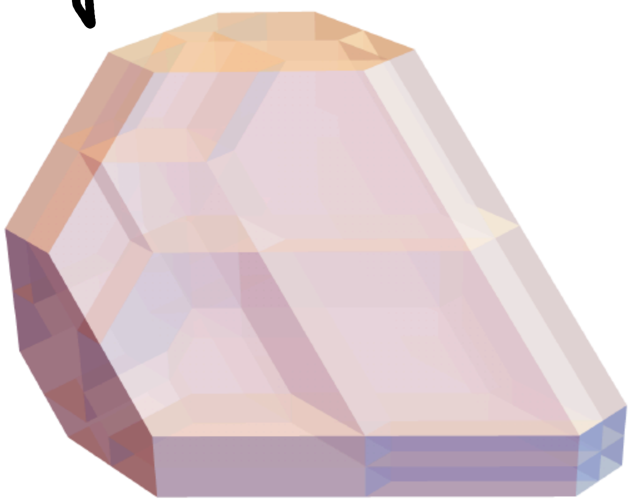
March 15th, 2023

56th STDC

The diagonal of the



multiplihedra

j.w. T. Mazuir

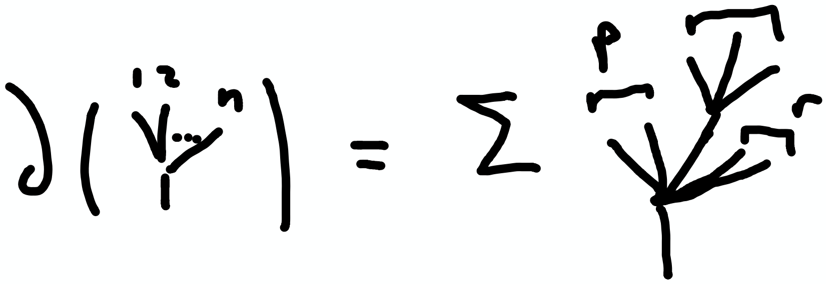
[1] Algebra

Definition 4.1 (A_∞-algebra). — An A_∞-*algebra* is the data of a dg module (A, ∂) together with operations

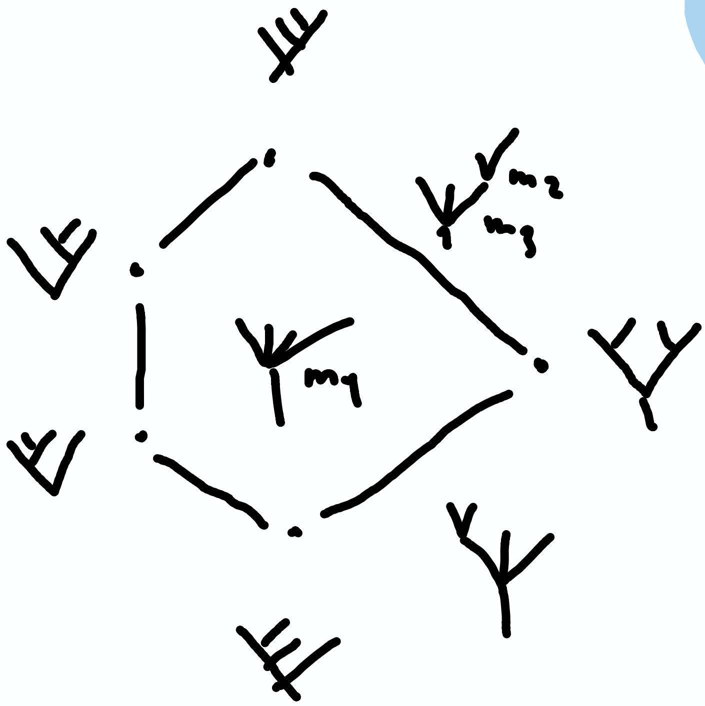
$$m_n : A^{\otimes n} \longrightarrow A, \quad n \geqslant 2$$

of degree $|m_n| = n-2$, satisfying the equations

$$[\partial, m_n] = - \sum_{\substack{p+q+r=n \\ 2 \leqslant q \leqslant n-1}} (-1)^{p+qr} m_{p+1+r}(\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r}), \quad n \geqslant 2.$$



"an algebra whose product m_2 is associative up to homotopy"



