Kashiwara–Vergne solutions degree by degree

Guillaume Laplante-Anfossi, j/w Zsuzsanna Dancso, Iva Halacheva & Marcy Robertson

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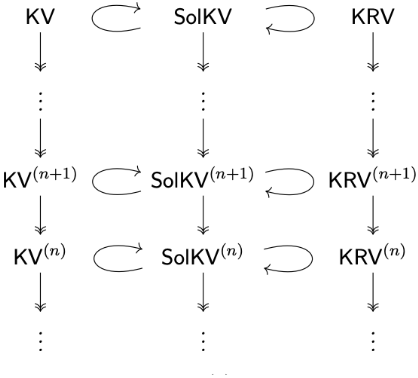

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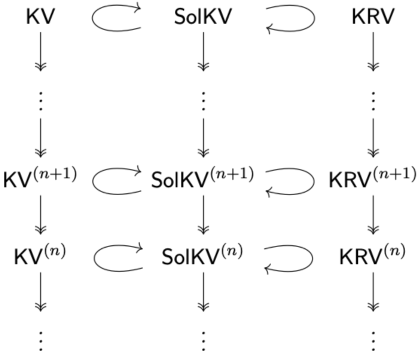

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$$\mathrm{bch}(x,y) = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] + \cdots$$

and $[x,y] = xy - yx$.

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Originally stated in the context of convolutions on Lie groups (Kashiwara–Vergne, '78), solved by Alekseev–Meinrenken ('06), reformulated by Alekseev–Torossian ('12).

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Definition

A *tangential derivation* of $\mathbb{L}$ is a Lie derivation $u : \mathbb{L} \to \mathbb{L}$ for which $u(x) = [x, u_1]$ and $u(y) = [y, u_2]$ for some $u_1, u_2 \in \mathbb{L}$.

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Tangential derivations form a Lie algebra, which integrates to the group TAut of *tangential automorphisms* of $\mathbb{L}$.

Non-commutative Jacobian

Each element $a \in \mathsf{A}$ has a unique decomposition of the form

$$a = a_0 + \partial_x(a)x + \partial_y(a)y$$

for some $a_0 \in \mathbb{C}$ and $\partial_x(a), \partial_y(a) \in \mathsf{A}$.

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Definition

The non-commutative *divergence* map $j : \mathfrak{tder} \to \mathrm{cyc}$ is the linear map defined on a tangential derivation $u = (u_1, u_2)$ by

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It integrates to the *non-commutative Jacobian* $J : \mathrm{TAut} \to \mathrm{cyc}.$

Kashiwara–Vergne solutions

Definition

A *Kashiwara–Vergne solution*, or KV solution for short, is a pair

$$(F, r) \in \mathrm{TAut} \times z^2\mathbb{C}[[z]]$$

satisfying the two equations

$$F(e^x e^y) = e^{x+y} \tag{SolKV1}$$

$$J(F) = \mathrm{tr}(r(x+y) - r(x) - r(y)). \tag{SolKV2}$$

We denote the set of KV solutions by SolKV.

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The tangential automorphism F uniquely determines the power series r, and the assignment $F \mapsto r$ is called the *Duflo map*.

Everything up to degree n

We denote by $\mathbb{L}_{\leq n} := \mathbb{L}/\mathbb{L}_{\geq n+1}$ the quotient of the Lie algebra $\mathbb{L}$ by the ideal of elements of degree greater than n. We define the degree n quotients of A and cyc by

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The degree n quotient of the group of *tangential automorphisms* is

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Note that there is a natural surjective group homomorphism

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The *non-commutative Jacobian* is homogeneous and induces a map

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KV solutions up to degree n

Definition

A *KV-solution up to degree n* is a pair

$$(F, r) \in \mathrm{TAut}_{\leq n} \times z^2\mathbb{C}[[z]]/z^{n+1}$$

satisfying the two equations

$$F(e^x e^y) = e^{x+y} \quad \text{in } \mathbb{L}_{\leq n} \tag{1}$$

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- Does every KV-solution up to degree n extends to a full KV-solution?

