First L_∞-isomorphism

(1)

A cyclic unital A_∞ algebra fixed $\quad C_*^{Hoch}(A) = \bigoplus_{n \geq 1} A \otimes A^{\otimes n}$

$L = (C_*^{Hoch}(A)[d], d_{Hoch} \overset{=}{}b, B, \langle -,- \rangle_{Muk})$

$B : L \to L[-1]$ circle action $\leftarrow B$ self adjoint wrt to $\langle -,- \rangle_{Muk}$

Set $L^{Tate} := (L((u)), b + uB)$ $\quad$ b degree -1

$L_+ := (L[[u]], b + uB)$ $\quad$ B degree $+1$

$L_- := (L[u^{-1}], b + uB)$

$|u| = -2$

$h_A^+ = \bigoplus_{k \geq 1} \mathrm{Sym}^k L_-)[[\hbar, \lambda]][1]$ $\quad$ $a = a_0 | \ldots | a_n \in L$ $\rightsquigarrow |a| = \sum |a_i|_A + n$

$h_A = \mathbb{K}[[\hbar, \lambda]][1] \oplus h_A^+ = (\mathrm{Sym}^\bullet L_-)[[\hbar, \lambda]][1]$

$h_A^+ \xrightarrow{\sim} \widehat{h_A} = \bigoplus_{l \geq 0,\ k \geq 1} \mathrm{Hom}^{cont}(\mathrm{Sym}^k(L_+[1]), \mathrm{Sym}^l(L_-))[[\hbar, \lambda]]$

deg -1 deg -1 deg $|\hbar| + 1 \Rightarrow |\hbar| \overset{!}{=} -2$

h_A has differential $b + uB + \hbar\Delta$ & Lie bracket

$\{x, y\} = \Delta(xy) - (\Delta x)y - (-1)^{|x|} x(\Delta y)$ $\leftarrow$ almost derivation

$\Delta(xy) = \langle Bx_0, y_0 \rangle$ $\quad$ for x_0, y_0 the degree 0 part wrt to polynomial grading on L_-

BV operator of degree 1 def. by B.

parity$(a_0 + a_1 u^{-1} + \ldots + a_i u^{-i})$ $\equiv |a_0| + \ldots + |a_i| + n \mod 2$

$\rightsquigarrow$ now: replace $h_A, \widehat{h_A}$ with q-iso dglas whose differential does not involve Δ

$\mathbb{Z}$-graded

Take h_A^{triv} as same underlying $\mathbb{Z}/2$ graded vspc (?) with differential $b + uB$ & trivial $\{-,-\} = 0$.

Thm A splitting $L \to L_+$ of the nc-Hodge filtration provides $h_A \xrightarrow{\sim} h_A^{triv}$ L_∞-q-iso.

Splitting of Hodge filtration: (2)

$S : L \to L_+$, $S = id + S_1 u + S_2 u^2 + \dots$, $S_i \in \mathrm{End}(L)$

It is called symplectic if $\langle S\alpha, S\beta \rangle_{res} = \langle \alpha, \beta \rangle_{Muk}$

with $\langle \sum_k x_k u^k, \sum_\ell y_\ell u^\ell \rangle_{res} = \sum_{k,\ell} (-1)^\ell \langle x_k, x_\ell \rangle u^{k+\ell} \in k[[u]]$

& homologically sympl. if $H_*(S)$ is symplectic.

In particular, if symplectic, coeffic. in higher powers of u vanish:

$\sum_{j=0}^{N} (-1)^j \langle S_j \alpha, S_{N-j} \beta \rangle_{Muk} = 0 \quad \forall N \geq 1, \ \alpha, \beta \in L$

$\rightsquigarrow$ extend by u-linearity to iso $S : (L((s)), b) \xrightarrow{\sim} L^{Tate}$

Lemma If $s : H_*(L) \to H_*(L_+)$ is a splitting (by def assumed to be sympl.) it lifts to a splitting $S : L \to L_+$ which is homologically sympl.

$\rightsquigarrow$ For (chain level) splitting we can extend to quasi-iso $S : (L[[u]], b) \longrightarrow (L[[u]], b + uB)$

with inverse $R = id + R_1 u + R_2 u^2 + \dots$

$R \in \mathrm{End}(L)$, $\sum_{i+j=k} R_i S_j = \begin{cases} id & k=0 \\ 0 & k \geq 1 \end{cases}$

$\rightsquigarrow$ get $[b, S_n] = -B S_{n-1}$ $\quad [b, R_n] = R_{n-1} B$ by chain maps

recall $\Delta : h_A \to h_A$, $\Delta|_{\mathrm{Sym}^2 L_-} (x, y) = \langle B x_0, y_0 \rangle_{Muk}$
$=: \Omega(x, y)$