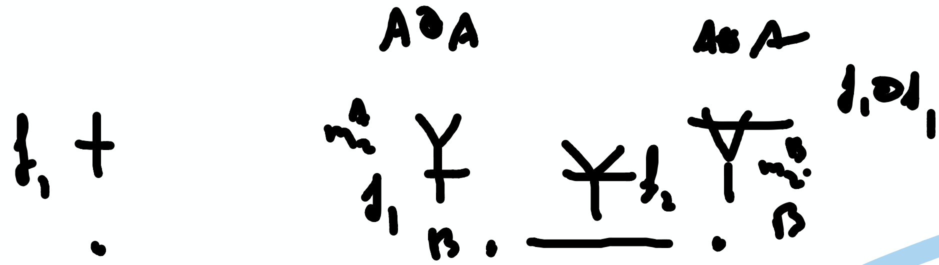
Definition 4.2 (A_∞-morphism). — An A_∞-*morphism* $F : A \rightsquigarrow B$ between two A_∞-algebras $(A, \{m_n\})$ and $(B, \{m'_n\})$ is a family of linear maps

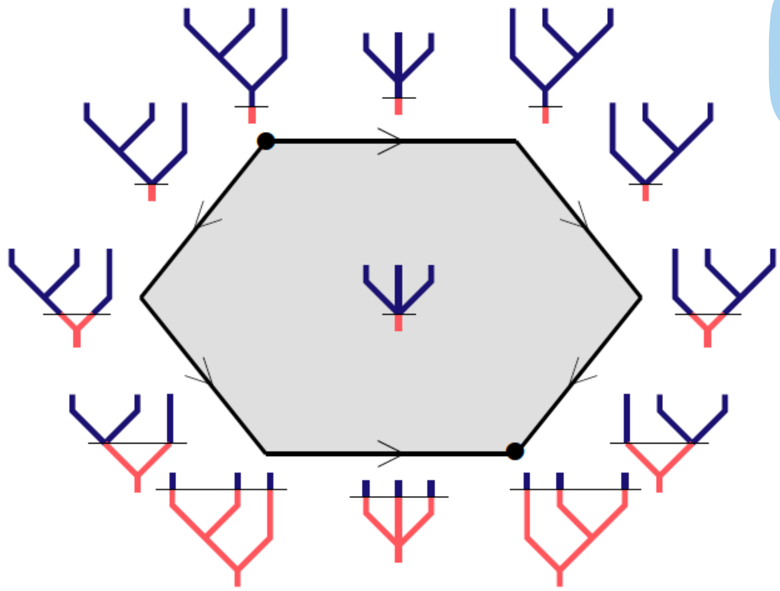
$$f_n : A^{\otimes n} \longrightarrow B, \ n \geqslant 1$$

of degree $|f_n| = n - 1$, satisfying the equations

$$[\partial, f_n] = \sum_{\substack{p+q+r=n \\ q\geqslant 2}} (-1)^{p+qr} f_{p+1+r}(\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r}) - \sum_{\substack{i_1+\cdots+i_k=n \\ k\geqslant 2}} (-1)^{\varepsilon} m'_k(f_{i_1} \otimes \cdots \otimes f_{i_k}),$$

for $n \geqslant 1$, where $\varepsilon = \sum_{u=1}^{k} (k-u)(1-i_u)$.





"a morphism which preserves homotopy associativity"

Rem: If $m_n = 0$, $n \geq 3$ ($f_n = 0$, $n \geq 3$) these are just As-alg, As-morphisms

$A \otimes B$ $\qquad m_1^{A \otimes B} = m_1^A \otimes id + id \otimes m_1^B$

$$m_2^{A \otimes B} = m_2^A \otimes m_2^B$$

similar for morphisms

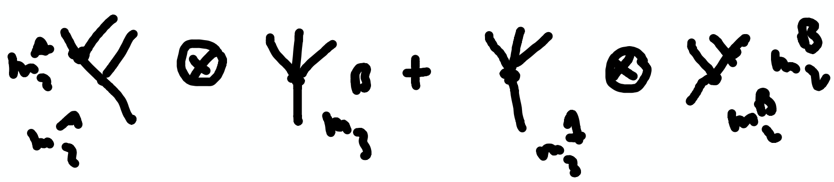
$\Rightarrow$ (As-alg, $\otimes$) monoidal category

