Notes on Wheeled PROPs, Totalisations, and the $\mathfrak{h}_A, \widehat{\mathfrak{h}}_A$ dg Lie Algebras

1 PROPs

Definition 1.1 ([MMS09, after equation (17)], [KV21, Proposition 6.2])**.** A **PROP** is a family $P = \{P(m,n)\}_{m,n\geq 0}$ of (Σ_m, Σ_n)-bimodules, equipped with:

(i) *Horizontal composition*: maps

$$\otimes\colon P(m_1,n_1) \otimes P(m_2,n_2) \to P(m_1+m_2,\, n_1+n_2),$$

(ii) *Vertical composition*: maps

$$\circ\colon P(m,n) \otimes P(n,k) \to P(m,k),$$

(iii) a *unit* $\mathbf{1} \in P(1,1)$,

satisfying the following axioms:

1. **Associativity of $\otimes$:** for all triples $p_i \in P(m_i,n_i)$, $1 \leq i \leq 3$, we have

$$(p_1 \otimes p_2) \otimes p_3 = p_1 \otimes (p_2 \otimes p_3).$$

2. **Associativity of $\circ$:** for $p \in P(m,n)$, $q \in P(n,k)$, $r \in P(k,l)$, we have

$$(p \circ q) \circ r = p \circ (q \circ r).$$

3. **Unit axioms:** for all $p \in P(m,n)$, we have

$$\mathbf{1}^{\otimes m} \circ p = p = p \circ \mathbf{1}^{\otimes n},$$

where $\mathbf{1}^{\otimes k} := \mathbf{1} \otimes \cdots \otimes \mathbf{1} \in P(k,k)$ denotes the k-fold horizontal product.

4. **Interchange law:** for any $p_i \in P(m_i,n_i)$ and $q_i \in P(n_i,k_i)$ with $i = 1,2$, we have

$$(p_1 \circ q_1) \otimes (p_2 \circ q_2) = (p_1 \otimes p_2) \circ (q_1 \otimes q_2).$$

5. **Equivariance:** for any $\sigma \in \Sigma_m$, $\tau \in \Sigma_n$, $\sigma' \in \Sigma_{m'}$, $\tau' \in \Sigma_{n'}$, $p \in P(m,n)$, $q \in P(m',n')$, we have

$$(\sigma \cdot p \cdot \tau) \otimes (\sigma' \cdot q \cdot \tau') = (\sigma \oplus \sigma') \cdot (p \otimes q) \cdot (\tau \oplus \tau'),$$

and for any $\sigma \in \Sigma_m$, $\rho \in \Sigma_n$, $\tau \in \Sigma_k$, $p \in P(m,n)$, $q \in P(n,k)$, we have

$$(\sigma \cdot p \cdot \rho) \circ (\rho^{-1} \cdot q \cdot \tau) = \sigma \cdot (p \circ q) \cdot \tau.$$

Here, $\sigma \oplus \sigma'$ is the permutation in $\Sigma_{m+m'}$ which acts by σ on the first m numbers and by σ' on the remaining m' numbers.

Example 1.2 ([MMS09, Example 2.1.1]). For a graded vector space V, the *endomorphism PROP* End_V is defined by

$$\mathrm{End}_V(m,n) := \mathrm{Hom}(V^{\otimes n}, V^{\otimes m}).$$

We think of an element $f \in \mathrm{End}_V(m,n)$ as a linear map with n inputs and m outputs. The structure maps are:

(i) *Symmetric group actions:* Σ_m acts on the left and Σ_n acts on the right by permuting tensor factors. Explicitly, for $\sigma \in \Sigma_m$, $\tau \in \Sigma_n$, and $f \in \mathrm{End}_V(m,n)$,

$$(\sigma \cdot f \cdot \tau)(v_1 \otimes \cdots \otimes v_n) := \sigma_*\big(f(v_{\tau^{-1}(1)} \otimes \cdots \otimes v_{\tau^{-1}(n)})\big),$$

where $\sigma_*(w_1 \otimes \cdots \otimes w_m) := w_{\sigma^{-1}(1)} \otimes \cdots \otimes w_{\sigma^{-1}(m)}$.

(ii) *Horizontal composition:* for $f \in \mathrm{End}_V(m_1,n_1)$ and $g \in \mathrm{End}_V(m_2,n_2)$,

$$f \otimes g \colon V^{\otimes n_1+n_2} \to V^{\otimes m_1+m_2}$$

is the tensor product of linear maps:

$$(f \otimes g)(v_1 \otimes \cdots \otimes v_{n_1+n_2}) := f(v_1 \otimes \cdots \otimes v_{n_1}) \otimes g(v_{n_1+1} \otimes \cdots \otimes v_{n_1+n_2}).$$

(iii) *Vertical composition:* for $f \in \mathrm{End}_V(m,n)$ and $g \in \mathrm{End}_V(n,k)$,

$$f \circ g \colon V^{\otimes k} \to V^{\otimes m}$$

is the ordinary composition of linear maps:

$$(f \circ g)(v_1 \otimes \cdots \otimes v_k) := f\big(g(v_1 \otimes \cdots \otimes v_k)\big).$$

(iv) *Unit:* $\mathbf{1} := \mathrm{id}_V \in \mathrm{End}_V(1,1) = \mathrm{Hom}(V,V)$.