$\rightsquigarrow$ define homotopy trivialization of Ω to trivialize Δ

$H : L_- \otimes L_- \to K$, on $H_{ij} : u^{-i} L_- \otimes u^{-j} L_- \to K$

with $H_{ij}(u^{-i} x, u^{-j} y) = \langle (-1)^i \sum_{\ell=0}^{j} S_\ell R_{i+j+1-\ell} x, y \rangle$

on $L_- \otimes L_-$ $H(\alpha, \beta) = -\langle S \tau_{\geq 1} R(\alpha), \beta \rangle_{res}$

$\tau_{\leq 1} : L^{Tate} \to u L_+$ proj. to positive part

Exc $\forall \alpha, \beta \in L_-$ $H((b+uB)\alpha, \beta) + (-1)^{|\alpha|} H(\alpha, (b+uB)\beta)$
$= -\Omega(\alpha, \beta)$ (3)

ie H is a htpy trivialisation: $\Omega = [b+uB, H]$

Can symmetrize H to H^{sym} $Sym^2 L_- \to K$
$H^{sym}(x \cdot y) = \frac{1}{2}(H(x,y) + (-1)^{|x||y|} H(y,x))$

Ω is symmetric so $[b+uB, H^{sym}]$.

$\rightsquigarrow$ extend H^{sym} to second order diff op $\Delta^{H^{sym}}$ $Sym(L_-) \to Sym(L_-)$
which htpy to Δ as both are second order operators
& agree on $Sym^{\leq 2}(L_-)$.

Exc If S is a symplectic chain level splitting $\Rightarrow H = H^{sym}$

Lemma: If S is homologically symplectic chain level splitting
$\Rightarrow \exists$ odd operator $\delta : L_- \otimes L_- \longrightarrow K$ s.t.
$[b+uB, \delta] = H - H^{sym}$, ie H & H^{sym} are homotopic

Proof: Sanity check ✓ But not used in talk...

Constructing $K: h_A \to h_A^{tw}$

Observe $b + uB + \hbar t \Delta + d_{dR} + \hbar \Delta^{H^{sym}}$ is a differential
on $h_A \otimes \Omega^\bullet_{[0,1]}$ &
$t \cdot \{-,-\} + \{-,-\}^{H^{sym}}$ a Lie bracket
$\leftarrow$ failure of $\Delta^{H^{sym}}$ to be first order
diff operator on $Sym\, h_A$.

Elements in $h_A \otimes \Omega^\bullet_{[0,1]}$ are of the form
$$a \otimes f + a \otimes g dx \xmapsto{ev_t} a \otimes f(t) + a \otimes \int_0^t g(x) dx$$

$\rightsquigarrow h_A(t) := ev_{t*}(h_A \otimes \Omega^\bullet_{[0,1]})$

⇝ as a vectorspace : $h_A(t) \cong h_A(t')$ $\forall t, t'$ (4)

<u>Graph conventions</u>:

G / leaves , V_G vertices , L_G leaves

- $g : V_G \to \mathbb{Z}_{\geq 0}$ labels
- $g(G) = \sum_{v \in V} g(v) + \dim H_1(G)$ Euler genus
- $n(v)$ <u>valency</u>

[- If $2g(v) - 2 + n(v) > 0$ G is <u>stable</u>]

- for $\# V = m$ a <u>marking</u> is a bij $\{1, \dots, m\} \xrightarrow{\cong} V_G$

↳ iso of marked graphs preserve marking

- $\Gamma(g, n)$ iso classes of graphs of genus g with n leaves

[- $\Gamma((g, n))$ furthermore ensure they're stable]

- subscript $\Gamma((g, n))_m$ indicates $\# V_G = m$
- write $\overline{\Gamma((g, n))}_m$ for marked graphs

Want $K_m : \mathrm{Sym}^m (h_A[1]) \to h_A^{\mathrm{triv}}[1]$

⇝ amenable to K ↖ shift as L is shifted

$$K_m = \sum_{g, n} \sum_{(G, f) \in \overline{\Gamma(g, n)}_m} \frac{1}{|\mathrm{Aut}(G, f)|} \cdot K_{(G, f)} \cdot \hbar^{g - \sum_{i=1}^m g(f(i))}$$

is moved later & added here

⇝ $g_i = g(f(i))$, $n_i = n(f(i))$

$$K_{(G, f)} : \bigotimes_{i=1}^m \left(\mathrm{Sym}^{n_i} L_- \cdot \hbar^{g_i} \right) \to \mathrm{Sym}^n L_- \hbar^g$$

← desymmetrication

For $\underset{x_1 \dots x_j}{\gamma} \in \mathrm{Sym}^{n_i}(L_-)$ write $\tilde{\gamma} = \sum_{\sigma \in \Sigma_j} \varepsilon \, x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(j)} \in L^{\otimes j}$

