Steenrod operations via higher Bruhat orders

Guillaume Laplante-Anfossi, j/w Nicholas J. Williams

The University of Melbourne

NMSU Geometry & Topology Seminar - December 1, 2023

Table of contents

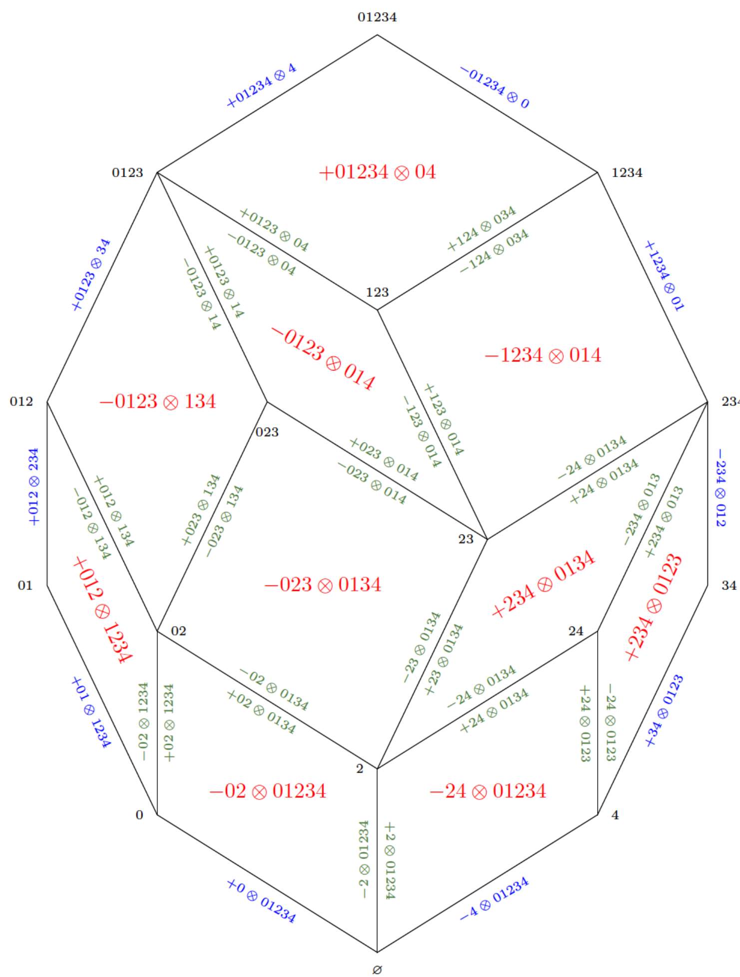
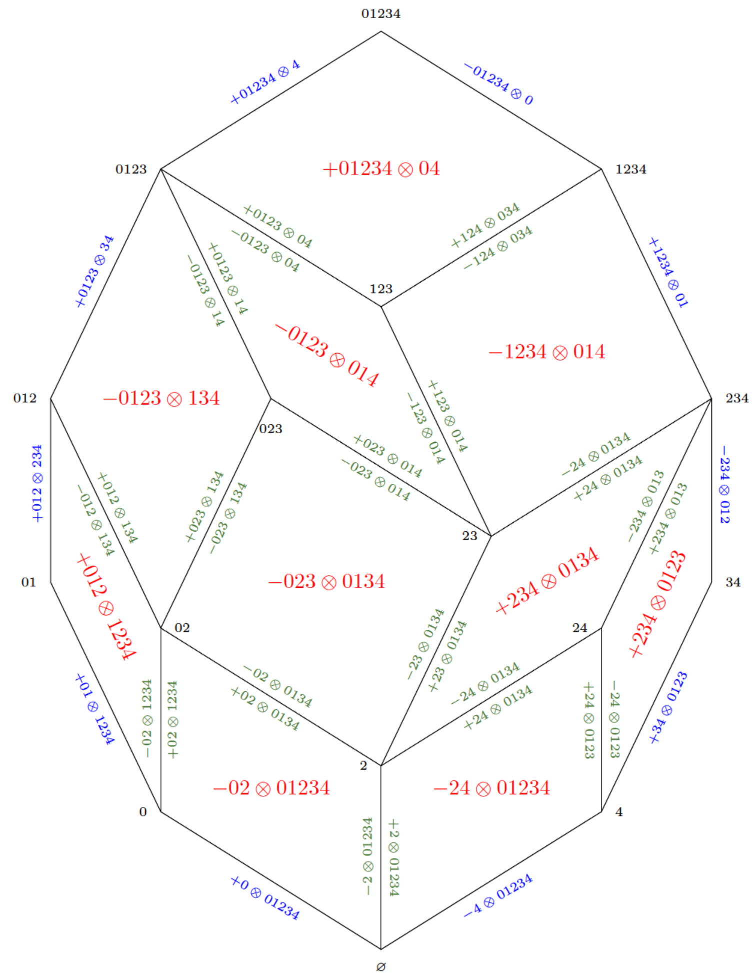


Table of contents



Cellular diagonals

Consider the standard simplex $\mathbb{A}^n$ in $\mathbb{R}^{n+1}$. The diagonal

$$\begin{array}{rccc} \boldsymbol{\Delta} : & \mathbb{A}^n & \rightarrow & \mathbb{A}^n \times \mathbb{A}^n \\ & x & \mapsto & (x, x) \end{array}$$

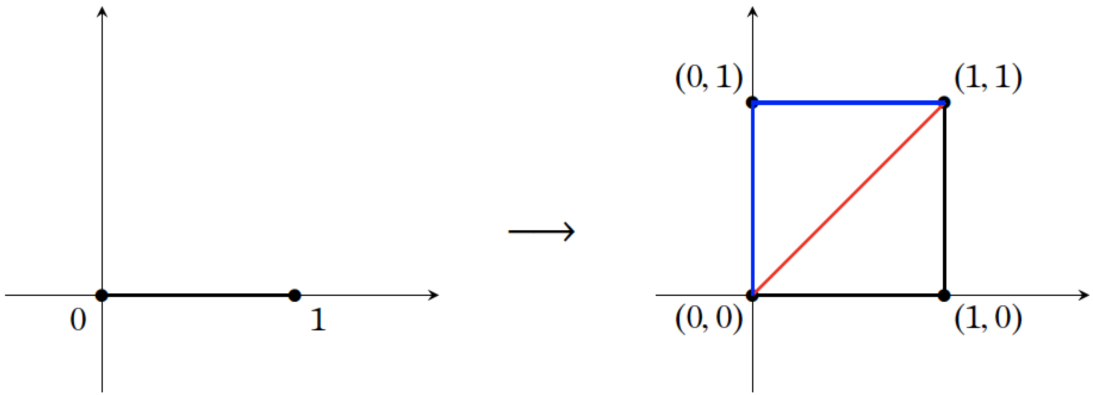
is *not* cellular.