2 Wheeled PROPs

2.1 Description via contractions

Definition 2.1 ([MMS09, after equation (17)], [KV21, Section 2.2]). A wheeled PROP structure on a Σ-bimodule P is given by the following data:

(i) horizontal composition $\otimes \colon P(m_1,n_1) \otimes P(m_2,n_2) \to P(m_1+m_2, n_1+n_2)$;

(ii) a unit $\mathbf{1} \in P(1,1)$;

(iii) *contraction maps*

$$\xi^i_j \colon P(m,n) \to P(m-1,n-1), \quad 1 \le i \le m,\ 1 \le j \le n,$$

satisfying the following axioms:

1. **Contractions commute:** for $i \neq i'$ and $j \neq j'$, we have

$$\xi^{i'}_{j'} \circ \xi^i_j = \xi^{i''}_{j''} \circ \xi^{i'}_{j'}$$

where i'', j'' are i, j reindexed after the first contraction removes a slot.

2. **Equivariance:** for $\sigma \in \Sigma_m$ and $\tau \in \Sigma_n$, we have

$$\xi^i_j \circ (\sigma, \tau) = (\sigma^{(\sigma^{-1}(i))}, \tau^{(j)}) \circ \xi^{\sigma^{-1}(i)}_{\tau(j)},$$

   where $\sigma^{(k)}$ denotes σ with the k-th letter removed.

3. **Compatibility with horizontal composition:** for $p \in P(m_1, n_1)$ and $q \in P(m_2, n_2)$, we have

$$\xi^i_j(p \otimes q) = \begin{cases} \xi^i_j(p) \otimes q & \text{if } 1 \leq i \leq m_1 \text{ and } 1 \leq j \leq n_1, \\ p \otimes \xi^{i-m_1}_{j-n_1}(q) & \text{if } m_1 < i \leq m_1 + m_2 \text{ and } n_1 < j \leq n_1 + n_2. \end{cases}$$

A wheeled prop is in particular a prop. The vertical composition $\circ\colon P(n,l) \otimes P(m,n) \to P(m,l)$ is recovered from the above data by the equation [KV21, equation (2.1)]

$$\mu \circ \nu := (\xi^1_{l+1})^n(\mu \otimes \nu), \tag{1}$$

where the contraction ξ^1_{l+1} is applied n times, each time connecting the first output of μ to the first input of ν. The *dioperadic compositions*, connecting one output to one input across different factors, are expressed as

$$p \circ^j_i q := \xi^{m_1+j}_i(p \otimes q), \quad 1 \leq j \leq n_1,\ 1 \leq i \leq m_2.$$

Example 2.2 ([MMS09, Example 2.1.1])**.** Let V be a finite-dimensional graded vector space. The endomorphism PROP End_V of Example 1.2 carries the structure of a wheeled PROP, where the contraction maps are defined as follows. Using the canonical isomorphism $\mathrm{End}_V(m,n) = \mathrm{Hom}(V^{\otimes n}, V^{\otimes m}) \cong (V^*)^{\otimes n} \otimes V^{\otimes m}$, write an arbitrary element as a tensor $f_1 \otimes \cdots \otimes f_n \otimes v_1 \otimes \cdots \otimes v_m$ with $f_k \in V^*$ and $v_k \in V$. Then, for $1 \leq i \leq m$ and $1 \leq j \leq n$, define

$$\xi^i_j(f_1 \otimes \cdots \otimes f_n \otimes v_1 \otimes \cdots \otimes v_m) := f_j(v_i) \cdot (f_1 \otimes \cdots \widehat{f_j} \cdots \otimes f_n \otimes v_1 \otimes \cdots \widehat{v_i} \cdots \otimes v_m).$$

This is indepedent of the choice of basis and defines a linear map

$$\xi^i_j\colon\ \mathrm{End}_V(m,n) \to \mathrm{End}_V(m-1, n-1).$$

When $m = n = 1$, this reduces to $\xi^1_1(f \otimes v) = f(v)$, which is exactly the trace $\mathrm{End}(V) \to \mathbf{k}$.

3 Totalisation and the dg Lie algebra structure

Definition 3.1 ([MV09, Section 1.1])**.** The **totalisation** of a Σ-bimodule $P = \{P(m,n)\}$ is the graded vector space

$$\mathrm{Tot}(P) := \prod_{m,n \geq 0} P(m,n)^{\Sigma_m \times \Sigma_n},$$

where $P(m,n)^{\Sigma_m \times \Sigma_n} = \{x \in P(m,n) : \sigma \cdot x \cdot \tau = x \text{ for all } \sigma \in \Sigma_m, \tau \in \Sigma_n\}$ is the subspace of (Σ_m, Σ_n)-invariants.

An element $\mu \in \mathrm{Tot}(P)$ is thus a collection $\mu = (\mu_{m,n})_{m,n \geq 0}$ where $\mu_{m,n} \in P(m,n)^{\Sigma_m \times \Sigma_n}$.

Theorem 3.2 ([MV09, Proposition 5]). Let P be a dg wheeled PROP. There exists a product

$$\bullet\colon \operatorname{Tot}(P) \otimes \operatorname{Tot}(P) \to \operatorname{Tot}(P)$$

making $(\operatorname{Tot}(P), d, \bullet)$ a dg associative algebra.