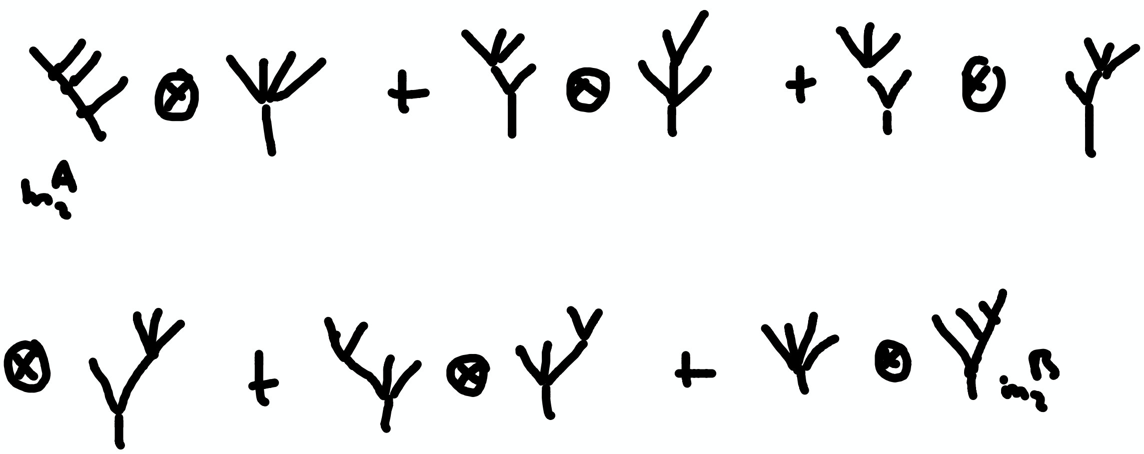
Q: Can this structure be lifted to A_∞-alg ?

First step: construct $\otimes$

(A, m_n^A)

(B, m_n^B)

$m_3^{A\otimes B} =$ 

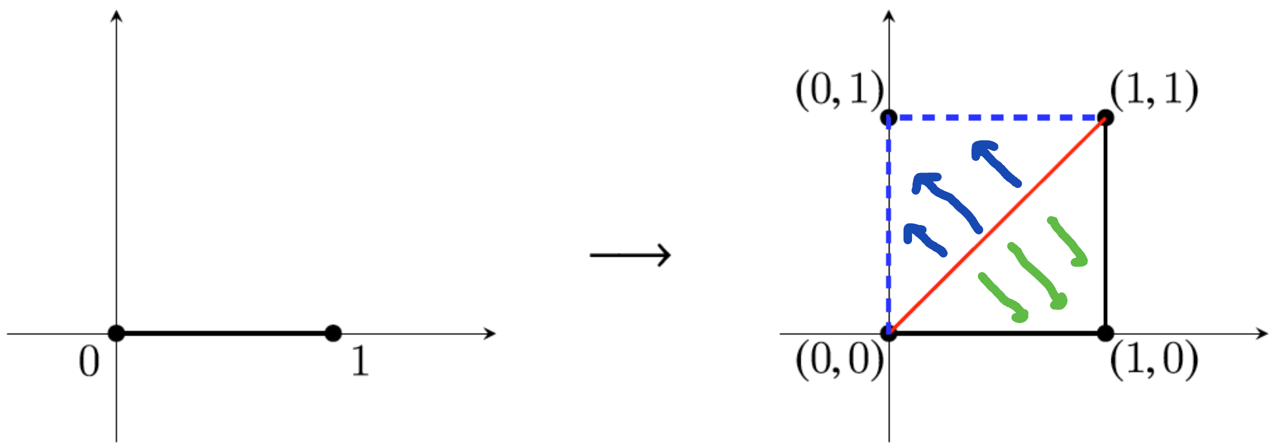
$m_4^{A\otimes B} =$ 

$m_5^{A\otimes B} = \dots$??

2 Geometry

Diagonal of a polytope $P \subset \mathbb{R}^n$

$$\begin{aligned} P &\longrightarrow P \times P \\ x &\longmapsto (x, x) \end{aligned}$$ is not cellular!