Cellular diagonals

Consider the standard simplex $\mathbb{\Delta}^n$ in $\mathbb{R}^{n+1}$. The diagonal

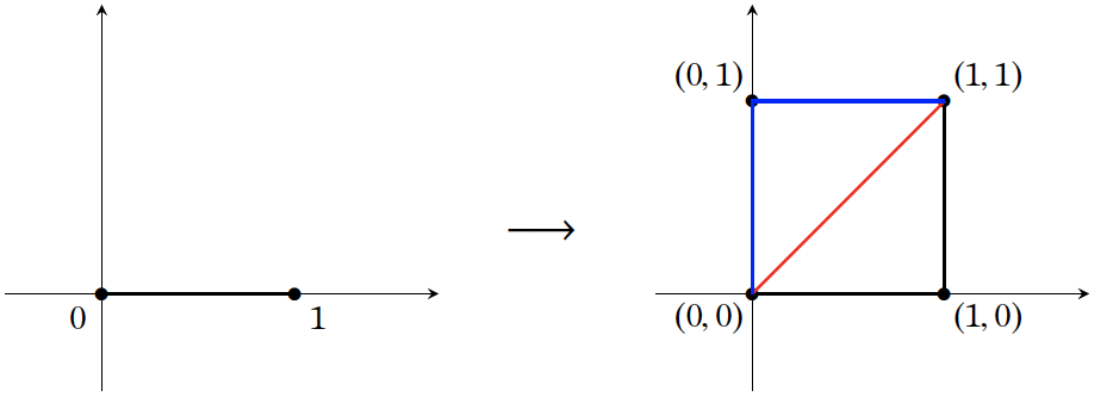
$$\begin{array}{cccc} \Delta & : & \mathbb{\Delta}^n & \to & \mathbb{\Delta}^n \times \mathbb{\Delta}^n \\ & & x & \mapsto & (x, x) \end{array}$$

is *not* cellular.



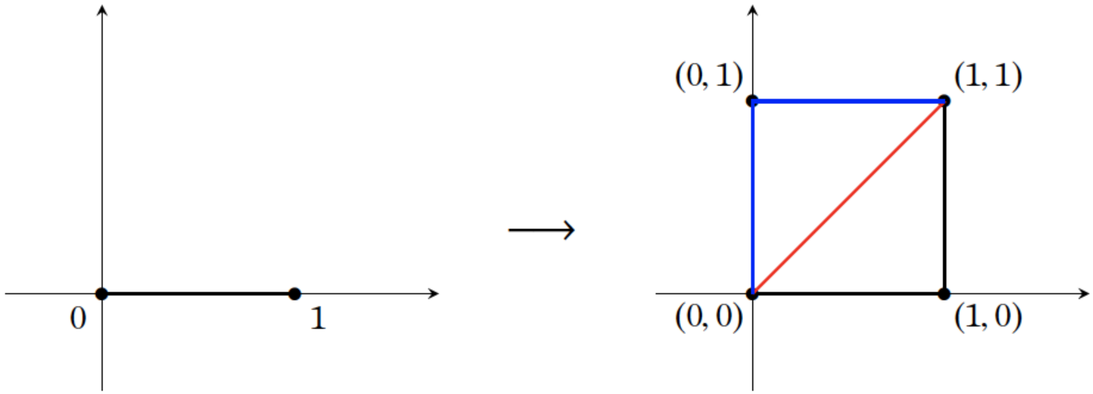
Cellular diagonals

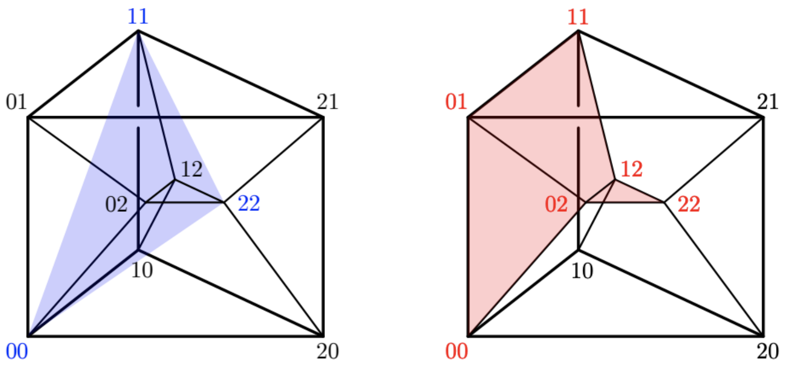
One is thus looking for a *cellular approximation*



Cellular diagonals

One is thus looking for a *cellular approximation* (blue above/red below)





Alexander–Whitney diagonal

One such approximation is given by the *Alexander–Whitney map*.

$$\Delta_0(01) \quad = \quad 0 \otimes 01 + 01 \otimes 1$$

Alexander–Whitney diagonal

One such approximation is given by the *Alexander–Whitney map*.

$$\Delta_0(01) \quad = \quad 0 \otimes 01 + 01 \otimes 1$$

However, choosing such a map one breaks the $\mathbb{S}_2$ symmetry of Δ.

$$T\Delta_0(01) \quad = \quad 01 \otimes 0 + 1 \otimes 01$$

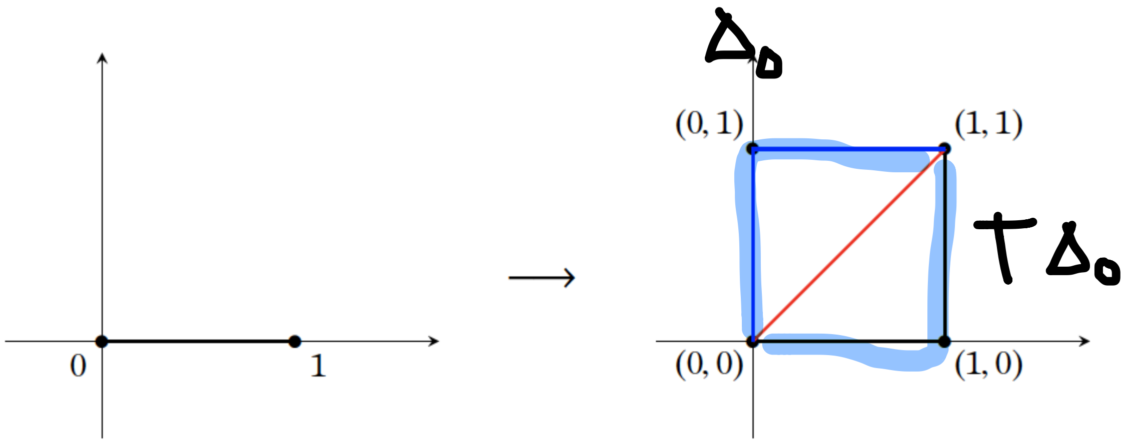
Alexander–Whitney diagonal

One such approximation is given by the *Alexander–Whitney map*.