Remark 3.3. Intuitively, the product $\mu \bullet \nu$ is defined by summing over all ways of connecting outputs of ν to inputs of μ using the contraction maps ξ^i_j of the wheeled PROP, and then taking the horizontal composition of the remaining free inputs and outputs. The (Σ_m, Σ_n)-invariance of μ and ν ensures that this sum is well-defined independently of the ordering of legs. The fact that it is associative comes from the fact that horizontal and vertical compositions are.

Any associative algebra is a Lie algebra for the commutator bracket, therefore we have a Lie algebra structure on the totalisation of P.

Corollary 3.4 ([MV09, Theorem 8]). For any dg wheeled PROP P, the commutator

$$[\mu, \nu] := \mu \bullet \nu - (-1)^{|\mu||\nu|} \nu \bullet \mu$$

defines a dg Lie bracket on $\operatorname{Tot}(P)$. In particular, $(\operatorname{Tot}(P), d, [-,-])$ is a dg Lie algebra.

A detailed proof of this fact is given in Appendix A.1.

4 The Căldararu–Tu dg Lie algebras

Definition 4.1. An A_∞**-algebra** is a graded vector space A together with multilinear maps $m_n\colon A^{\otimes n} \to A$ of degree $|m_n| = 2 - n$ for $n \geq 1$, such that for each $n \geq 1$,

$$\sum_{\substack{i+j=n+1\\ i,j\geq 1}} \sum_{k=0}^{n-j} (-1)^{k(j-1)+j\sum_{l=1}^{k}|a_l|} m_i(a_1, \dots, a_k, m_j(a_{k+1}, \dots, a_{k+j}), a_{k+j+1}, \dots, a_n) = 0.$$

A dg associative algebra is the special case where $m_n = 0$ for all $n \geq 3$.

Definition 4.2. A **cyclic** A_∞**-algebra** is a finite-dimensional A_∞-algebra $(A, \{m_n\})$ equipped with a non-degenerate symmetric bilinear pairing $\langle -,- \rangle\colon A \otimes A \to \mathbf{k}$ of degree d such that for all $a_1, \dots, a_{n+1} \in A$,

$$\langle m_n(a_1, \dots, a_n),\, a_{n+1}\rangle = (-1)^{|a_{n+1}|(|a_1|+\cdots+|a_n|)} \langle m_n(a_{n+1}, a_1, \dots, a_{n-1}),\, a_n\rangle.$$

Let $(A, \{m_n\}, \langle -,- \rangle)$ be a cyclic A_∞-algebra. The non-degenerate pairing induces an isomorphism $\phi\colon A \xrightarrow{\sim} A^*[d]$ by $\phi(a) = \langle a, - \rangle$, and one uses this to identify $\operatorname{End}_A(m,n) = \operatorname{Hom}(A^{\otimes n}, A^{\otimes m})$ with $(A^*)^{\otimes n} \otimes A^{\otimes m}$. The contraction maps ξ^i_j on End_A are defined exactly as in Example 2.2, making End_A a wheeled PROP.

We continue with the two dg Lie algebras constructed in [CT24, Section 4] associated to a cyclic A_∞-algebra. Throughout this section, let $(A, \{m_n\}_{n\geq 1}, \langle -,- \rangle)$ be a cyclic A_∞-algebra over a field $\mathbf{k}$ of characteristic zero, of Calabi–Yau dimension d, satisfying the smoothness condition (†) of [CT24, Section 1.2]: A is smooth, finite-dimensional, unital, and satisfies the Hodge-de Rham degeneration property.

4.1 Hochschild chains and the Tate space

Let $C_*(A)$ denote the Hochschild chain complex of A, defined by

$$C_n(A) := A^{\otimes n+1}, \quad n \geq 0,$$

with the Hochschild differential b of degree -1 given by

$$\begin{aligned} b(a_0 \otimes \cdots \otimes a_n) \;&:=\; \sum_{k\geqslant 1} \sum_{i=0}^{n-k+1} \pm a_0 \otimes \cdots \otimes a_{i-1} \otimes m_k(a_i, \ldots, a_{i+k-1}) \otimes a_{i+k} \otimes \cdots \otimes a_n \\ &\quad + \sum_{k\geqslant 1} \sum_{i=n-k+2}^{n} \pm m_k(a_i, \ldots, a_n, a_0, \ldots, a_{i+k-n-2}) \otimes a_{i+k-n-1} \otimes \cdots \otimes a_{i-1}. \end{aligned}$$

Denote by $L := C_*(A)[d]$ the Hochschild chain complex shifted by the Calabi–Yau dimension d of A [CT24, Section 3.4]. The complex L carries two additional structures [CT24, Section 3.2]:

(i) the *Connes operator* $B : L \to L$ of degree $+1$, defined by

$$B(a_0 \otimes \cdots \otimes a_n) \;:=\; \sum_{i=0}^{n} \pm 1 \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

which gives L the structure of a *complex with circle action* [CT24, Section 3.2], satisfying

$$b^2 = 0, \qquad B^2 = 0, \qquad bB + Bb = 0;$$

(ii) the *Mukai pairing* $\langle -, - \rangle_{\mathrm{Muk}} \colon L \otimes L \to \mathbf{k}$, a symmetric bilinear form of even degree with respect to which B is self-adjoint:

$$\langle Bx, y \rangle_{\mathrm{Muk}} = (-1)^{|x|} \langle x, By \rangle_{\mathrm{Muk}}.$$

An explicit formula is given in [She20, Definition 5.19].