Definition 1.1. A *cellular diagonal* of a polytope P is a continuous map $P \to P \times P$ such that

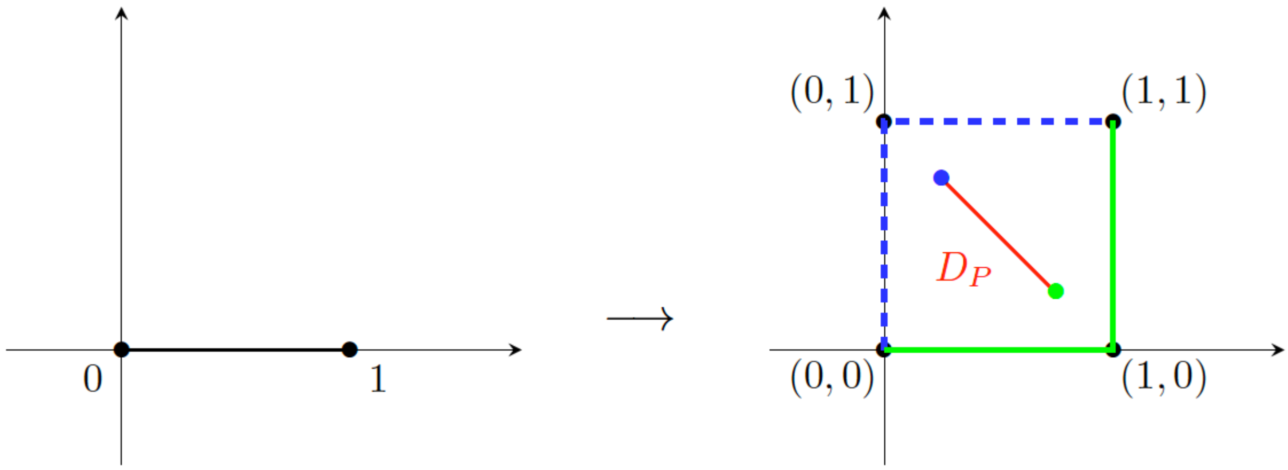
(1) its image is a union of $\dim P$-faces of $P \times P$ (i.e. it is *cellular*),
(2) it agrees with the thin diagonal on the vertices of P, and
(3) it is homotopic to the thin diagonal, relative to the image of the vertices.

A cellular diagonal is said to be *face-coherent* if its restriction to a face of P is itself a cellular diagonal for that face.

Universal construction (Billera-Sturmfels-Fulton)

Definition 9. The *diagonals polytope* D_P of a polytope P is the fiber polytope $\Sigma(P \times P, P)$ of the projection

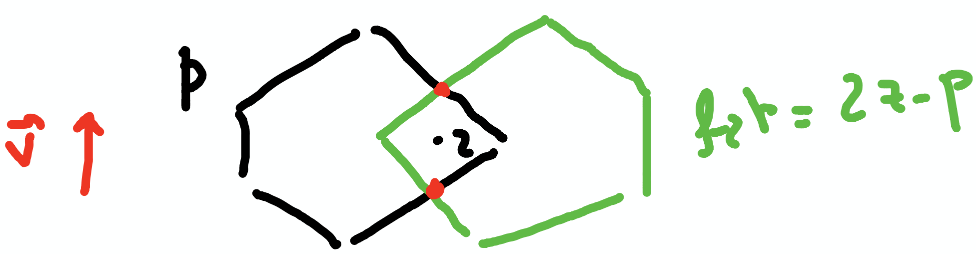
$$\begin{aligned} P \times P &\longrightarrow P \\ (x, y) &\longmapsto \tfrac{x+y}{2}. \end{aligned}$$



Thm (Masuda-Tonks-Thomas-Vallette)

Each vertex of D_P defines a cellular diagonal

Take-away: a vector $\vec{v}$ which is generic wrt to P defines a cellular diagonal!

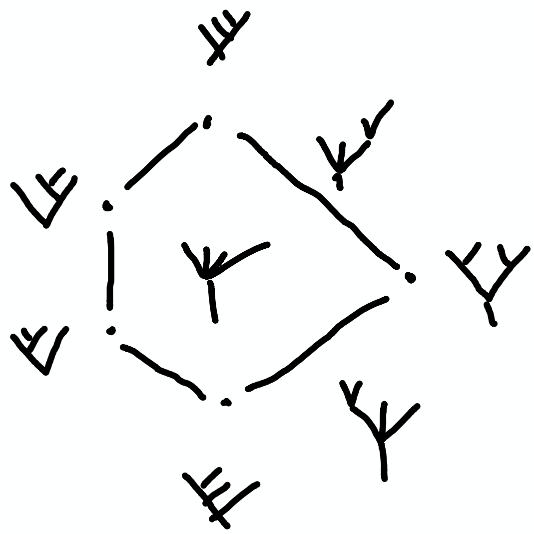


$$
\begin{array}{cccl}
\triangle_{(P,\vec{v})} & : & P & \to & P \times P \\
 & & z & \mapsto & \big(\min_{\vec{v}}(P \cap \rho_z P),\, \max_{\vec{v}}(P \cap \rho_z P)\big) \, .
\end{array}
$$

