Diagonals of permutahedra and associahedra

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Cellular diagonals of polytopes

DEF. thin diagonal of a set X = map $\delta : \left\{ \begin{array}{ccc} X & \to & X \times X \\ x & \mapsto & (x, x) \end{array} \right.$.

DEF. cellular diagonal of a d-polytope $\mathbb{P}$ = continuous map $\Delta : \mathbb{P} \to \mathbb{P} \times \mathbb{P}$ such that

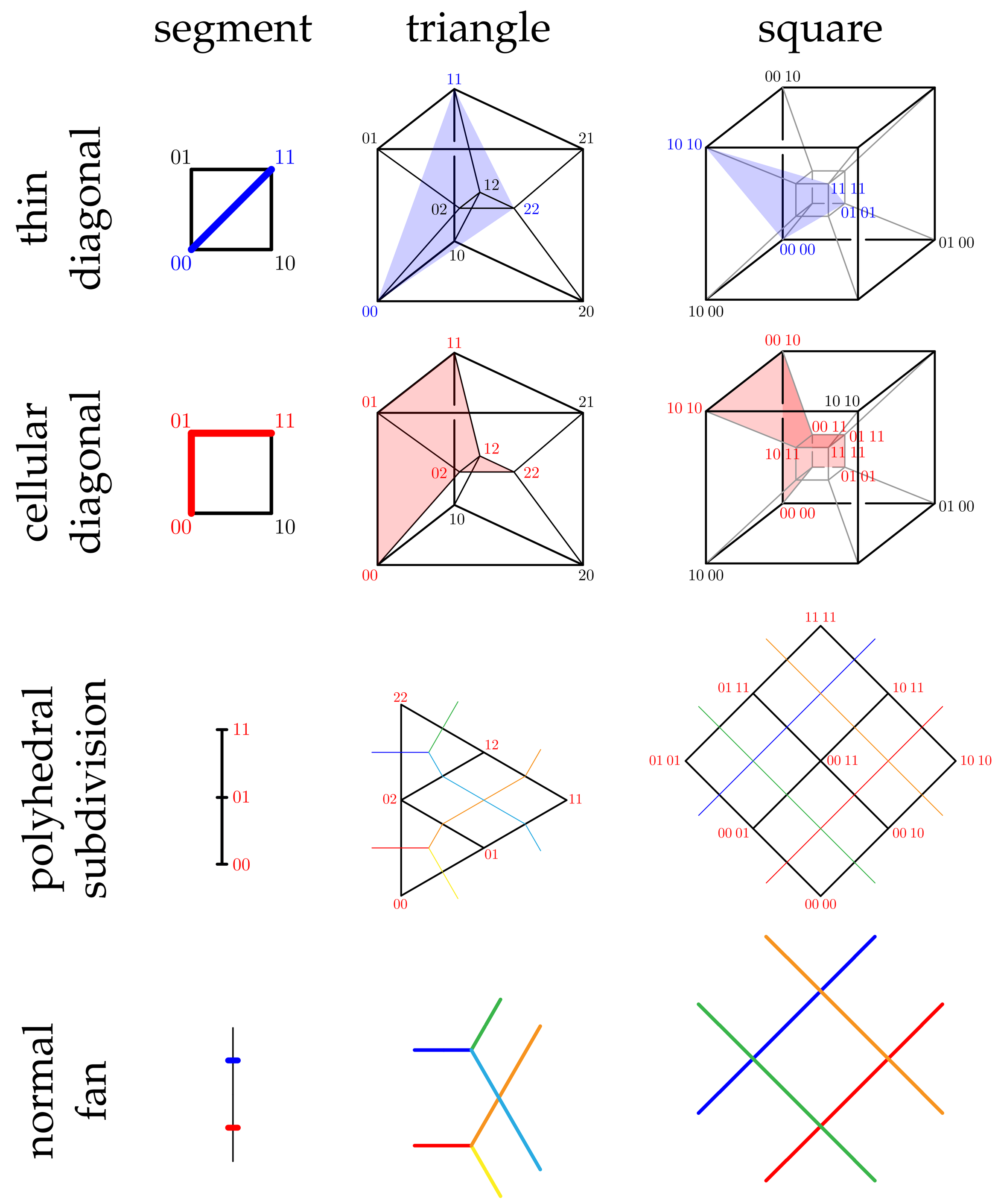
- its image is a union of d-dimensional faces of $\mathbb{P} \times \mathbb{P}$,
- it agrees with δ on the vertices of $\mathbb{P}$,
- it is homotopic to δ, relative to the image of the vertices of $\mathbb{P}$.

