1\. I. Rel-C^∞-structures

Global Kuranishi

Def. (Rel-C^∞ maps).

write $Y \to S$ as Y/S.

$\mathcal{U} \subset \mathbb{R}^n \times S$, $V \subset \mathbb{R}^m \times T$. (S.T are top spaces)

A rel C^∞ map: $\mathcal{U}/S \to V/T$

consists : $\varphi : S \to T$.

$\tilde{\varphi} : \mathcal{U} \to V$, $\tilde{\varphi} = (F(x,s), \varphi(s))$

s.t. F is smooth along $\mathbb{R}^n$.

Def. (Rel C^∞ mflds) Y/S,

a chart of Y/S is an open set $\mathcal{U} \subset Y$,

& $\varphi : \mathcal{U} \to \mathbb{R}^n \times S$.

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\varphi"] \arrow[d, hook] & \mathbb{R}^n \times S \arrow[r] & S \\ Y \arrow[urr, bend right] & & \end{tikzcd}$$

$(\mathcal{U}, \varphi)$ and (V, ψ) are compatible if

$(\psi \circ \varphi^{-1}, id_S) : \varphi(\mathcal{U} \cap V)/S \to \psi(\mathcal{U} \cap V)/S$

is a rel-C^∞ diff.

A rel-C^∞-str on Y/S is a max atlas of rel-C^∞ compatible charts for Y/S.

2.

Def. ~~rel~~ rel-C^∞ maps $Y/S \to Z/T$

locally $\mathbb{R}^n \times S$ ~~to~~ $\mathbb{R}^n \times$ ~~S~~ T

$$\begin{matrix} \mathbb{R}^n \times S & & \mathbb{R}^n \times T \\ \cup & & \cup \\ U/S & \to & V/T \end{matrix}$$

Def. A rel-C^∞ v.b. on Y/S is a ($\mathbb{R}/\mathbb{C}$) bundle E/Y, w/ rel-C^∞ transition maps

3.

Def. (Rel-C^∞ global Kuranishi charts)

A rel-C^∞ global Kuranishi chart $K = (G, \mathcal{T}/\mathcal{B}, \mathcal{E}, s)$

a i). a rel-C^∞ manifold, ("thickening") $\mathcal{T}/\mathcal{B}$.

$\mathcal{B}$ is a sm mfd.

ii). a rel-C^∞ v.b. $\mathcal{T}/\mathcal{B}$. obstruction bundle.

rel-C^∞ section s of $\mathcal{E}$. obstruction section.

iii). cpt Lie gp. G. symmetry group.

$G \curvearrowright \mathcal{T}/\mathcal{B}$ and $\mathcal{E}$. finite stabilizer.

s is G-equivariant.

$G \times \mathcal{T} / G \times \mathcal{B} \to \mathcal{T}/\mathcal{B}$. rel-$C^\infty$.

same for $\mathcal{E}$.

oriented if $\mathcal{T}$ & $\mathcal{E}$ are oriented, G preserves orientation.

If. $s^{-1}(0)/G \xrightarrow{\simeq} Z$. K is a global Kuranishi chart for Z.

$\mathrm{vdim}_K Z = \dim \mathcal{T} - \dim G - \operatorname{rank} \mathcal{E}$.

Def. (Rel-C^∞ equivalence). $K = (G, \mathcal{T}/\mathcal{B}, \mathcal{E}, s)$.

(i). $U = \mathrm{Nbd}(s^{-1}(0)) \subset \mathcal{T}$.

$K \sim (G, U/\mathcal{B}, \mathcal{E}|_U, s|_U)$.

4. (ii). $p: W \to \mathcal{T}/\mathcal{B}$. G-eq.

~~replace~~ $K \sim (G,\ W/\mathcal{B},\ p^*\mathcal{E} \oplus p^*W,\ p^*\sigma \oplus \Delta_W)$

(iii). G' bundle. $q: P \to \mathcal{T}/\mathcal{B}$.