$\boxed{3}$ Geometry informs Algebra

A_∞-alg are algebras over an operad

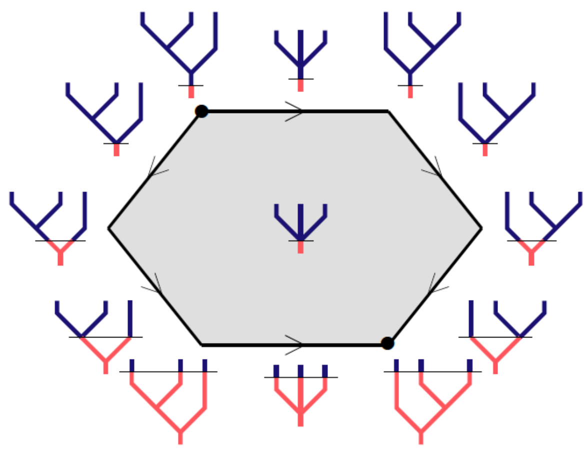
$$A_\infty(n) = C_\cdot^{cell}(K_n)$$



associahedron

A_∞-morphisms are encoded by an operadic bimod

$$M_\infty(n) = C_\cdot^{cell}(J_n)$$



multiplihedron

Prop:

Suppose that you have

1) cellular diagonals for K_n, J_n

2) top. cell. compatible operadic structures

Then, we have a universal $\otimes$

proof: $A_\infty \xrightarrow{\Delta} A_\infty \otimes A_\infty \xrightarrow{f \otimes g} \mathrm{End}_A \otimes \mathrm{End}_B \to \mathrm{End}_{A \otimes B}$ $\square$

<u>Thm</u> (LA-Mazuir)

- The family of vectors $\vec{v} = (v_1, \dots, v_{n+1}) \in \mathbb{R}^{n+1}$ satisfying $v_i \geq 2 v_{i+1}$ $1 \leq i \leq n$, $v_{n+1} > 0$ define

  1) cellular diagonals on K_n, J_n

  2) compatible operadic structures

  thus universal $\otimes$ products

- Moreover, they are given by an explicit universal combinatorial formula

$$\mathrm{Im}\, \Delta(K_n, \vec{v}) = \bigcup_{\max F \leq \min G} F \times G \qquad \mathrm{Im}\, \Delta(J_n, \vec{v}) = \bigcup_{\max F \leq \min G} F \times G$$

THEOREM 2. — *The cellular image of the diagonal map $\triangle_n : \mathrm{J}_n \to \mathrm{J}_n \times \mathrm{J}_n$ introduced in Definition 2.12 admits the following description. For $\mathcal{N}$ and $\mathcal{N}'$ two 2-colored nestings of the linear graph with n vertices, we have that*

$$\begin{aligned}(\mathcal{N}, \mathcal{N}') \in \operatorname{Im} \triangle_n \iff \; & \forall (I, J) \in D(n),\ \exists B \in B(\mathcal{N}),\ |B \cap I| > |B \cap J| \ \textit{or} \\ & \exists Q \in Q(\mathcal{N}),\ |(Q \cup \{n\}) \cap I| > |(Q \cup \{n\}) \cap J| \ \textit{or} \\ & \exists B' \in B(\mathcal{N}'),\ |B' \cap I| < |B' \cap J| \ \textit{or} \\ & \exists Q' \in Q(\mathcal{N}'),\ |(Q' \cup \{n\}) \cap I| < |(Q' \cup \{n\}) \cap J|.\end{aligned}$$

$$\triangle_2([\bullet\bullet]) = (\bullet\bullet) \times [\bullet\bullet] \cup [\bullet\bullet] \times (\bullet\bullet)$$