Following [CT24, Section 3.4], we introduce a formal variable u of even degree and define the *Tate space* $L^{\mathrm{Tate}} = L_+ \oplus L_-$ by

$$L_- := L[u^{-1}] = L \otimes_{\mathbf{k}} \mathbf{k}[u^{-1}], \qquad L_+ := L[\![u]\!] = L \otimes_{\mathbf{k}} \mathbf{k}[\![u]\!],$$

each equipped with the differential $b + uB$.

4.2 The dg Lie algebra $\mathfrak{h}_A$

The space L_+ is naturally dual to L_- via the *residue pairing* [CT24, Section 3.2]

$$\langle x \cdot u^k,\, y \cdot u^l \rangle_{\mathrm{res}} := \begin{cases} (-1)^l \langle x, y \rangle_{\mathrm{Muk}} & \text{if } k + l = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.3 ([CT24, Section 4.1])**.** Let $\hbar$ be a formal variable of even degree and λ a formal variable of degree 2. The *graded-symmetric algebra* on L_- is

$$\mathrm{Sym}(L_-) := \bigoplus_{k\geq 0} (L_-^{\otimes k})^{\Sigma_k},$$

the direct sum of the spaces of symmetric tensors. Here, the Define

$$\mathfrak{h}_A := \mathrm{Sym}(L_-)[[\hbar, \lambda]][1],$$

where [1] denotes a shift in cohomological degree by +1, i.e. $(\mathfrak{h}_A)^i := \mathrm{Sym}(L_-)[[\hbar, \lambda]]^{i-1}$. The differential on $\mathfrak{h}_A$ is $D_1 := b + uB + \hbar\Delta$, where:

(i) $b + uB$ extends to $\mathrm{Sym}(L_-)$ as a derivation, i.e. for $x, y \in L_-$,

$$(b + uB)(x \cdot y) = (b + uB)(x) \cdot y + (-1)^{|x|} x \cdot (b + uB)(y);$$

(ii) $\Delta\colon \mathrm{Sym}(L_-) \to \mathrm{Sym}(L_-)$ is the *BV operator*, defined as the unique second-order operator satisfying:

- $\Delta(1) = 0$ on $\mathrm{Sym}^0(L_-) = \mathbf{k}$,
- $\Delta(x) = 0$ for all $x \in \mathrm{Sym}^1(L_-) = L_-$,
- $\Delta(x \cdot y) = \langle Bx, y\rangle_{\mathrm{Muk}}$ for all $x, y \in \mathrm{Sym}^1(L_-) = L_-$.

Δ is determined on all of $\mathrm{Sym}(L_-)$ via the relation

$$\Delta(x \cdot y \cdot z) = \Delta(x \cdot y) \cdot z + (-1)^{|x|} x \cdot \Delta(y \cdot z) - \Delta(x) \cdot y \cdot z - (-1)^{|x|} x \cdot \Delta(y) \cdot z - (-1)^{|x|+|y|} x \cdot y \cdot \Delta(z).$$

The Lie bracket on $\mathfrak{h}_A$ measures the failure of Δ to be a derivation for the symmetric product: for $x, y \in \mathfrak{h}_A$,

$$\{x, y\} := \Delta(x \cdot y) - (\Delta x) \cdot y - (-1)^{|x|} x \cdot (\Delta y).$$

We use the notation $\{-,-\}$ rather than $[-,-]$ to distinguish this bracket from the commutator bracket of Section 3 and to follow the notation of [CT24].

Proposition 4.4 ([CT24, Section 4.1], [Get94])**.** The triple $(\mathfrak{h}_A, D_1, \{-,-\})$ is a dg Lie algebra.

Proof. We verify the three axioms of a dg Lie algebra. Throughout, degrees are computed in $\mathrm{Sym}(L_-)$.

Step 1: $\Delta^2 = 0$. For $x, y \in L_-$,

$$\Delta^2(x \cdot y) = \Delta\big(\langle Bx, y\rangle_{\mathrm{Muk}}\big) = \langle Bx, y\rangle_{\mathrm{Muk}} \cdot \Delta(1) = 0,$$

since $\langle Bx, y\rangle_{\mathrm{Muk}} \in \mathbf{k}$ is a scalar and Δ vanishes on $\mathrm{Sym}^0(L_-) = \mathbf{k}$ and $\mathrm{Sym}^1(L_-)$. Hence $\Delta^2 = 0$.

Step 2: $\{-,-\}$ is a graded Lie bracket. We must show that $\{-,-\}$ is graded antisymmetric and satisfies the graded Jacobi identity. By [Get94, Proposition 1.2], for any graded-commutative algebra equipped with an operator Δ of odd degree satisfying $\Delta^2 = 0$, the bracket defined by

$$\{x, y\} := \Delta(x \cdot y) - (\Delta x) \cdot y - (-1)^{|x|} x \cdot (\Delta y)$$

satisfies both axioms. In our case $\mathrm{Sym}(L_-)$ is graded-commutative and $\Delta^2 = 0$ by Step 1, so the result applies.