$K \sim (G \times G',\ P/\mathcal{B},\ q^*\mathcal{E},\ q^*\sigma)$

(iv). $\mathcal{T}/\mathcal{B} \to \mathcal{B}'/\mathcal{B}$ submersion.

$K \sim (G,\ \mathcal{T}/\mathcal{B}',\ \mathcal{E},\ \sigma)$

$\tau_{\mathcal{E}/G} \in H^*_G(\mathcal{E})$ Thom class.

$\sigma: \mathcal{T} \to \mathcal{E}$.

Def. (Virtual fundamental class).

$K = (G, \mathcal{T}/\mathcal{B}, \mathcal{E}, \sigma)$: oriented rel-C^∞ global Kuranishi chart.

$[Z]^{vir}_K \in \check{H}^d(Z, \mathbb{Q})^\vee$ ...

$$\check{H}^{\mathrm{vdim}\, Z}(Z;\mathbb{Q}) \xrightarrow{\sigma^* \tau_{\mathcal{E}/G} \cup} H_c^{\dim(\mathcal{T}/G)}(\mathcal{T}/G;\mathbb{Q}) \xrightarrow{[\mathcal{T}/G]} \mathbb{Q}$$

$$\check{H}^{\mathrm{vdim}\, Z}(Z;\mathbb{Q}) \cong \varinjlim_{U \supset Z} H^*(U,\mathbb{Q})$$

5. III. G. K. S. for Moduli of open curves.

$\overline{\mathcal{M}}^{J,\beta}_{g,h;k,l}(X,L)$

k → interior marking

l → boundary marking.

· Base.

$\mathcal{B} := \overline{\mathcal{M}}^{*}_{g,h,0,0}(\mathbb{C}P^N, \mathbb{R}P^N; m) \in \overline{\mathcal{M}}(\cdots)$

moduli of curves s.t. doubling is regular non-degenerate embedding.

Prop. $\mathcal{B}$ is a sm mfd w/ corner.

$\mathcal{B} \to \overline{\mathcal{M}}^{*}_{\tilde{g}}(\mathbb{C}P^N, 2m)$ (the doubling map) is smooth.

$\mathcal{G} \simeq PGL_{\mathbb{R}}(N+1) \curvearrowright \mathcal{B}$.

$G := PO(N+1)$ $(\simeq O/\{\pm 1\})$.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{ev} & \mathcal{B} \times \mathbb{C}P^N \\ \downarrow & & \\ \mathcal{B} & & \end{array}$$

$\mathcal{Z} \subset \mathcal{B}$: $\mathcal{Z}$: $u : (C, \partial C) \to (X, L)$

$\int_{C'} u^* \omega \geq 0$ for $\forall$ irreducible components C' of C

8.. $\int_{C'} u^* \omega \geq \hbar$, $\forall$ unstable irreducible component C' of C.

Def. A good covering :

$\mathcal{U} = \{(\mathcal{U}_i, D_i, \chi_i)\}_{i \in \Lambda}$. finite set.

- $\mathcal{U}_i$: open subset of Z, $D_i \subset X \setminus L$. submfd codim 2. w/. boundary. s.t. $\forall$ $(u, C, \iota) \in \mathcal{U}_i$. ($\iota \in \mathcal{B}$, $u: C \to X$)
  - (i). $u^{-1}(D_i)$ consists $3d$. non-nodal points
    $$\#(u^{-1}(D_i) \cap C') = 3 \langle [\Omega], u_* [C'] \rangle$$
  - (ii). $u(C) \cap \partial D_i = \phi$. $u \pitchfork D_i$.
- $\chi_i : Z \to [0,1]$ supported in D_i.
- $\forall [u, C]. \in \mathcal{M}^{J, \beta}_{g,h}(X, L; \beta)$, $\exists$. $\iota: (C, \partial C) \to (\mathbb{C}P^N, \mathbb{R}P^N)$ & $i \in \Lambda$ s.t. $(u, C, \iota) \in \mathcal{U}_i$. $\chi_i(u, C, \iota) > 0$.

Thm. $\exists$. Global Kuranishi Chart for $\mathcal{M}^{J,\beta}_{g,h,0,0}(X, L)$.