$$\Delta_0(01) \quad = \quad 0 \otimes 01 + 01 \otimes 1$$

However, choosing such a map one breaks the $\mathbb{S}_2$ symmetry of Δ.

$$T\Delta_0(01) \quad = \quad 01 \otimes 0 + 1 \otimes 01$$



Steenrod squares

N. E. Steenrod resolves homotopically this symmetry break by a family of *cup-i coproducts*

$$\Delta_i \colon \mathrm{C}_\bullet(\mathbb{\Delta}^n) \to \mathrm{C}_\bullet(\mathbb{\Delta}^n) \otimes \mathrm{C}_\bullet(\mathbb{\Delta}^n)$$

Steenrod squares

N. E. Steenrod resolves homotopically this symmetry break by a family of *cup-i coproducts*

$$\Delta_i \colon \mathrm{C}_\bullet(\mathbb{\Delta}^n) \to \mathrm{C}_\bullet(\mathbb{\Delta}^n) \otimes \mathrm{C}_\bullet(\mathbb{\Delta}^n)$$

which satisfy the *homotopy formula*

$$\partial\Delta_i - (-1)^i \Delta_i \partial = (1 + (-1)^i T)\Delta_{i-1} \ .$$

That is, each Δ_i is an homotopy between Δ_{i-1} and $T\Delta_{i-1}$.

Steenrod squares

N. E. Steenrod resolves homotopically this symmetry break by a family of *cup-i coproducts*

$$\Delta_i\colon \mathrm{C}_\bullet(\mathbb{A}^n) \to \mathrm{C}_\bullet(\mathbb{A}^n) \otimes \mathrm{C}_\bullet(\mathbb{A}^n)$$

which satisfy the *homotopy formula*

$$\partial\Delta_i - (-1)^i \Delta_i \partial = (1 + (-1)^i T)\Delta_{i-1} \ .$$

That is, each Δ_i is an homotopy between Δ_{i-1} and $T\Delta_{i-1}$. These give rise to *Steenrod squares*

$$\mathrm{Sq}_i\colon H^p(X; \mathbb{Z}/2\mathbb{Z}) \to H^{2p-i}(X; \mathbb{Z}/2\mathbb{Z}).$$

Steenrod "square"

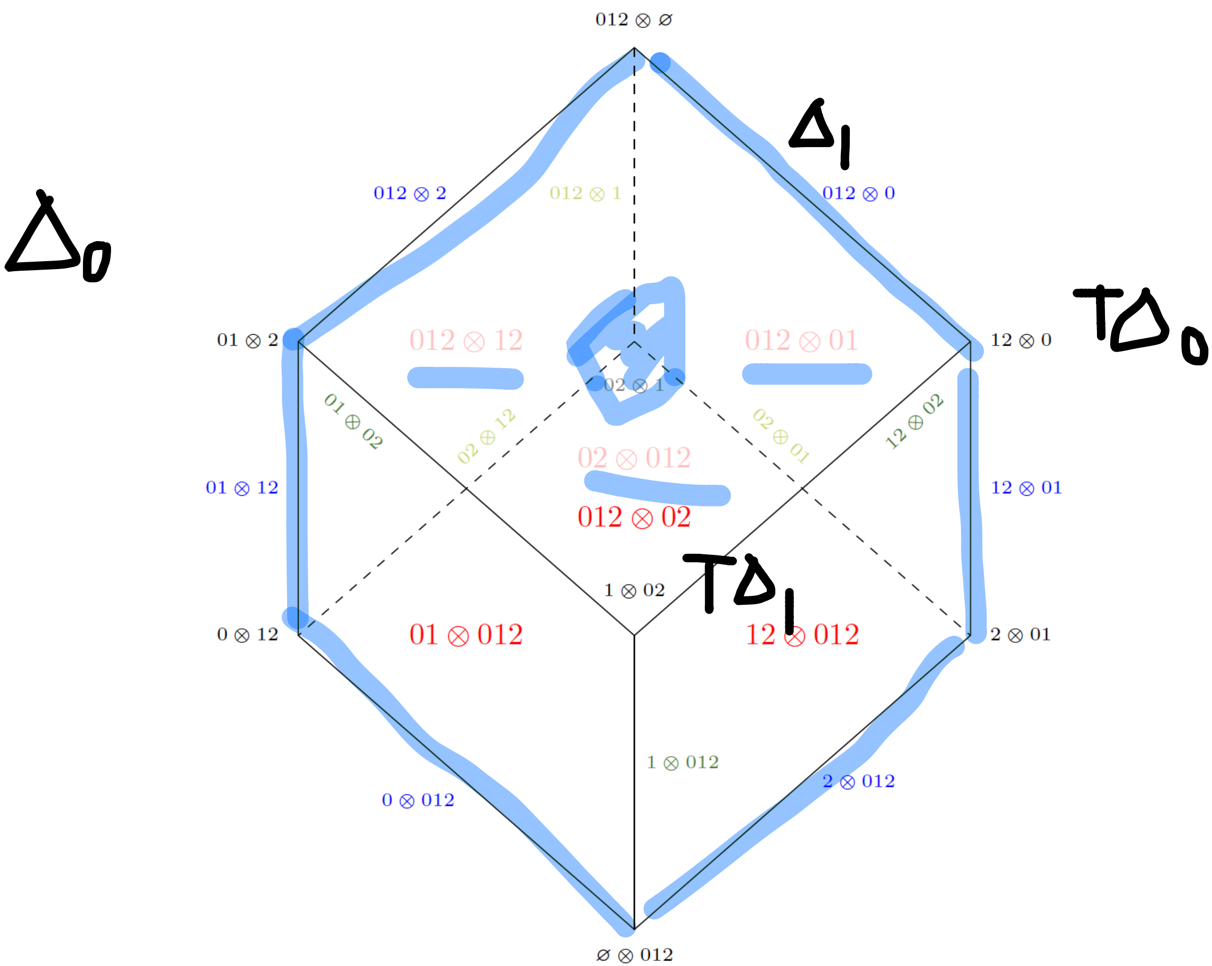
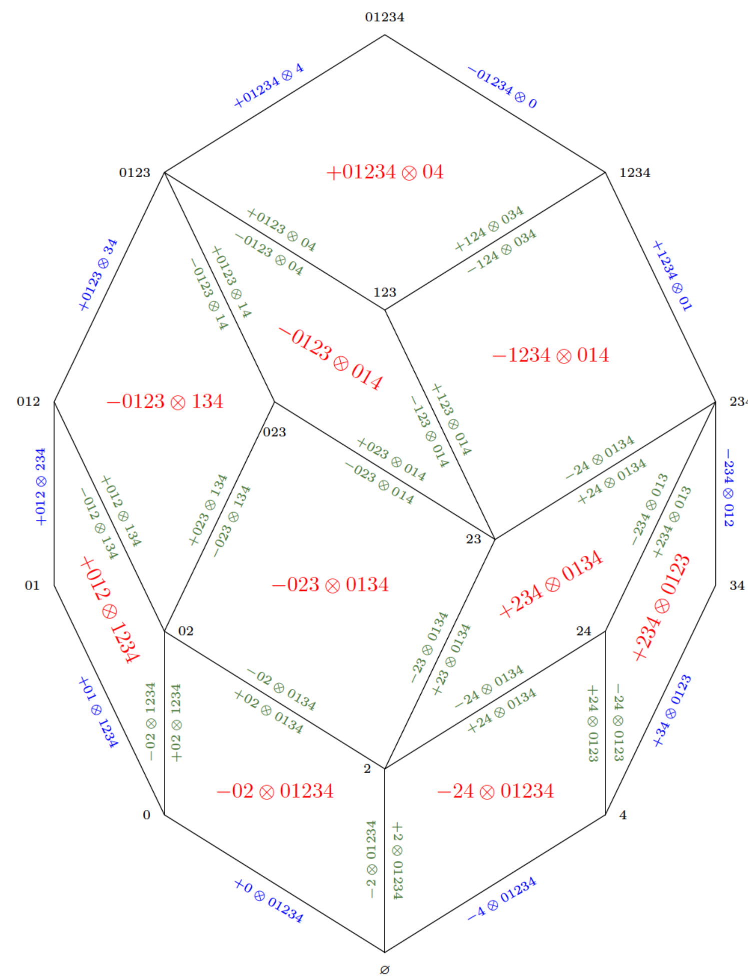