REM. The image of Δ is a union of pairs of faces $\mathbb{F} \times \mathbb{G}$ of $\mathbb{P} \times \mathbb{P}$. By drawing the polytopes $(\mathbb{F} + \mathbb{G})/2$, we can visualize $\triangle_{(\mathbb{P}, \boldsymbol{v})}$ as a polytopal subdivision of $\mathbb{P}$.

THM. [LA'22] For $\boldsymbol{v} \in \mathbb{R}^d$ generic wrt $\mathbb{P}$, the $(-\boldsymbol{v}, \boldsymbol{v})$-optimal vertex of the fiber polytope of the projection $\left\{ \begin{array}{ccc} \mathbb{P} \times \mathbb{P} & \to & \mathbb{P} \\ (p, q) & \mapsto & \frac{p+q}{2} \end{array} \right.$ yields a cellular diagonal $\triangle_{(\mathbb{P}, \boldsymbol{v})}$ of $\mathbb{P}$.

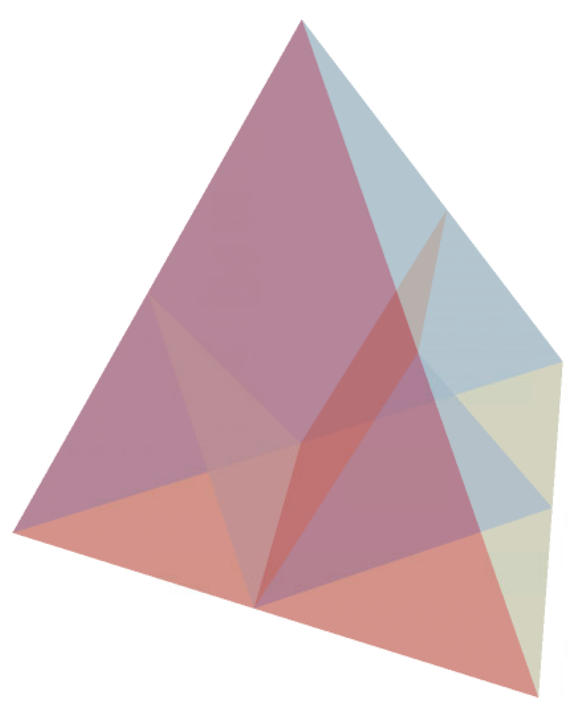
THM. [LA'22]

- Combinatorics of $\triangle_{(\mathbb{P}, \boldsymbol{v})}$ = combinatorics of the common refinement of two copies of the normal fan of $\mathbb{P}$ translated in direction $\boldsymbol{v}$.
- Faces of $\Delta_{\mathbb{P}, v} \subseteq$ pairs $(\mathbb{F}, \mathbb{G})$ of faces of $\mathbb{P}$ such that $\max_v(\mathbb{F}) \le \min_v(\mathbb{G})$. When this inclusion is an equality, the diagonal is called magical.



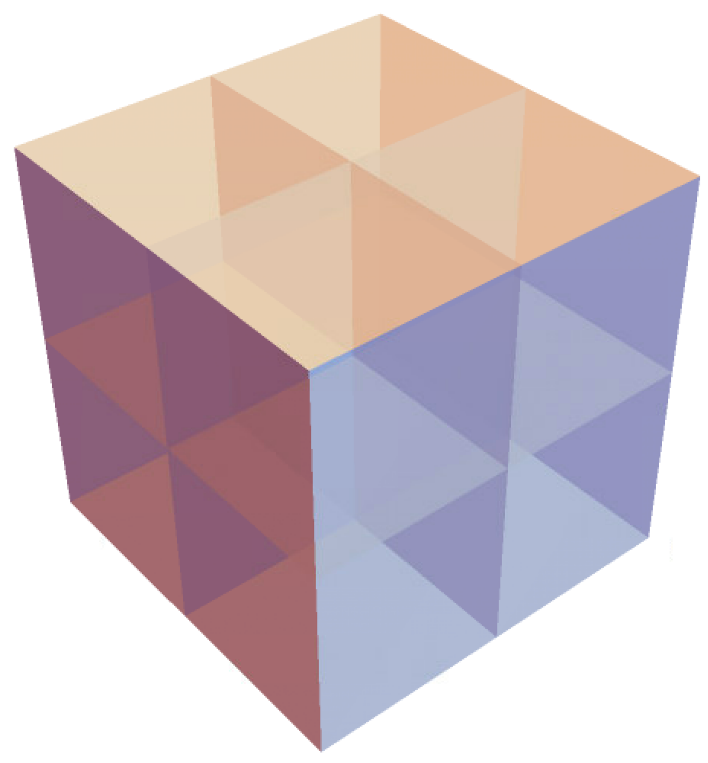
Face numbers of cellular diagonals of classical polytopes