⇝ half edges adjacent to v_i decorated by $\tilde{\gamma}_i$ sym, order doesn't matter

⇝ tensoring gives $\bigotimes_{i=1}^m \tilde{\gamma}_i \in L_-^{\otimes (\sum n_i)}$ ←

⇝ internal edges decorated by $H^{\mathrm{sym}} L^{\otimes 2} \to \mathbb{K}$

⇝ read of remaining comp (leaves) in any order, project $L_-^{\otimes n} \to \mathrm{Sym}^n L$

$\underline{\mathcal{U}_1} = \widetilde{\Gamma}(g,n)_1$ rank $H_1(G)$ = # loop edges
V & S are unique   (5)

1) For G s.t. $H_1(G)=0$:
G determined by leaves & $|\mathrm{Aut}(G)| = \#\text{leaves}! = n!$

⇝ G induces $\mathrm{Sym}^n L \longrightarrow \mathrm{Sym}^n L$

$$\begin{tikzcd} \mathrm{Sym}^n L \arrow[rr] \arrow[dr, "\text{antisym}"'] & & \mathrm{Sym}^n L \\ & L^{\otimes n} \arrow[ur, "\text{proj}"'] & \end{tikzcd}$$

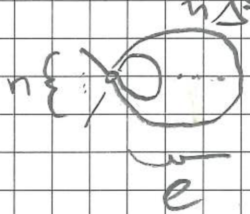
$x_1 \cdots x_n \longmapsto \sum_{\sigma\in S_n} \varepsilon(\sigma)\, x_{\sigma(1)}\otimes \cdots \otimes x_{\sigma(n)}$

$x_1\otimes \cdots \otimes x_n \longmapsto \sum_{\tau\in S_n} x_{\tau(1)} \cdots x_{\tau(n)}$ sym without $n!$

sym ∘ desym = id

⇒ proj ∘ antisym $= n! \cdot \mathrm{id}$ ⇒ star graphs action gets cancelled out by $\frac{1}{|\mathrm{Aut}\, G|}$ ⇒ only loop graphs contribute.

$\underline{H_1(G)>0}$: Same argument: looped graphs can treat leaves as fixed as contribution gets canceled out by $\frac{1}{|\mathrm{Aut}(G)|}$ ⇒ For graphs with ℓ loops & 1 vertex
$|\mathrm{Aut}\, G| = \ell!$



$\rightsquigarrow (\hbar\Delta^{sym})^\ell \Rightarrow \mathcal{U}_1 = \sum_{\ell\geq 0} \frac{1}{\ell!} (\hbar\Delta^{sym})^\ell$
$= \exp(\hbar\Delta^{sym})$

$\underline{\text{Exc}}$ $\mathcal{U}_1$ is a chain map: $(b+uB)\,\mathcal{U}_1 = \mathcal{U}_1(b+uB+\hbar\Delta)$

Use Theorem 8.2 of Fukaya: Cyclic symmetries & adic convergence in Lagrangian Floer theory:
$(h_A \hat{\otimes} \Omega^\bullet_{[0,1]}, \{m^t_{k,\beta}\}, \langle\cdot,\cdot\rangle)_{t\in s}$ is a pseudo isotopy:
→ β element of a discrete group $G_{\geq 0} \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}/2$
where we take $G = (0,0)$
→ m^t_k is the bracket for $k=2$ & differential for $k=1$
→ Problem: 2) & 4) only hold for A_∞-algebra structure m^t_k trivially, but formulas for L_∞-shifted

- Problem: need another family of maps
$$c^t_{k,\beta} : h_A(t)^{\otimes k} \to h_A(t) \quad + \text{degree rule.}$$
with $c^t_k(0,0) = 0 \Rightarrow c^t_k = 0 \to$ degrees don't work. But: in Fukaya paper, a normal A_∞-algebra structure is the motivation for the G-indexed A_∞ structure, he works for $G = (0,0)$; there are the "trivial" examples $\Rightarrow$ degrees also don't work unless we shift, which Fukaya does sometimes.

- m^t_k is smooth in the sense that for $\forall t$ & fixed x_1, x_2 $t \mapsto m^t_2(x_1, x_2)$ & $t \mapsto m^t_1(x_1)$ can be written as $\sum_i a_i(t) e_i$ finite sum for $a_i : [0,1] \to \mathbb{R}$ smooth & e_i basis of $h_A(t)$.
$\to$ immediate for $a_i = id$ as b & B are finite sums, Δ & Δ^{sym} defined over b & B & for fixed x_1, x_2, both & dd_R are finite sums.

<u>Thm</u> A pseudo isotopy induces an A_∞-homomorphism
$$(C, m^0_{k,\beta}, \langle -,- \rangle) \longrightarrow (C, m^1_{k,\beta}, \langle -,- \rangle)$$
respecting the bracket & admitting an inverse.