Table of contents



Packets

For $i+2 \leqslant n$, we write $\binom{[0,n]}{i+1} := \{S \subset [0,n] \mid |S| = i+1\}$.

Definition

The *packet* of $K = \{k_0 < k_2 < \cdots < k_{i+1}\} \in \binom{[0,n]}{i+2}$ is the set

$$P(K) := \{K \setminus k \mid k \in K\}$$

Packets

For $i+2 \leqslant n$, we write $\binom{[0,n]}{i+1} := \{S \subset [0,n] \mid |S| = i+1\}$.

Definition

The *packet* of $K = \{k_0 < k_2 < \cdots < k_{i+1}\} \in \binom{[0,n]}{i+2}$ is the set

$$P(K) := \{K \setminus k \mid k \in K\}$$

It is naturally ordered by the *lexicographic order*, where $K \setminus k_p < K \setminus k_q$ if and only if $p < q$, or its opposite, the *reverse lexicographic order*.

Packets

For $i+2 \leqslant n$, we write $\binom{[0,n]}{i+1} := \{S \subset [0,n] \mid |S| = i+1\}$.

Definition

The *packet* of $K = \{k_0 < k_2 < \cdots < k_{i+1}\} \in \binom{[0,n]}{i+2}$ is the set

$$P(K) := \{K \setminus k \mid k \in K\}$$

It is naturally ordered by the *lexicographic order*, where $K \setminus k_p < K \setminus k_q$ if and only if $p < q$, or its opposite, the *reverse lexicographic order*.

Example

The packet of 012 is $\{01, 02, 12\}$, lex is $01 < 02 < 12$.
The packet of 023 is $\{02, 03, 23\}$, rex is $23 < 03 < 02$.

Admissible orders

Definition

1. A total order α of $\binom{[0,n]}{i+1}$ is *admissible* if for all $K \in \binom{[0,n]}{i+2}$, the elements $P(K)$ appear in either lexicographic or reverse-lexicographic order under α.
2. Two orderings α and α' of $\binom{[0,n]}{i+1}$ are *equivalent* if they differ by a sequence of interchanges of pairs of adjacent elements that do not lie in a common packet.

Admissible orders

Definition

1. A total order α of $\binom{[0,n]}{i+1}$ is *admissible* if for all $K \in \binom{[0,n]}{i+2}$, the elements $P(K)$ appear in either lexicographic or reverse-lexicographic order under α.
2. Two orderings α and α' of $\binom{[0,n]}{i+1}$ are *equivalent* if they differ by a sequence of interchanges of pairs of adjacent elements that do not lie in a common packet.

Example

Consider the order $01 < 23 < 03 < 13 < 02 < 12$ of $\binom{[0,3]}{2}$.
It is admissible since $P(012)$, $P(013)$ in lex and $P(023)$, $P(123)$ in rex.
It is equivalent to $23 < 01 < 03 < 13 < 02 < 12$.

Higher Bruhat orders

Definition

The elements of the *higher Bruhat poset* $\mathcal{B}([0,n], i+1)$ are admissible orders of $\binom{[0,n]}{i+1}$, modulo equivalence.

Higher Bruhat orders

Definition

The elements of the *higher Bruhat poset* $\mathcal{B}([0,n], i+1)$ are admissible orders of $\binom{[0,n]}{i+1}$, modulo equivalence.

The *inversion set* $\mathrm{inv}(\alpha)$ of an admissible order α is the set of all $(i+2)$-subsets of $[0,n]$ whose packets appear in rex.

Higher Bruhat orders

Definition

The elements of the *higher Bruhat poset* $\mathcal{B}([0,n], i+1)$ are admissible orders of $\binom{[0,n]}{i+1}$, modulo equivalence.

The *inversion set* $\mathrm{inv}(\alpha)$ of an admissible order α is the set of all $(i+2)$-subsets of $[0,n]$ whose packets appear in rex.

Example

The inversion set of $\alpha = 01 < 23 < 03 < 13 < 02 < 12 \in \mathcal{B}([0,3],2)$ is $\mathrm{inv}(\alpha) = \{023, 123\}$.