simplex



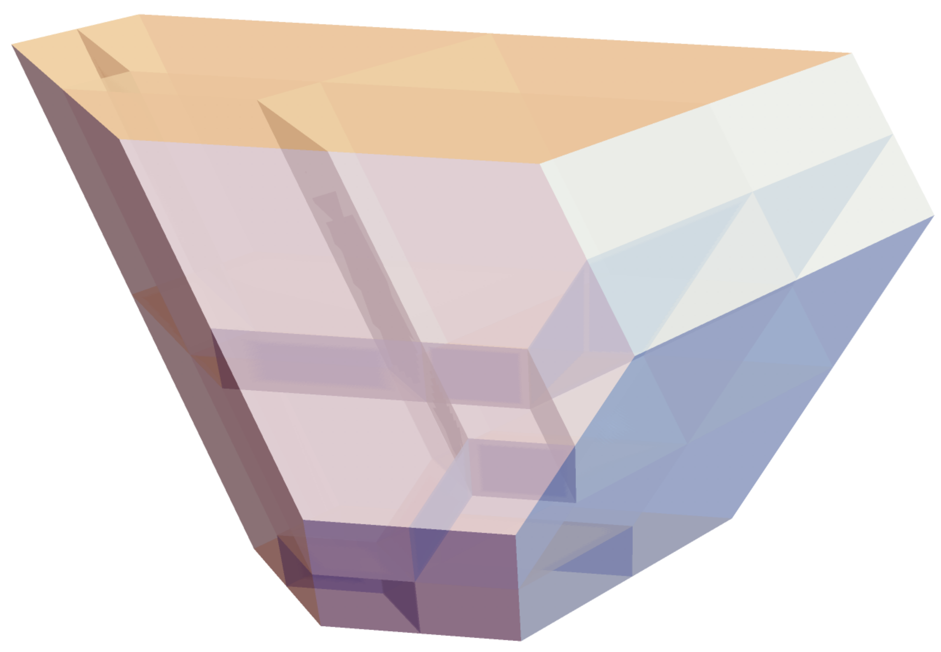
$$f_k = (k+1)\binom{n+1}{k+2}$$

cube



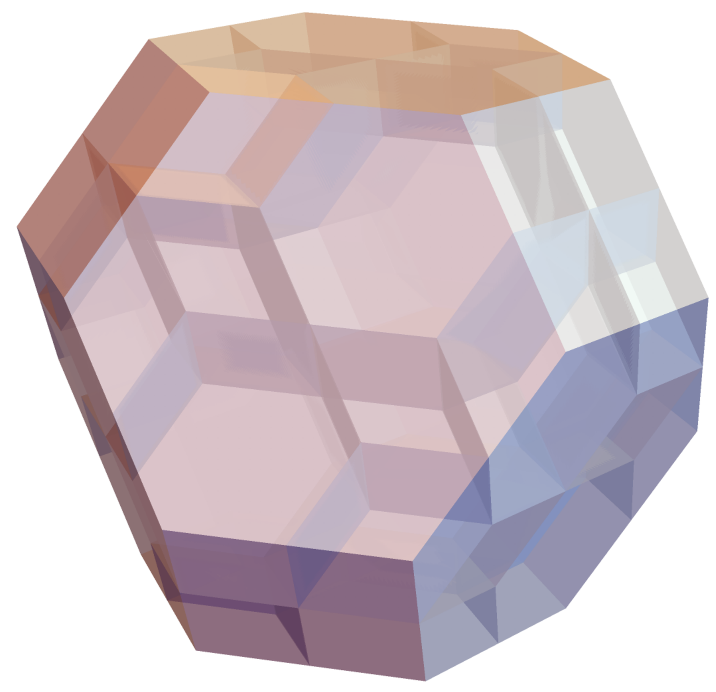
$$f_k = \binom{n-1}{k} 2^k 3^{n-1-k}$$

associahedron



$$f_k = \frac{2\binom{n-1}{k}\binom{4n+1-k}{n+1}}{(3n+1)(3n+2)}$$

permutahedron



$$f_0 = n!\,[z^n] \exp\Big(\sum_{m \ge 1} \frac{C_m\, z^m}{m}\Big)$$

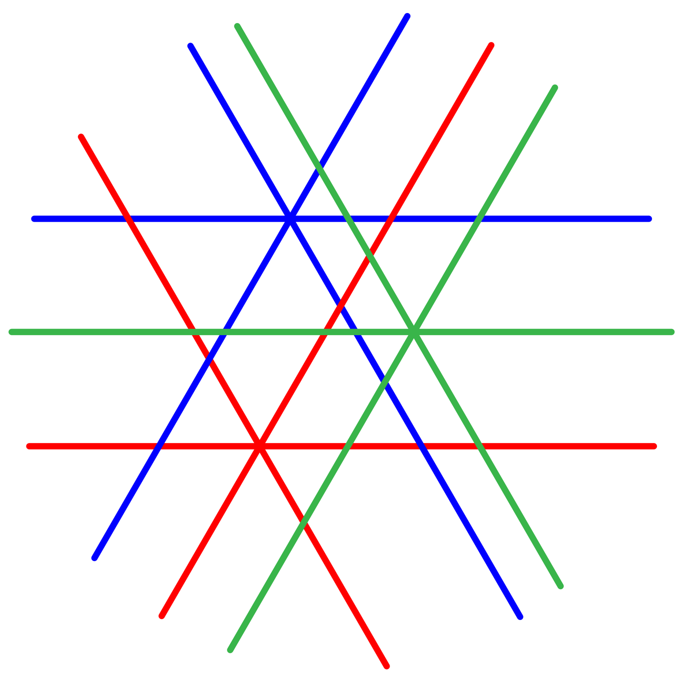
$$f_{n-1} = 2(n+1)^{n-2}$$

Diagonals of permutahedra

f-vector of $\triangle_{\mathrm{Perm}_n}$ = f-vector of two generically translated copies of the braid arrangement.

(ℓ, n)-partition forest = ℓ-tuple $\boldsymbol{F} := (F_1, \dots, F_\ell)$ of set partitions of $[n]$ whose intersection hypergraph is a hyperforest.

(ℓ, n)-braid arrangement $\mathcal{B}_n^\ell$ = union of ℓ generically translated copies of the braid arrangement.



THM. Flat poset of $\mathcal{B}_n^\ell$ $\simeq$ poset of (ℓ, n)-partition forests ordered by componentwise refinement.

THM. The Möbius polynomial $\mu_{\mathcal{B}_n^\ell}(x, y)$ is given by