Main result

Theorem (Dancso–Halacheva–L.-A.–Robertson, '23)

The truncation maps $\mathsf{SolKV}^{(n+1)} \to \mathsf{SolKV}^{(n)}$ *are surjections, and* SolKV *admits a tower decomposition*

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- degree by degree calculations of KV solutions always succeed,
- one could use symmetric KV solutions to simplify the computation of a class of Drinfel'd associators (Alekseev–Enriquez–Torossian '10),
- one could compute certain knot invariants degree by degree (Bar-Natan–Dancso '17).

In fact Alekseev–Torossian ('12) conjecture that all KV solutions arise from associators (verified up to degree 16).

Proof strategy

Idea: study KV solutions via their symmetries governed by

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where $\mathrm{bch}(x, y)$ denotes the Baker–Campbell–Hausdorff series.

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As shown by Alekseev–Torossian, both KV and KRV act freely and transitively on the left/right of SolKV by left/right multiplication by the inverse.

Proof strategy (continued)

Proposition

The elements of $\mathrm{KRV}^{(n)}$ *form a subgroup of* $\mathrm{TAut}_{\leq n}$. *Moreover,* $\mathrm{KRV}^{(n)}$ *acts freely and transitively on* $\mathrm{SolKV}^{(n)}$ *by left product with the inverse.*

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which is easy to prove.

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The proof of this last lemma uses crucially the following result, which builds on the topological interpretation of KV solutions as homomorphic expansions of welded foams (knotted surfaces in $\mathbb{R}^4$) by Bar-Natan–Dancso.

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The up-to-degree-n version of this theorem allows one to show that $\mathrm{KRV}^{(n)}$ is an algebraic matrix group, which is moreover unipotent.

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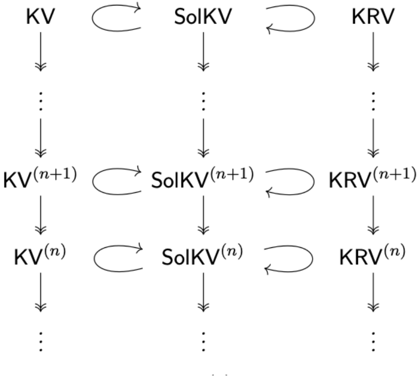

Kashiwara–Vergne Lie algebra

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$$u(x+y) = 0 \quad \text{(krv1)}$$
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Theorem

There is an isomorphism of Lie algebras $\mathrm{gr}(\mathrm{KRV}) \cong \mathfrak{krv}$.

Here, the bracket on $\mathrm{gr}(\mathrm{KRV})$ is induced by the group commutator, which respects the degree filtration.

Kashiwara–Vergne Lie algebra (continued)

In fact, this compatiblity holds at the level of tangential automorphisms. The *degree filtration* on TAut is given by

$$\mathcal{F}_n(\mathrm{TAut}) := \ker(\mathrm{TAut} \to \mathrm{TAut}_{\leq n-1}).$$

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There is an analogous result for the Grothendieck–Teichmüller group (Drinfel'd, '90).

Kashiwara–Vergne towers

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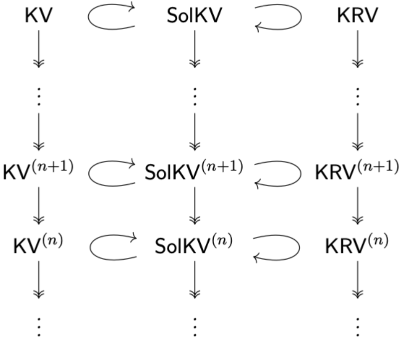
Theorem

Any $F^{(n)} \in \mathsf{SolKV}^{(n)}$ *induces an isomorphism* $\Psi_{F^{(n)}} : \mathsf{KV}^{(n)} \xrightarrow{\cong} \mathsf{KRV}^{(n)}$, *and the vertical arrows in the following commutative diagram are surjective.*