Higher Bruhat orders

Definition

The elements of the *higher Bruhat poset* $\mathcal{B}([0,n], i+1)$ are admissible orders of $\binom{[0,n]}{i+1}$, modulo equivalence. The poset structure is generated by the covering relations given by $[\alpha] \lessdot [\alpha']$ if $\mathrm{inv}(\alpha') = \mathrm{inv}(\alpha) \cup \{K\}$ for $K \in \binom{[0,n]}{i+2} \setminus \mathrm{inv}(\alpha)$.

Higher Bruhat orders

Definition

The elements of the *higher Bruhat poset* $\mathcal{B}([0,n], i+1)$ are admissible orders of $\binom{[0,n]}{i+1}$, modulo equivalence. The poset structure is generated by the covering relations given by $[\alpha] \lessdot [\alpha']$ if $\mathrm{inv}(\alpha') = \mathrm{inv}(\alpha) \cup \{K\}$ for $K \in \binom{[0,n]}{i+2} \setminus \mathrm{inv}(\alpha)$.

Example

$[01 < 02 < 03 < 23 < 13 < 12] \lessdot [01 < 23 < 03 < 13 < 02 < 12]$

Higher Bruhat orders

Definition

The elements of the *higher Bruhat poset* $\mathcal{B}([0,n], i+1)$ are admissible orders of $\binom{[0,n]}{i+1}$, modulo equivalence. The poset structure is generated by the covering relations given by $[\alpha] \lessdot [\alpha']$ if $\mathrm{inv}(\alpha') = \mathrm{inv}(\alpha) \cup \{K\}$ for $K \in \binom{[0,n]}{i+2} \setminus \mathrm{inv}(\alpha)$.

Example

$[01 < 02 < 03 < 23 < 13 < 12] \lessdot [01 < 23 < 03 < 13 < 02 < 12]$

- Lex is unique minimum for $\leq$, and rex is unique maximum.

Properties

Theorem (Manin–Schechtman, 1989)

There is a bijection between elements of $\mathcal{B}([0, n], i + 2)$ *and equivalence classes of maximal chains in* $\mathcal{B}([0, n], i + 1)$.

Properties

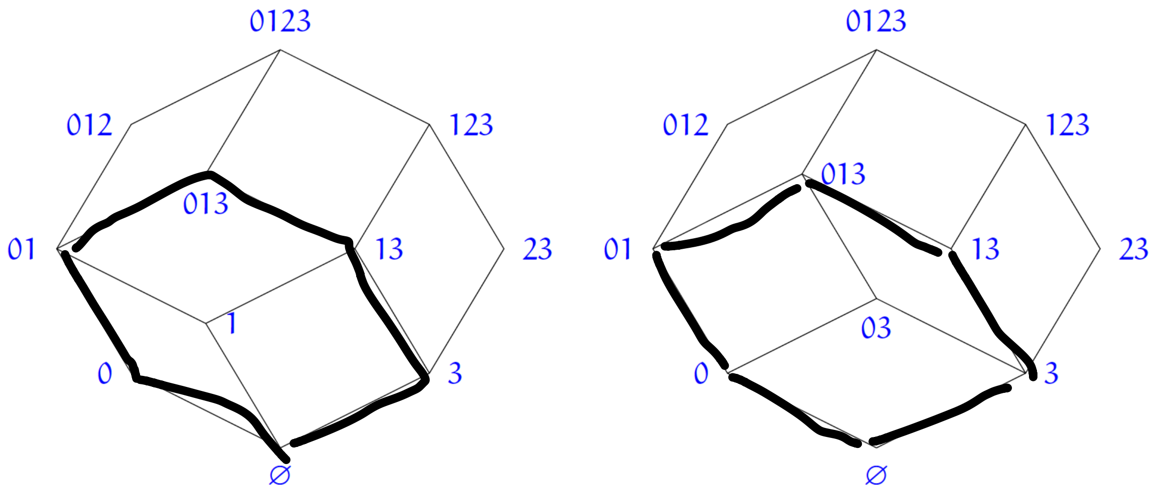
Theorem (Kapranov–Voevodsky, 1991; Thomas, 2002)

There is a bijection between elements of $\mathcal{B}([0,n], i+1)$ *and cubillages of the zonotope* $Z([0,n], i+1)$.

Properties

Theorem (Kapranov–Voevodsky, 1991; Thomas, 2002)

There is a bijection between elements of $\mathcal{B}([0, n], i + 1)$ *and cubillages of the zonotope* $Z([0, n], i + 1)$.

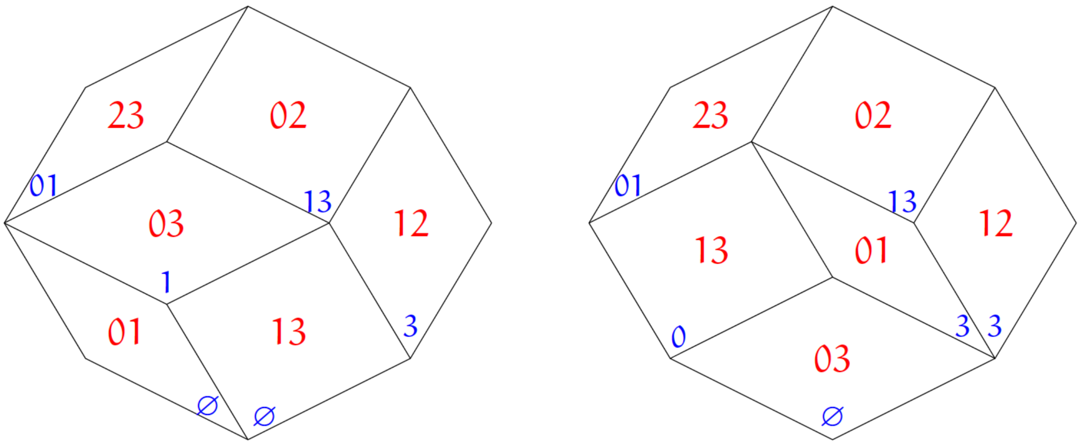


Cubillages of $Z([0, 3], 2)$ related by a flip.

