial description obtained in [13], via the universal formula, could provide the missing tool.

**Objective 1**. *Compute explicitly some mod $p$ homotopy groups of spheres with the help of the new combinatorial description of the diagonal of the associahedra obtained in* [13].

## References


[1] F. Ardila and J. Doker. Lifted generalized permutahedra and composition polynomials. *Advances in Applied Mathematics*, 50(4):607–633, 2013.

[2] M. Batanin, M. Markl, and J. Obradović. Minimal models for graphs-related operadic algebras. *ArXiv e-prints*, page arXiv:2002.06640, 2020.

[3] L. J. Billera and B. Sturmfels. Fiber polytopes. *Ann. of Math. (2)*, 135(3):527–549, 1992.

[4] L. J. Billera and B. Sturmfels. Iterated fiber polytopes. *Mathematika*, 41(2):348–363, 1994.

[5] L. J. Billera, B. Sturmfels, and M. M. Kapranov. Cellular strings on polytopes. *Proceedings of the American Mathematical Society*, 122(2):549–555, 1994.

[6] E. H. Brown, Jr. Twisted tensor products. I. *Ann. of Math. (2)*, 69:223–246, 1959.

[7] M. Dehling and B. Vallette. Symmetric homotopy theory for operads. *Algebraic & Geometric Topology*, 21(4):1595–1660, 2015.

[8] S. Forcey. Convex hull realizations of the multiplihedra. *Topology Appl.*, 156(2):326–347, 2008.

[9] N. Gadish. Möbius inversion in homotopy theory. MIT Topology Seminar, 2021.

[10] E. Hoffbeck and I. Moerdijk. Homology of infinity-operads. *ArXiv e-prints*, page arXiv:2105.11943, 2021.

[11] S. Ikonicoff. Unstable algebras over an operad. *Homology, Homotopy and Applications*, 23(1):119–141, 2021.

[12] T. V. Kadeišvili. On the theory of homology of fiber spaces. *Uspekhi Mat. Nauk*, 35(3(213)):183–188, 1980. International Topology Conference (Moscow State Univ., Moscow, 1979).

[13] G. Laplante-Anfossi. The diagonal of the operahedra. *ArXiv e-prints*, page arXiv:2110.14062, 2021.

[14] R. Lipshitz, P. Ozsváth, and D. P. Thurston. Diagonals and A-infinity tensor products. *ArXiv e-prints*, page arXiv:2009.05222, 2020.

[15] M. Markl and S. Shnider. Associahedra, cellular $W$-construction and products of $A_\infty$-algebras. *Trans. Amer. Math. Soc.*, 358(6):2353–2372 (electronic), 2006.

[16] N. Masuda, H. Thomas, A. Tonks, and B. Vallette. The diagonal of the associahedra. *Journal de l'École polytechnique*, 8:121–146, 2021.

[17] J. P. May. The cohomology of augmented algebras and generalized Massey products for DGA-algebras. *Transactions of the American Mathematical Society*, 122(2):334–340, 1966.

[18] T. Mazuir. Higher algebra of $A_\infty$ and $\Omega B$As-algebras in Morse theory I, II. *ArXiv e-prints*, pages arXiv:2102.06654,2102.08996, 2021.

[19] A. M. Medina-Mardones. An algebraic representation of globular sets. *Homology Homotopy Appl.*, 22(2):135–150, 2020.

[20] D. Poliakova. Homotopical algebra and combinatorics of polytopes. *Ph.D. Thesis*, http://web.math.ku.dk/noter/filer/phd21dp.pdf, 2021.

[21] A. Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, 6:1026–1106, 2009.

[22] A. Prouté. $A_\infty$-structures, modèle minimal de Baues-Lemaire et homologie des fibrations. *Ph.D. Thesis*, 1986.

[23] S. Saneblidze and R. Umble. Diagonals on the permutahedra, multiplihedra and associahedra. *Homology Homotopy Appl.*, 6(1):363–411, 2004.

[24] J. D. Stasheff. Homotopy associativity of $H$-spaces. I, II. *Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid.*, 108:293–312, 1963.

[25] J. D. Stasheff. *$H$-spaces from a homotopy point of view.* Lecture Notes in Mathematics, Vol. 161. Springer-Verlag, Berlin, 1970.

[26] N. E. Steenrod. Products of cocycles and extensions of mappings. *Ann. of Math. (2)*, 48:290–320, 1947.

[27] B. Vallette. Homology of generalized partition posets. *J. Pure Appl. Algebra*, 208(2):699–725, 2007.

[28] V. A. Voevodsky and M. M. Kapranov. Combinatorial-geometric aspects of polycategory theory: pasting schemes and higher bruhat orders. *Cahiers de topologie et géométrie différentielle catégoriques*, 32(1):11–27, 1991.

[29] B. Ward. Massey products for graph homology. *Int. Math. Res. Not. IMRN*, 2021.

[30] A. Y. Zhang. André-quillen homology of spectral Lie algebras with application to mod $p$ homology of labeled configuration spaces. *arXiv e-prints*, page arXiv:2110.08428, 2021.

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