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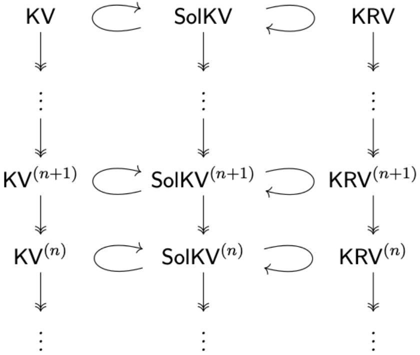

Higher genus KV solutions

Let $\mathbb{L}$ denote the free Lie algebra L on $2g+n$ generators x_i, y_i, z_j, $1 \leq i \leq g$, $1 \leq j \leq n$. Define its *tangential automorphisms* $\mathrm{TAut}(\mathbb{L}) := \{(F, f_1, \ldots, f_n) \in \mathrm{Aut}(\mathbb{L}) \times \mathbb{L}^{\oplus n} \mid F(z_j) = e^{-f_j} z_j e^{f_j}\}$.

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Definition

An *KV solution of type $(g, n+1)$* is an $F \in \mathrm{TAut}(\mathbb{L})$ such that

$$F(\sum_{i=1}^{g}[x_i, y_i] + \sum_{i=1}^{n} z_j) = \log(\Pi_{i=1}^{g}[e^{x_i}, e^{y_i}]\Pi_{j=1}^{n} e^{z_j}), \text{ and}$$

$$j(F) = \sum_{i=1}^{g} |h(x_i) + h(y_i)| + \sum_{j=1}^{n} |r_j(z_j)| - |r(\log(\Pi_{i=1}^{g}[e^{x_i}, e^{y_i}]\Pi_{j=1}^{n} e^{z_j}))|,$$

where $h(s) := \log((e^s - 1)/s)$, j is the divergence map and r is the Duflo function. The set of such solutions is denoted $\mathrm{SolKV}^{(g,n+1)}$.

Higher genus KV solutions (continued)

These higher genus KV solutions have their symmetry groups $\mathsf{KV}^{(g,n+1)}$ and $\mathsf{KRV}^{(g,n+1)}$. We have $\mathsf{SolKV} = \mathsf{SolKV}^{(0,3)}$, $\mathsf{KV} = \mathsf{KV}^{(0,3)}$ and $\mathsf{KRV} = \mathsf{KRV}^{(0,3)}$.

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Theorem (Alekseev–Kawazumi–Kuno–Naef,'20)

KV solutions of type $(g, n+1)$ are in bijection with formality isomorphisms of the Goldman–Turaev Lie bialgebra associated to a compact oriented surface of genus g with $n+1$ boundary components.

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Moreover, gluing and contraction of surfaces induce operations

- $\mathfrak{tder}^{(g_1,n_1+1)} \times \mathfrak{tder}^{(g_2,n_2+1)} \to \mathfrak{tder}^{(g_1+g_2,n_1+n_2+1)}$, and
- $\mathfrak{tder}^{(g,n+1)} \to \mathfrak{tder}^{(g+1,n)}$

which integrate to operations on tangential automorphisms.

Kashiwara–Vergne operads

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Theorem (Dancso–Halacheva–L.-A.–Robertson)

The set of genus zero KV solutions $\{\mathrm{SolKV}^{(0,n+1)}\}_{n\geq 1}$*, as well as their symmetry groups* $\{\mathrm{KV}^{(0,n+1)}\}_{n\geq 1}$ *and* $\{\mathrm{KRV}^{(0,n+1)}\}_{n\geq 1}$*, form colored operads in groups.*

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Since the Grothendieck–Teichmüller group GT embeds into $\mathrm{KV} = \mathrm{KV}^{(0,3)}$, one could expect an action on the tower $\{\mathrm{SolKV}^{(g,n+1)}\}_{g,n+1\geq 0}$ of higher genus KV solutions. This is the subject of ongoing work.

Conclusion

Thank you for your attention!