Properties

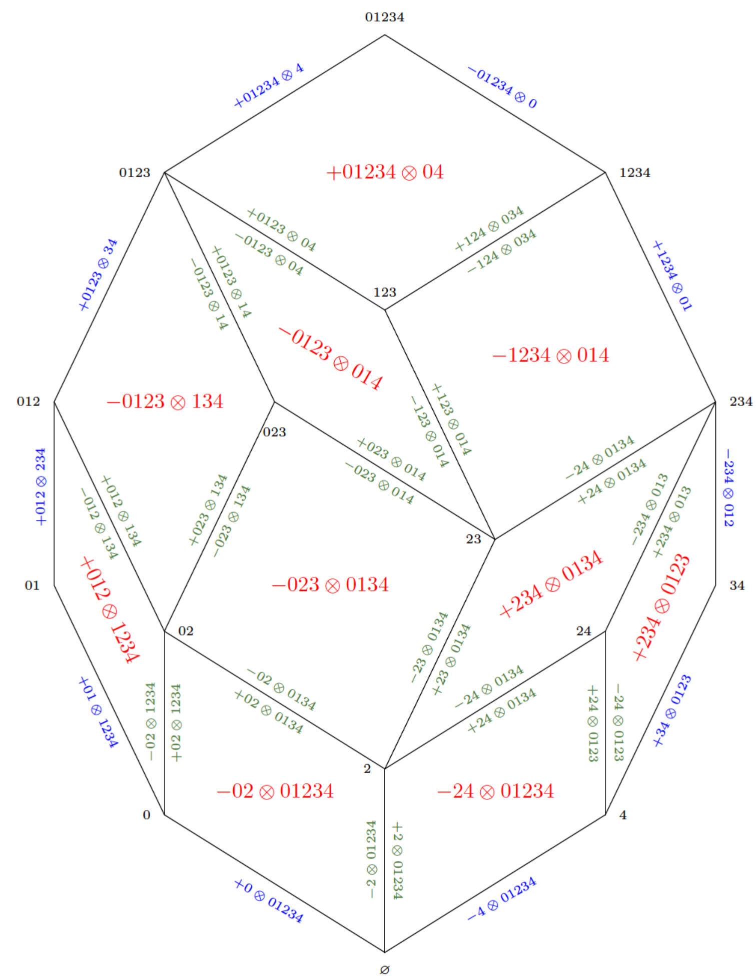
Theorem (Kapranov–Voevodsky, 1991; Thomas, 2002)

There is a bijection between elements of $\mathcal{B}([0, n], i + 1)$ *and cubillages of the zonotope* $Z([0, n], i + 1)$.



Same cubillages indexed by initial vertices and generating vectors.

Table of contents



Steenrod coproducts

An *overlapping partition* $\mathcal{L} = (L_0, L_1, \ldots, L_{i+1})$ of $[0, n]$ is a family of intervals $L_p = [l_p, l_{p+1}]$ such that $l_0 = 0$, $l_{i+2} = n$, and for each $0 < p < i + 1$ we have $l_p < l_{p+1}$.

012 ⟶ 01, 12

0123 012, 23

Steenrod coproducts

An *overlapping partition* $\mathcal{L} = (L_0, L_1, \ldots, L_{i+1})$ of $[0, n]$ is a family of intervals $L_p = [l_p, l_{p+1}]$ such that $l_0 = 0$, $l_{i+2} = n$, and for each $0 < p < i + 1$ we have $l_p < l_{p+1}$.

Definition

For $i \geqslant -1$, the *Steenrod cup-i coproduct* is the chain map $\Delta_i \colon \mathrm{C}_\bullet(\mathbb{\Delta}^n) \to \mathrm{C}_\bullet(\mathbb{\Delta}^n) \otimes \mathrm{C}_\bullet(\mathbb{\Delta}^n)$ defined by

$$\Delta_i([0, n]) := \sum_{\mathcal{L}} (-1)^{\varepsilon(\mathcal{L})} (L_0 \cup L_2 \cup \cdots) \otimes (L_1 \cup L_3 \cup \cdots) \; ,$$

where the sum is over all overlapping partitions of $[0, n]$ into $i + 2$ intervals.

Steenrod coproducts

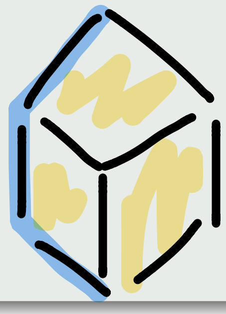
Example

For the 0-simplex Δ^0, we have $\Delta_0(0) = 0 \otimes 0$. For the 1-simplex Δ^1,

$$\Delta_0(01) = 0 \otimes 01 + 01 \otimes 1 \ ,$$
$$\Delta_1(01) = -01 \otimes 01 \ .$$

For the 2-simplex Δ^2, we have

$$\Delta_0(012) = 0 \otimes 012 + 01 \otimes 12 + 012 \otimes 2 \ ,$$
$$\Delta_1(012) = 012 \otimes 01 - 02 \otimes 012 + 012 \otimes 12 \ ,$$
$$\Delta_2(012) = 012 \otimes 012 \ .$$



Cubical subcomplex

The key observation is the following.

Cubical subcomplex

The key observation is the following. $[0,n]=S$

Proposition

There is a bijection between faces of $Z(S,|S|)$ excluding $\varnothing$ and S and basis elements of $C_\bullet(\Delta^n)\otimes C_\bullet(\Delta^n)$ which are supported on S.

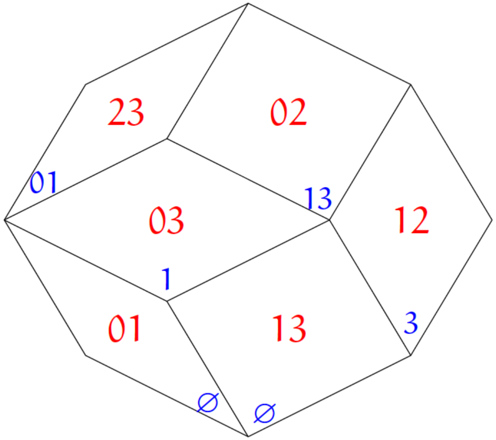
$A \otimes B$ $\quad$ $A \cup B = [0,n]$

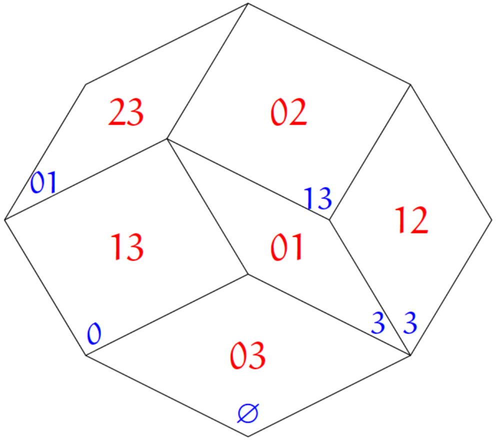
Cubical subcomplex

The key observation is the following.