U is L_∞-morphism

Δ^{sym} built of Δ : $[b + uB, \Delta^{sym}] = \Delta$

$\{x,y\}^{\Delta^{sym}} = \Delta^{sym}(xy) - \Delta^{sym}(x)y - x\Delta^{sym}(y)$

Write V_1 for the coderivation extension Δ^{sym}

& V_2 for that of $\{-,-\}^{\Delta^{sym}}$, $V_1, V_2 : S(L_+(1))$

Fukaya paper gives family of $(A_\infty ..)$-morphisms ← V_i are on $T^c(L)$

$U'(t)$ st $\dfrac{d U'_m(t)}{dt} = U'_m(t) V_1 + U'_{m-1}(t) V_2$

← show $U'(t) = U(t)$

& $U'(0) = id$ as initial condition

$$U_m(t) = \sum_{g,n} \sum_{(G,f) \in P(g,n)_m} \frac{1}{|Aut(G,f)|} U((G,f)(t)) \cdot \hbar^{g - \sum_{i=1}^m g(f(i))}$$

Where $U(G,f)(t)$ is like $U(G,f)$ but using edges tH^{sym} & $t\Delta^{sym}$ on self-loops.

⇝ differentiation should give $\hbar U'(t)_m V_2$ for s-loop

& $U'_{m-1}(t) V_2$ for loops.

Use groupoid integration from Fiorenza 2006

"Sums over graphs & integration are discrete groupoids"

D_m groupoid of marked labelled graphs (G,f) with m vertices

D'_m groupoid with (G,f,e) as above & e non-self loop

$$D_m \xleftarrow{F} D_m' \xrightarrow{C} D_{m-1}$$

F: forget e

C: contract e for e non self loop, mark contr. edge by 1, ret accordingly to f

view $(G,f) \mapsto U_{(G,f)}(t) \hbar^{g - \sum_{i=1}^m g(f(i))}$ $D_m \to V$ ← vector space

& $\dfrac{1}{|Aut(G,f)|}$ as a measure on iso classes of objects

$$\Rightarrow \sum_{g,n} \sum_{(G,f) \in \Gamma(g,n)_m} \frac{1}{|Aut(G,f)|} U_{(G,f)}(t) \hbar^{g - \sum_{i=1}^m g(f(i))} = \int_{D_m} U_m(t)\, d\mu$$

Observe that every edge gives one factor t

$$\Rightarrow \frac{d}{dt}\int_{D_m} U_m(t)\, d\mu = \int_{D_m} |E_G| t^{|E_G|-1} U_m\, d\mu$$

recall $v_2(x_1 \dots x_n) = \sum_{i=1}^{n-1} (-1)^{\varepsilon} x_1 \dots \{x_i, x_{i+1}\}^{sym} \dots x_n$

$$= \sum_{i=1}^{n-1} (-1)^{\varepsilon} x_1 \dots (\Delta^{sym}(x_i x_{i+1}) - \Delta^{sym}(x_i) x_{i+1} - x_{i+1}\Delta^{sym}(x_i)) \dots x_n$$

with Δ^{sym}

$v_1 \in$

$v_2 : Sym\, h_4[1] \longrightarrow Sym\, h_4[1]$

Observe for $\frac{d}{dt} U'(t)$ & in general for U

$$\int_{D'_{m-1}} U_m(t) v_1 + \int_{D_{m-1}} U_{m-1}(t) v_2\, d\mu = \int_{D'_m} U_m(t) v_1\, d\mu + \int_{D'_m} C^* U_{m-1}(t)\, d\mu$$

$$= \int_{D_m}^{F^*} (U_m(t) v_1 + C^* U_{m-1}(t)\, d\mu_1$$

Compute $F_*(U_m(t)v_1 + C^*(U_{m-1}(t) v_2))(G, f)$

$$= \sum_{e \in E_G} (U_m(t) v_1 + C^*(U_{m-1}(t) v_2))(G, f, e)$$

$$= \sum_{e \in E_G} (t^{|E_G|-1}(U_m v_1 + C^*(U_{m-1} v_2))(G, f, e)$$

$$= \sum_{e \in E_G^{loop}} t^{|E_G|-1} U_m\, v_1(G,f) + \sum_{e \in E_G^{non-loop}} t^{|E_G|-1} U_m\, v_2(G,f)$$

$$= \left(\sum_{e \in E_G^{loop}} t^{|E_G|-1} U_m - \sum_{e \in E_G^{non-loop}} t^{|E_G|-1} U_m\right)(G,f)$$

v_i act on $Sym\, h_4[1]$ not $Sym\, L$ $\rightsquigarrow$ ignore

$$= |E_G| t^{|E_G|-1} U_m(G,f)$$

$\Rightarrow U_m(t) = U'_m(t)$ as they fulfill same ODE //