$$(xy)^{n-1-\ell n} \sum_{\boldsymbol{F} \le \boldsymbol{G}} \prod_{i \in [\ell]} x^{\#F_i} y^{\#G_i} \prod_{p \in G_i} (-1)^{\#F_i[p]-1} (\#F_i[p] - 1)!\,,$$

where $\boldsymbol{F} \le \boldsymbol{G}$ ranges over all intervals of the (ℓ, n)-partition forest poset, and $F_i[p]$ denotes the restriction of the partition F_i to the part p of G_i.

Diagonals of associahedra

THM. [MTTV'21]

k-faces of $\Delta_{\mathbb{A}\mathrm{sso}(n)} \longleftrightarrow$ $(\mathbb{F}, \mathbb{G})$ faces of $\mathbb{A}\mathrm{sso}(n)$ with $\dim(\mathbb{F}) + \dim(\mathbb{G}) = k$ and $\max(\mathbb{F}) \le \min(\mathbb{G})$.

THM. For any $n, k \ge 1$, the number of k-faces of $\Delta_{\mathbb{A}\mathrm{sso}(n)}$ is

$$\sum_{S \le T \in \mathrm{Tam}(n)} \binom{\mathrm{des}(S) + \mathrm{asc}(T)}{k} = \frac{2\binom{n-1}{k}\binom{4n+1-k}{n+1}}{(3n+1)(3n+2)}$$

Techniques: generating functions — quadratic method — reparametrization — Lagrange inversion.

More details?

Refined product formulas for Tamari intervals
A. Bostan, F. Chyzak & V. Pilaud
arXiv:2303.10986

Cellular diagonals of permutahedra
B. Delcroix-Oger, G. Laplante-Anfossi, V. Pilaud & K. Stoeckl
arXiv:2308.12119