Proposition

There is a bijection between faces of $Z(S, |S|)$ excluding $\varnothing$ and S and basis elements of $C_\bullet(\mathbb{\Delta}^n) \otimes C_\bullet(\mathbb{\Delta}^n)$ which are supported on S.





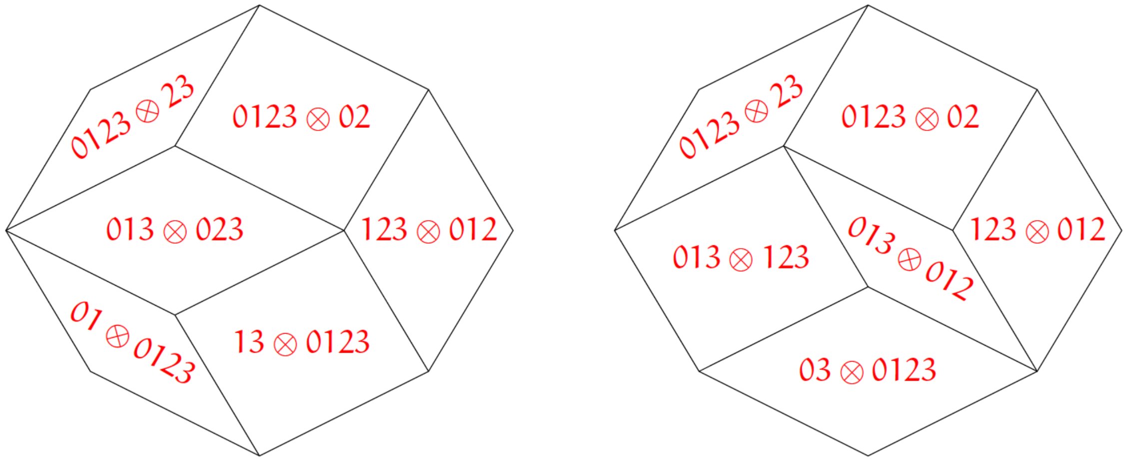
Two cubillages indexed by initial vertices and generating vectors.

Cubical subcomplex

The key observation is the following.

Proposition

There is a bijection between faces of $Z(S, |S|)$ excluding $\varnothing$ and S and basis elements of $C_\bullet(\mathbb{A}^n) \otimes C_\bullet(\mathbb{A}^n)$ which are supported on S.



The same cubillages indexed by basis elements of $C_\bullet(\mathbb{A}^3) \otimes C_\bullet(\mathbb{A}^3)$.

Main results

We are now in position to connect higher Bruhat orders and Steenrod coproducts.

Main results

We are now in position to connect higher Bruhat orders and Steenrod coproducts.

Theorem (L.-A.–Williams, 2023)

For every element $U = \mathrm{inv}(\alpha) \in \mathcal{B}([0,n], i+1)$, *there is a coproduct*

$$\Delta_i^U \colon \mathsf{C}_\bullet(\mathbb{\Delta}^n) \to \mathsf{C}_\bullet(\mathbb{\Delta}^n) \otimes \mathsf{C}_\bullet(\mathbb{\Delta}^n)$$

which gives a homotopy between Δ_{i-1} *and* $\Delta_{i-1}^{\mathrm{op}}$. *If* $U_{\min}$ *and* $U_{\max}$ *are the maximal and minimal elements of* $\mathcal{B}([0,n], i+1)$, *then* $\{\Delta_i^{U_{\min}}, \Delta_i^{U_{\max}}\} = \{\Delta_i, \Delta_i^{\mathrm{op}}\}$.

$\Delta_i^{op} = T\Delta_i$

Main results

We are now in position to connect higher Bruhat orders and Steenrod coproducts.

Theorem (L.-A.–Williams, 2023)

For every element $U = \mathrm{inv}(\alpha) \in \mathcal{B}([0,n], i+1)$, *there is a coproduct*

$$\Delta_i^U \colon \mathsf{C}_\bullet(\mathbb{\Delta}^n) \to \mathsf{C}_\bullet(\mathbb{\Delta}^n) \otimes \mathsf{C}_\bullet(\mathbb{\Delta}^n)$$

which gives a homotopy between Δ_{i-1} *and* $\Delta_{i-1}^{\mathrm{op}}$. *If* $U_{\min}$ *and* $U_{\max}$ *are the maximal and minimal elements of* $\mathcal{B}([0,n], i+1)$, *then* $\{\Delta_i^{U_{\min}}, \Delta_i^{U_{\max}}\} = \{\Delta_i, \Delta_i^{\mathrm{op}}\}$.

- Moreover, every coproduct on $\mathsf{C}_\bullet(\mathbb{\Delta}^n)$ giving a homotopy between Δ_{i-1} and $\Delta_{i-1}^{\mathrm{op}}$ arises in this way, so long as it does not contain redundant terms.

Main results

- It follows that from any covering relation $U \lessdot V$ in $\mathcal{B}([0,n], i+1)$, one can construct a chain homotopy between Δ_i^U and Δ_i^V.

Main results

- It follows that from any covering relation $U \lessdot V$ in $\mathcal{B}([0,n], i+1)$, one can construct a chain homotopy between Δ_i^U and Δ_i^V.

Theorem (L.-A.–Williams, 2023)

Any coproduct Δ_i^U defines a Steenrod square Sq_i^U in the cohomology of a simplicial complex, and for any two $U, V \in \mathcal{B}([0,n], i+1)$ we have $\mathrm{Sq}_i^U = \mathrm{Sq}_i^V$.

Main results

- It follows that from any covering relation $U \lessdot V$ in $\mathcal{B}([0,n], i+1)$, one can construct a chain homotopy between Δ_i^U and Δ_i^V.

Theorem (L.-A.–Williams, 2023)

Any coproduct Δ_i^U defines a Steenrod square Sq_i^U in the cohomology of a simplicial complex, and for any two $U, V \in \mathcal{B}([0,n], i+1)$ we have $\mathrm{Sq}_i^U = \mathrm{Sq}_i^V$.

We show that

- for cohomology of simplicial complexes non-trivial coproducts from different elements of the higher Bruhat orders exist,

Main results

- It follows that from any covering relation $U \lessdot V$ in $\mathcal{B}([0,n], i+1)$, one can construct a chain homotopy between Δ_i^U and Δ_i^V.