Step 3: D_1 is a differential. We must show $D_1^2 = 0$. Expanding:

$$D_1^2 = b^2 + u(bB + Bb) + u^2 B^2 + \hbar\big[(b + uB)\Delta + \Delta(b + uB)\big] + \hbar^2\Delta^2.$$

The terms $b^2 = 0$, $bB + Bb = 0$, and $B^2 = 0$ hold by the identities satisfied by the Hochschild differential b and the Connes operator B stated in Section 4.1., and $\Delta^2 = 0$ by Step 1. For the remaining term, we must show $(b + uB)\Delta + \Delta(b + uB) = 0$. Since Δ is determined by its values on $\mathrm{Sym}^2(L_-)$, it suffices to check on generators $x \cdot y \in \mathrm{Sym}^2(L_-)$. We compute $(b + uB)\Delta(x \cdot y)$ first. By definition of Δ on Sym^2,

$$\Delta(x \cdot y) = \langle Bx, y\rangle_{\mathrm{Muk}} \in \mathbf{k}.$$

The Mukai pairing $\langle -, -\rangle_{\mathrm{Muk}}$ satisfies the graded Leibniz rule with respect to $b + uB$ as it is a graded map of complexes from $L \otimes L \to k$ and L has differential $b + uB$. This means it satisfies

$$(b + uB)\langle \alpha, \beta\rangle_{\mathrm{Muk}} = \langle (b + uB)\alpha, \beta\rangle_{\mathrm{Muk}} + (-1)^{|\alpha|}\langle \alpha, (b + uB)\beta\rangle_{\mathrm{Muk}}$$

for any $\alpha, \beta \in L_-$, and since $|Bx| = |x| + 1$, we get

$$(b + uB)\Delta(x \cdot y) = \langle (b + uB)Bx, y\rangle_{\mathrm{Muk}} + (-1)^{|x|+1}\langle Bx, (b + uB)y\rangle_{\mathrm{Muk}}. \tag{2}$$

We now compute $\Delta(b + uB)(x \cdot y)$. Since $b + uB$ is a derivation of the product in $\mathrm{Sym}(L_-)$,

$$(b + uB)(x \cdot y) = (b + uB)x \cdot y + (-1)^{|x|}x \cdot (b + uB)y.$$

Applying Δ to this expression and using the definition of Δ on Sym^2 together with the fact that Δ vanishes on Sym^1, we get

$$\begin{aligned}\Delta(b + uB)(x \cdot y) &= \Delta\big((b + uB)x \cdot y\big) + (-1)^{|x|}\Delta\big(x \cdot (b + uB)y\big)\\ &= \langle B(b + uB)x, y\rangle_{\mathrm{Muk}} + (-1)^{|x|}\langle Bx, (b + uB)y\rangle_{\mathrm{Muk}}.\end{aligned}$$

Hence, we have

$$\Delta(b + uB)(x \cdot y) = \langle B(b + uB)x, y\rangle_{\mathrm{Muk}} + (-1)^{|x|}\langle Bx, (b + uB)y\rangle_{\mathrm{Muk}}. \tag{3}$$

Adding (2) and (3), we obtain:

$$\begin{aligned}\big[(b + uB)\Delta + \Delta(b + uB)\big](x \cdot y) = \langle &(b + uB)Bx + B(b + uB)x,\, y\rangle_{\mathrm{Muk}}\\ &+ \big[(-1)^{|x|+1} + (-1)^{|x|}\big]\langle Bx, (b + uB)y\rangle_{\mathrm{Muk}}.\end{aligned}$$

The second line vanishes since $(-1)^{|x|+1} + (-1)^{|x|} = 0$. For the first line, we have expanded $(b + uB)B + B(b + uB) = bB + Bb + uB^2 + uB^2 = 0 + 2uB^2 = 0$, using $bB + Bb = 0$ and $B^2 = 0$. Hence

$$\big[(b + uB)\Delta + \Delta(b + uB)\big](x \cdot y) = 0,$$

which completes the proof that $D_1^2 = 0$. □

Exercise 4.5. The fact that D_1 is a derivation for the bracket $\{-, -\}$ is left as an exercise. For more details see [CT24, Section 4.1] and [Get94, Proposition 1.2].

4.3 The dg Lie algebra $\widehat{\mathfrak{h}}_A$

Definition 4.6 ([CT24, Section 4.2])**.** We work over a field $\mathbf{k}$ of characteristic zero. Define the notation

$$L_{k,l} := \mathrm{Hom}^c\big(\mathrm{Sym}^k(L_+[1]),\, \mathrm{Sym}^l(L_-)\big),$$

where Hom^c denotes continuous homomorphisms with respect to the u-adic topology on $L_+ = L[\![u]\!]$. The degree of an element $\Phi \in L_{k,l}$ is denoted $|\Phi|$. Define the graded vector space

$$\widehat{\mathfrak{h}}_A := \bigoplus_{k\geq 1,\, l\geq 0} L_{k,l}[[\hbar, \lambda]],$$

where $\hbar$ and λ are the same formal variables as in Definition 4.3.

Remark 4.7. The u-adic topology on $L_+ = L[\![u]\!]$ is the standard $\mathfrak{m}$-adic topology with respect to the ideal $\mathfrak{m} = uL[\![u]\!]$, with neighbourhood basis of 0 given by $\{u^N L[\![u]\!]\}_{N\geq 0}$.