$$\begin{aligned}\triangle_3([\bullet\bullet\bullet]) = \; & ((\bullet\bullet)\bullet) \times [\bullet\bullet\bullet] \\ & \cup [\bullet\bullet\bullet] \times (\bullet(\bullet\bullet)) \cup (\bullet\bullet\bullet) \times [\bullet(\bullet\bullet)] \\ & \cup (\bullet\bullet\bullet) \times (\bullet[\bullet\bullet]) \cup [\bullet(\bullet\bullet)] \times (\bullet[\bullet\bullet]) \cup [(\bullet\bullet)\bullet] \times ([\bullet\bullet]\bullet) \\ & \cup [(\bullet\bullet)\bullet] \times (\bullet\bullet\bullet) \cup ([\bullet\bullet]\bullet) \times (\bullet\bullet\bullet)\end{aligned}$$

$$\begin{aligned}& \triangle_4([\bullet\bullet\bullet\bullet]) \\ & = (((\bullet\bullet)\bullet)\bullet) \times [\bullet\bullet\bullet\bullet] \cup [\bullet\bullet\bullet\bullet] \times (\bullet(\bullet(\bullet\bullet))) \cup ((\bullet\bullet\bullet)\bullet) \times [\bullet(\bullet\bullet)\bullet] \\ & \cup ([\bullet\bullet][\bullet\bullet]) \times (\bullet\bullet(\bullet\bullet)) \cup ((\bullet\bullet\bullet)\bullet) \times [\bullet(\bullet\bullet\bullet)] \cup ([\bullet\bullet]\bullet\bullet) \times (\bullet\bullet(\bullet\bullet)) \\ & \cup (\bullet(\bullet\bullet)\bullet) \times [\bullet(\bullet\bullet\bullet)] \cup ([\bullet\bullet\bullet]\bullet) \times (\bullet(\bullet\bullet)\bullet) \cup ((\bullet\bullet)\bullet\bullet) \times [\bullet\bullet(\bullet\bullet)] \\ & \cup ([\bullet\bullet\bullet]\bullet) \times (\bullet(\bullet\bullet\bullet)) \cup [((\bullet\bullet)\bullet)\bullet] \times ([\bullet\bullet\bullet]\bullet) \cup [\bullet\bullet(\bullet\bullet)] \times (\bullet(\bullet[\bullet\bullet])) \\ & \cup [(\bullet\bullet)(\bullet\bullet)] \times ([\bullet\bullet][\bullet\bullet]) \cup [\bullet(\bullet\bullet)\bullet] \times (\bullet([\bullet\bullet]\bullet)) \cup ((\bullet\bullet)\bullet\bullet) \times ([\bullet\bullet][\bullet\bullet]) \\ & \cup [\bullet(\bullet\bullet)\bullet] \times (\bullet(\bullet\bullet\bullet)) \cup [\bullet((\bullet\bullet)\bullet)] \times (\bullet[\bullet\bullet\bullet]) \cup [(\bullet\bullet)\bullet\bullet] \times ([\bullet\bullet](\bullet\bullet)) \\ & \cup (\bullet(\bullet\bullet)\bullet) \times (\bullet[\bullet\bullet\bullet]) \cup ((\bullet\bullet\bullet)\bullet) \times (\bullet[\bullet\bullet\bullet]) \cup [(\bullet\bullet)\bullet\bullet] \times (\bullet\bullet(\bullet\bullet)) \\ & \cup ([(\bullet\bullet)\bullet]\bullet) \times ([\bullet\bullet]\bullet\bullet) \cup [\bullet(\bullet\bullet\bullet)] \times (\bullet[\bullet(\bullet\bullet)]) \cup [((\bullet\bullet)\bullet)\bullet] \times ([\bullet\bullet]\bullet\bullet) \\ & \cup [\bullet(\bullet\bullet\bullet)] \times (\bullet(\bullet[\bullet\bullet])) \cup (\bullet[\bullet\bullet]\bullet) \times (\bullet(\bullet\bullet\bullet)) \cup (([\bullet\bullet]\bullet)\bullet) \times (\bullet\bullet\bullet\bullet) \\ & \cup (\bullet\bullet\bullet\bullet) \times [\bullet(\bullet(\bullet\bullet))] \cup ([(\bullet\bullet)\bullet]\bullet) \times (\bullet\bullet\bullet\bullet) \cup (\bullet\bullet\bullet\bullet) \times (\bullet[\bullet(\bullet\bullet)]) \\ & \cup [((\bullet\bullet)\bullet)\bullet] \times (\bullet\bullet\bullet\bullet) \cup (\bullet\bullet\bullet\bullet) \times (\bullet(\bullet[\bullet\bullet])) \cup ([\bullet\bullet](\bullet\bullet)) \times (\bullet\bullet[\bullet\bullet]) \\ & \cup [(\bullet\bullet\bullet)\bullet] \times ([\bullet(\bullet\bullet)]\bullet) \cup [(\bullet\bullet)(\bullet\bullet)] \times (\bullet\bullet[\bullet\bullet]) \cup [(\bullet\bullet\bullet)\bullet] \times (\bullet([\bullet\bullet]\bullet)) \\ & \cup [(\bullet\bullet\bullet)\bullet] \times (\bullet(\bullet\bullet)\bullet) \cup ((\bullet\bullet)\bullet\bullet) \times (\bullet\bullet[\bullet\bullet]) \cup [(\bullet\bullet\bullet)\bullet] \times (\bullet(\bullet\bullet\bullet)) \\ & \cup ([\bullet(\bullet\bullet)]\bullet) \times (\bullet[\bullet\bullet]\bullet) \cup ((\bullet\bullet\bullet)\bullet) \times (\bullet[\bullet\bullet]\bullet) \cup [((\bullet\bullet)\bullet)\bullet] \times (\bullet[\bullet\bullet]\bullet)\end{aligned}$$