Theorem (L.-A.–Williams, 2023)

Any coproduct Δ_i^U defines a Steenrod square Sq_i^U in the cohomology of a simplicial complex, and for any two $U, V \in \mathcal{B}([0,n], i+1)$ we have $\mathrm{Sq}_i^U = \mathrm{Sq}_i^V$.

We show that

- for cohomology of simplicial complexes non-trivial coproducts from different elements of the higher Bruhat orders exist,
- whereas for singular cohomology only the Steenrod coproducts are possible.

Construction

For $U \in \mathcal{B}([0, n], i+1)$ and a set of generating vectors $L \in \binom{[0,n]}{i+1}$, we define $A_L^U \subset [0, n] \setminus L$, by asserting that $a \in [0, n] \setminus L$ is in A_L^U if and only if either

- $L \cup \{a\} \in U$ and a is an even gap, or
- $L \cup \{a\} \notin U$ and a i an odd gap.

Here, a is an even (odd) gap if $\#\{l \in L \mid a < l\}$ is even (odd).

Construction

For $U \in \mathcal{B}([0,n], i+1)$ and a set of generating vectors $L \in \binom{[0,n]}{i+1}$, we define $A_L^U \subset [0,n] \setminus L$, by asserting that $a \in [0,n] \setminus L$ is in A_L^U if and only if either

- $L \cup \{a\} \in U$ and a is an even gap, or
- $L \cup \{a\} \notin U$ and a i an odd gap.

Here, a is an even (odd) gap if $\#\{l \in L \mid a < l\}$ is even (odd).

Definition

We define the cup-i coproduct $\Delta_i^U \colon \mathrm{C}(\mathbb{A}^n) \to \mathrm{C}(\mathbb{A}^n) \otimes \mathrm{C}(\mathbb{A}^n)$ by the formula

$$\Delta_i^U([0,n]) := \sum_{L \in \binom{[0,n]}{i+1}} \pm L \cup A_L^U \otimes L \cup B_L^U,$$

where $B_L^U := [0,n] \setminus (L \cup A_L^U)$.

Proof of the homotopy formula

Proposition

For any $U \in \mathcal{B}([0,n], i+1)$, *we have*

$$\partial \Delta_i^U - (-1)^i \Delta_i^U \partial = (1 + (-1)^i T)\Delta_{i-1} .$$

Steenrod

Proof of the homotopy formula

Proposition

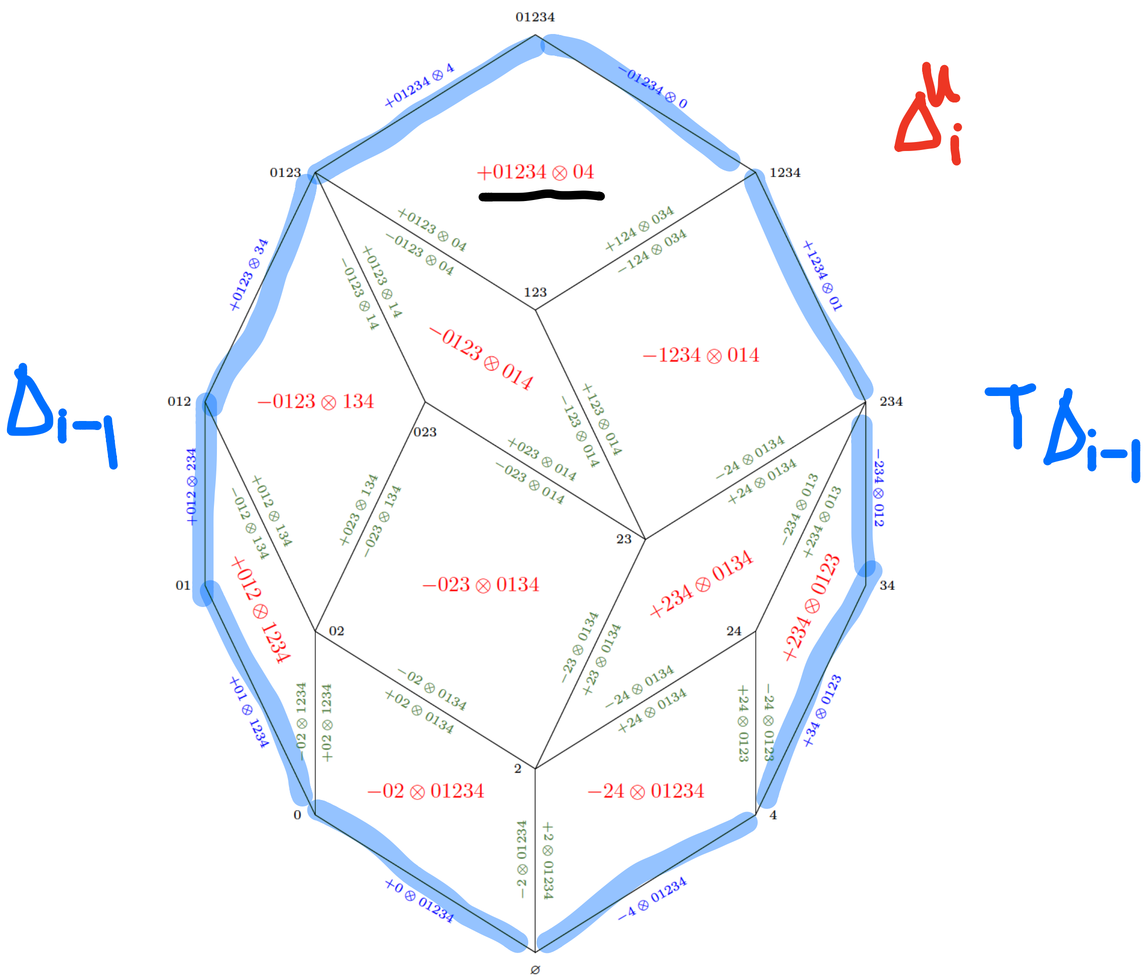
For any $U \in \mathcal{B}([0,n], i+1)$, *we have*

$$\partial\Delta_i^U - (-1)^i\Delta_i^U\partial = (1 + (-1)^i T)\Delta_{i-1} \ .$$

Proof.

When we expand $\partial\Delta_i^U$ we see that terms from shared facets of cubes lying inside $Z([0,n], i+1)$ cancel, and that we are left with terms of the form $F \setminus k$ and terms corresponding to Δ_i and $T\Delta_i$. The former terms cancel with those from $(-1)^i\Delta_i^U\partial$. □

Proof of the homotopy formula



Conclusion

Thank you for your attention!