4.3.1 The differential on $\widehat{\mathfrak{h}}_A$

The differential on $\widehat{\mathfrak{h}}_A$ is $D_2 := \eth + \iota + \hbar\Delta$, where each of the three components is defined as follows [CT24, Section 4.2].

The operator $\eth$. For $\Phi \in L_{k,l}$, the operator $\eth$ is the commutator action of $b + uB$:

$$\eth(\Phi) := (b + uB) \circ \Phi + (-1)^{|\Phi|}\Phi \circ (b + uB).$$

The operator Δ. The BV operator Δ on $\mathrm{Sym}(L_-)$ from Definition 4.3 extends to an operator $\Delta\colon L_{k,l} \to L_{k,l-2}$ by post-composition:

$$\Delta(\Phi) := \Delta \circ \Phi, \quad \Phi \in L_{k,l}.$$

The operator ι. To define ι we first introduce, for each $\beta \in L_+[1]$, the *contraction operator* $C_\beta\colon \mathrm{Sym}^l(L_-) \to \mathrm{Sym}^{l-1}(L_-)$ defined by

$$C_\beta(\alpha_1 \cdots \alpha_l) := \sum_{i=1}^{l} (-1)^{|\alpha_i|(|\alpha_1|+\cdots+|\alpha_{i-1}|)} \langle u\beta, \alpha_i\rangle_{\mathrm{res}} \cdot \alpha_1 \cdots \widehat{\alpha_i} \cdots \alpha_l,$$

where $\langle -, -\rangle_{\mathrm{res}}$ is the residue pairing as in Section 4.1. Using these contraction operators, the operator $\iota\colon L_{k,l} \to L_{k+1,l-1}$ is defined by

$$\iota(\Phi)(\beta_1 \cdots \beta_{k+1}) := \sum_{j=1}^{k+1} (-1)^{|\beta_j|(|\beta_1|+\cdots+|\beta_{j-1}|+|\Phi|)} C_{\beta_j}\Phi(\beta_1 \cdots \widehat{\beta_j} \cdots \beta_{k+1}).$$

Lemma 4.8 ([CT24, Lemma 4.2])**.** The operator $D_2 = \eth + \iota + \hbar\Delta$ satisfies $D_2^2 = 0$.

4.3.2 The Lie bracket on $\widehat{\mathfrak{h}}_A$

Denote by $\Theta\colon L_- \to L_+[1]$ the *circle action map* defined by [CT24, Section 4.2]

$$\Theta\left(\sum_{k\leq 0}\alpha_k u^k\right) := B(\alpha_0),$$

i.e. Θ extracts the constant term $\alpha_0 \in L$ and applies the Connes operator B. This is a chain map of degree zero since $\Theta\eth(\alpha) = Bb\alpha_0 = -bB\alpha_0 = \eth\Theta(\alpha)$, using $bB + Bb = 0$. For a symmetric tensor $\gamma_1\cdots\gamma_n \in \mathrm{Sym}^n(V)$ and a subset $A = \{a_1,\dots,a_s\} \subset \{1,\dots,n\}$ with $a_1 < \cdots < a_s$, write $\gamma_A := \gamma_{a_1}\cdots\gamma_{a_s}$. For every $r \geq 1$, we define the composition operation [CT24, Section 4.2]

$$\circ_r\colon L_{k_2,l_2}\otimes L_{k_1,l_1} \to L_{k_1+k_2-r,\,l_1+l_2-r}, \quad l_1, k_2 \geq r,$$

by

$$(\Psi\circ_r\Phi)(\beta_1\cdots\beta_{k_1+k_2-r}) := \sum_{P,Q}\sum_{I,J}\epsilon_{I,J}\,\epsilon_{P,Q}\,(-1)^{|\Psi||\Phi(\beta_I)_P|}\,\Phi(\beta_I)_P\cdot\Psi\big(\Theta^{\otimes r}(\Phi(\beta_I)_Q)\otimes\beta_J\big),$$

where:

- $I \sqcup J = \{1,\dots,k_1+k_2-r\}$ is a shuffle of type (k_1, k_2-r), meaning $I = \{i_1 < \cdots < i_{k_1}\}$ and $J = \{j_1 < \cdots < j_{k_2-r}\}$ with $I\cap J = \emptyset$ and $I \cup J = \{1,\dots,k_1+k_2-r\}$. As defined above we have $\beta_I := \beta_{i_1}\cdots\beta_{i_{k_1}} \in \mathrm{Sym}^{k_1}(L_+[1])$ and $\beta_J := \beta_{j_1}\cdots\beta_{j_{k_2-r}} \in \mathrm{Sym}^{k_2-r}(L_+[1])$,
- $\Theta^{\otimes r}(\Phi(\beta_I)_Q) \in \mathrm{Sym}^{l_1-r}(L_+[1])$ is defined by applying $\Theta\colon L_- \to L_+[1]$ to each factor of $\Phi(\beta_I)_Q = \phi_{q_1}\cdots\phi_{q_{l_1-r}} \in \mathrm{Sym}^{l_1-r}(L_-)$:

$$\Theta^{\otimes r}(\Phi(\beta_I)_Q) := \Theta(\phi_{q_1})\cdots\Theta(\phi_{q_{l_1-r}}) \in \mathrm{Sym}^{l_1-r}(L_+[1]).$$

Definition 4.9 ([CT24, Section 4.2])**.** The Lie bracket on $\widehat{\mathfrak{h}}_A$ is defined by

$$\{\Psi,\Phi\}_\hbar := (-1)^{|\Psi|}\sum_{r\geq 1}\big(\Psi\circ_r\Phi - (-1)^{|\Psi||\Phi|}\Phi\circ_r\Psi\big)\hbar^{r-1},$$

where the degrees $|\Phi|$ and $|\Psi|$ are computed in unshifted $\widehat{\mathfrak{h}}_A$.