Pairs $(F, G) \in \operatorname{Im} \triangle_{(P,\vec{v})}$	Polytopes	0	1	2	3	4	5	6	[OEI22]
	Assoc.	1	2	6	22	91	408	1938	A000139
$\dim F + \dim G = \dim P$	Multipl.	1	2	8	42	254	1678	11790	to appear
	Permut.	1	2	8	50	432	4802	65536	A007334
	Assoc.	1	3	13	68	399	2530	16965	A000260
$\dim F = \dim G = 0$	Multipl.	1	3	17	122	992	8721	80920	to appear
	Permut.	1	3	17	149	1809	28399	550297	A213507

FIGURE 8. Number of pairs of faces in the cellular image of the diagonal of the associahedra, multiplihedra and permutahedra of dimension $0 \leqslant \dim P \leqslant 6$, induced by any good orientation vector.

New combinatorics!

Q: Is (∞-A_∞-alg, $\otimes$) monoidal?

Thm (Markl–Schnider, '06)

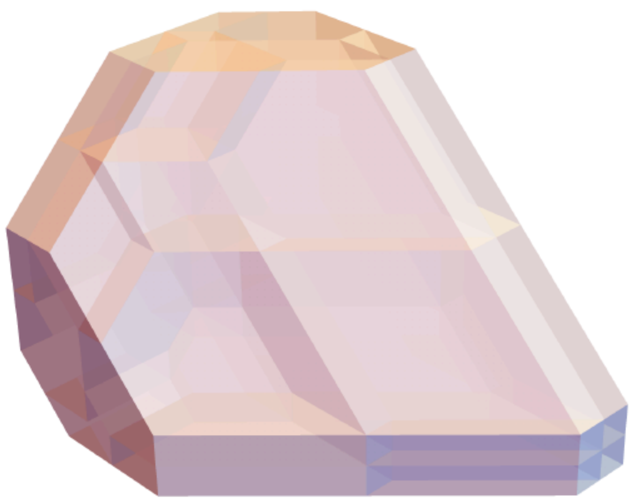
There is no coassociative diagonal for A_∞ ($\Rightarrow$ for M_∞)

Thm (LA-Mazuir)

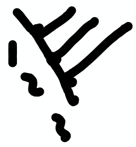
There is no diagonal on M_∞ which is compatible with composition of A_∞-morphisms

Thank you for

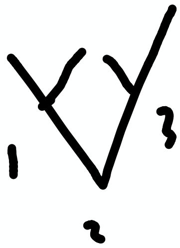
your attention!



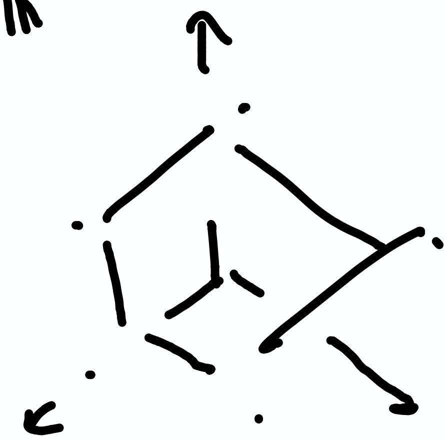
Loday's associahedron



$(1, 2, 3) \in \mathbb{R}^3$



$(1, 4, 1)$



Forcey-Loday multiplihedron