Theorem 4.10 ([CT24, Theorem 4.3])**.** The triple $(\widehat{\mathfrak{h}}_A,\ \eth + \iota + \hbar\Delta,\ \{-,-\}_\hbar)$ is a dg Lie algebra.

Proof. To show $(\widehat{\mathfrak{h}}_A, D_2, \{-,-\}_\hbar)$ we must verify that $\{-,-\}_\hbar$ is graded antisymmetric, satisfies the Jacobi identity, together with the fact that $D_2^2 = 0$ and D_2 is a derivation of $\{-,-\}_\hbar$. The property $D_2^2 = 0$ is the content of Lemma 4.8. The remaining properties are proved in [CT24, Theorem 4.3]. □

4.4 Relation to the totalisation

Exercise 4.11. Verify the following chain of isomorphisms. In characteristic zero, the quotient map $\pi\colon V^{\otimes k} \to \mathrm{Sym}^k(V)$ should induce an isomorphism

$$\mathrm{Hom}(\mathrm{Sym}^k(V), W) \xrightarrow{\sim} \mathrm{Hom}(V^{\otimes k}, W)^{\Sigma_k}, \qquad f \mapsto f\circ\pi,$$

and applying this we get

$$\mathrm{Hom}(\mathrm{Sym}^k(V), \mathrm{Sym}^l(W)) \cong \mathrm{Hom}(V^{\otimes k}, W^{\otimes l})^{\Sigma_k \times \Sigma_l}.$$

Setting $V = L_+[1]$ and $W = L_-$, this would give

$$\mathrm{Hom}^c\big(\mathrm{Sym}^k(L_+[1]),\, \mathrm{Sym}^l(L_-)\big) \cong \mathrm{Hom}^c\big(L_+[1]^{\otimes k},\, L_-^{\otimes l}\big)^{\Sigma_k \times \Sigma_l},$$

and therefore, setting $M^c_{k,l} := \mathrm{Hom}^c(L_+[1]^{\otimes k}, L_-^{\otimes l})$,

$$\widehat{\mathfrak{h}}_A \cong \bigoplus_{k \geq 1,\, l \geq 0} (M^c_{k,l})^{\Sigma_k \times \Sigma_l}[[\hbar, \lambda]] = \mathrm{Tot}\big(\{M^c_{k,l}\}_{k,l}\big) \otimes \mathbf{k}[[\hbar, \lambda]].$$

Exercise 4.12. Do we actually have $M^c_{k,l} = M_{k,l} := \mathrm{Hom}(L_+[1]^{\otimes k}, L_-^{\otimes l})$?

Exercise 4.13. Is is true that the Lie bracket $\{-,-\}_\hbar$ on $\widehat{\mathfrak{h}}_A$ is the totalisation bracket of Corollary 3.4?

A Technical proofs

There is a morphism of operads from the Lie operad to the associative operad, given by antisymmetrizing the product. In other words, any associative algebra is a Lie algebra for the commutator bracket.

Lemma A.1. For any dg wheeled PROP P, the commutator

$$[\mu, \nu] := \mu \bullet \nu - (-1)^{|\mu||\nu|} \nu \bullet \mu$$

defines a dg Lie bracket on $\mathrm{Tot}(P)$. In particular, $(\mathrm{Tot}(P), d, [-,-])$ is a dg Lie algebra.

Proof. Since $(\mathrm{Tot}(P), \bullet)$ is a dg associative algebra by Theorem 3.2, we verify the three dg Lie algebra axioms.

Graded antisymmetry We want to show that $[\mu, \nu] = -(-1)^{|\mu||\nu|}[\nu, \mu]$. By definition,

$$[\mu, \nu] = \mu \bullet \nu - (-1)^{|\mu||\nu|} \nu \bullet \mu$$

and

$$[\nu, \mu] = \nu \bullet \mu - (-1)^{|\nu||\mu|} \mu \bullet \nu = \nu \bullet \mu - (-1)^{|\mu||\nu|} \mu \bullet \nu.$$

Therefore

$$\begin{aligned}
[\mu, \nu] + (-1)^{|\mu||\nu|}[\nu, \mu] &= \mu \bullet \nu - (-1)^{|\mu||\nu|} \nu \bullet \mu + (-1)^{|\mu||\nu|}\big(\nu \bullet \mu - (-1)^{|\mu||\nu|} \mu \bullet \nu\big) \\
&= \mu \bullet \nu - (-1)^{|\mu||\nu|} \nu \bullet \mu + (-1)^{|\mu||\nu|} \nu \bullet \mu - (-1)^{2|\mu||\nu|} \mu \bullet \nu \\
&= \mu \bullet \nu - \mu \bullet \nu = 0,
\end{aligned}$$

where in the last step we used $(-1)^{2|\mu||\nu|} = 1$.

Graded Jacobi identity We want to show that

$$[\mu, [\nu, \rho]] = [[\mu, \nu], \rho] + (-1)^{|\mu||\nu|}[\nu, [\mu, \rho]].$$

We expand each of the three terms explicitly and compare. Using $[\nu,\rho]=\nu\bullet\rho-(-1)^{|\nu||\rho|}\rho\bullet\nu$:

$$\begin{aligned}[\mu,[\nu,\rho]] &= \mu\bullet[\nu,\rho]-(-1)^{|\mu|(|\nu|+|\rho|)}[\nu,\rho]\bullet\mu\\ &= \mu\bullet\nu\bullet\rho-(-1)^{|\nu||\rho|}\mu\bullet\rho\bullet\nu\\ &\quad -(-1)^{|\mu|(|\nu|+|\rho|)}\nu\bullet\rho\bullet\mu+(-1)^{|\mu|(|\nu|+|\rho|)+|\nu||\rho|}\rho\bullet\nu\bullet\mu.\end{aligned}$$

Next, we note that $[\mu,\nu]=\mu\bullet\nu-(-1)^{|\mu||\nu|}\nu\bullet\mu$:

$$\begin{aligned}[[\mu,\nu],\rho] &= [\mu,\nu]\bullet\rho-(-1)^{(|\mu|+|\nu|)|\rho|}\rho\bullet[\mu,\nu]\\ &= \mu\bullet\nu\bullet\rho-(-1)^{|\mu||\nu|}\nu\bullet\mu\bullet\rho\\ &\quad -(-1)^{(|\mu|+|\nu|)|\rho|}\rho\bullet\mu\bullet\nu+(-1)^{|\mu||\nu|+(|\mu|+|\nu|)|\rho|}\rho\bullet\nu\bullet\mu.\end{aligned}$$

Finally, we know that $[\mu,\rho]=\mu\bullet\rho-(-1)^{|\mu||\rho|}\rho\bullet\mu$:

$$\begin{aligned}(-1)^{|\mu||\nu|}[\nu,[\mu,\rho]] &= (-1)^{|\mu||\nu|}\nu\bullet[\mu,\rho]-(-1)^{|\mu||\nu|+|\nu|(|\mu|+|\rho|)}[\mu,\rho]\bullet\nu\\ &= (-1)^{|\mu||\nu|}\nu\bullet\mu\bullet\rho-(-1)^{|\mu||\nu|+|\mu||\rho|}\nu\bullet\rho\bullet\mu\\ &\quad -(-1)^{|\mu||\nu|+|\nu|(|\mu|+|\rho|)}\mu\bullet\rho\bullet\nu+(-1)^{|\mu||\nu|+|\mu||\rho|+|\nu|(|\mu|+|\rho|)}\rho\bullet\mu\bullet\nu.\end{aligned}$$

Compatibility with the differential We want to show the graded Leibniz rule $d[\mu,\nu]=[d\mu,\nu]+(-1)^{|\mu|}[\mu,d\nu]$. We start from

$$d[\mu,\nu]=d\big(\mu\bullet\nu-(-1)^{|\mu||\nu|}\nu\bullet\mu\big)=d(\mu\bullet\nu)-(-1)^{|\mu||\nu|}d(\nu\bullet\mu).$$

Using the Leibniz rule for $\bullet$ given by Theorem 3.2:

$$\begin{aligned}d(\mu\bullet\nu)&=(d\mu)\bullet\nu+(-1)^{|\mu|}\mu\bullet(d\nu),\\ d(\nu\bullet\mu)&=(d\nu)\bullet\mu+(-1)^{|\nu|}\nu\bullet(d\mu).\end{aligned}$$

We substitute to get:

$$\begin{aligned}d[\mu,\nu] &= (d\mu)\bullet\nu+(-1)^{|\mu|}\mu\bullet(d\nu)-(-1)^{|\mu||\nu|}\big[(d\nu)\bullet\mu+(-1)^{|\nu|}\nu\bullet(d\mu)\big]\\ &= \big[(d\mu)\bullet\nu-(-1)^{|\mu||\nu|+|\nu|}\nu\bullet(d\mu)\big]+(-1)^{|\mu|}\big[\mu\bullet(d\nu)-(-1)^{|\mu||\nu|}(d\nu)\bullet\mu\big].\end{aligned}$$

Note that $|d\mu|=|\mu|+1$, so the definition of $[d\mu,\nu]$ gives

$$[d\mu,\nu]=(d\mu)\bullet\nu-(-1)^{|d\mu||\nu|}\nu\bullet(d\mu)=(d\mu)\bullet\nu-(-1)^{(|\mu|+1)|\nu|}\nu\bullet(d\mu)=(d\mu)\bullet\nu-(-1)^{|\mu||\nu|+|\nu|}\nu\bullet(d\mu).$$

In addition, $|d\nu|=|\nu|+1$, so

$$[\mu,d\nu]=\mu\bullet(d\nu)-(-1)^{|\mu||d\nu|}(d\nu)\bullet\mu=\mu\bullet(d\nu)-(-1)^{|\mu|(|\nu|+1)}(d\nu)\bullet\mu=\mu\bullet(d\nu)-(-1)^{|\mu||\nu|+|\mu|}(d\nu)\bullet\mu.$$

Therefore

$$d[\mu,\nu]=[d\mu,\nu]+(-1)^{|\mu|}[\mu,d\nu],$$

which is the graded Leibniz rule. □

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