NOTES ON COSTELLO'S THEOREM

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ABSTRACT. The goal of this lecture is to explain Costello's construction of an open-closed topological conformal field theory associated to an A-infinity algebra, and its conjectural application to the Fukaya category of a symplectic manifold.

1. OPEN-CLOSED TCFTS

1.1. **Props.** We consider *linear* props, that is props in the dg-category of chain complexes.

Definition 1. A *prop* (or *product and permutation category*) is a strict symmetric monoidal dg-category whose objects are generated by a single object.

We can identify the objects of a prop P with the natural numbers, and its tensor product with addition. We use the notation $\mathrm{P}(m,n)$ for the space of morphisms between m and n.

We say that a functor $\Phi : \mathrm{P} \to \mathrm{Ch}$ is (weak; strong) *symmetric monoidal* if there are maps $\Phi(m) \otimes \Phi(n) \to \Phi(m+n)$ which are (quasi-isomorphisms; isomorphisms) natural in m and n, and compatible with the symmetries of P and Ch.

Definition 2. An (weak; strong) *algebra* over a prop P is a (weak; strong) symmetric monoidal functor $\mathrm{P} \to \mathrm{Ch}$.

1.2. **Moduli space of curves.** We will consider now Riemann surfaces Σ with *open-closed boundary*. The boundary of such a surface is partitioned into three types: the closed boundaries, the open boundary intervals, and the free boundaries (see Fig. 1):

- The *closed boundaries* are parametrised and labelled either incoming or outgoing,
- The *open boundaries* are parametrised intervals embedded in some boundary component, they are also parametrised and labelled either incoming or outgoing,
- The *free boundaries* are the complement of the closed boundaries and the open boundary intervals, and are either circles or intervals.

Each connected component of Σ is required to have at least one free or incoming closed boundary.

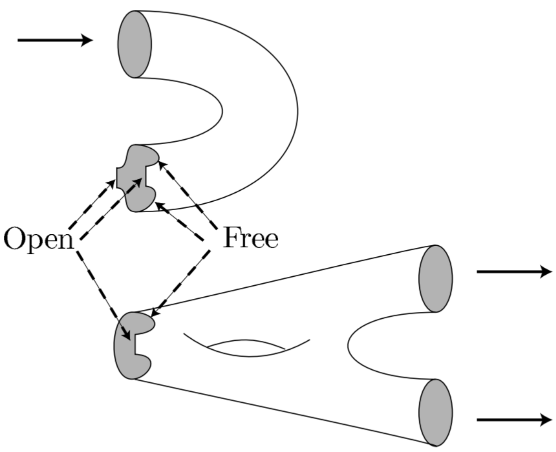

FIGURE 1. A Riemann surface with open-closed boundary. Boundaries can be incoming or outgoing, but this is not illustrated. Illustration from [4].

We consider here a prop with two colors ("open" and "closed"). The definition is similar, except that composition of morphisms is defined only when the colors match.

Definition 3. We let $\mathrm{OC}((m_1,n_1),(m_2,n_2))$ denote the singular chains on the moduli space of open-closed Riemann surfaces with m_1 incoming open boundaries, n_1 incoming closed boundaries, m_2 outgoing open boundaries, and n_2 outgoing closed boundaries.

We consider the full subcategory O whose objects have no closed part, and the subcategory C whose objects have no open part, and whose morphisms are chains on Riemann surfaces with only closed boundaries.

Lemma 1. *The family of chain complexes* OC*, endowed with disjoint union and gluing of open-to-open and closed-to-closed boundaries forms a prop. The families* O *and* C *are sub-props.*

Definition 4. An *open-closed TCFT* (resp. *open TCFT*, resp. *closed TCFT*) is a weak algebra over the prop OC (resp. the prop O, resp. the prop C).

2. Costello's theorem

The props O and C naturally embed into OC, giving a diagram

$$i : \mathrm{O} \hookrightarrow \mathrm{OC} \hookleftarrow \mathrm{C} : j.$$

Every morphism of props ι gives rise to a "restriction-induction" adjunction $\iota_! \dashv \iota^*$ between the respective categories of algebras. In particular, we have the following diagram

$$\begin{tikzcd} \mathrm{O}-\mathrm{Alg} \arrow[r, bend left, "i_!"] & \mathrm{OC}-\mathrm{Alg} \arrow[l, bend left, "i^*"] \arrow[r, bend left, "j^*"] & \mathrm{C}-\mathrm{Alg} \arrow[l, bend left, "j_!"] \end{tikzcd}$$

However, the left adjoint functors $i_!$ and $j_!$ do not preserve quasi-isomorphisms in general; therefore there is no such adjunction at the level of weak algebras. We take instead the left derived functor $\mathbb{L}i_!$, which preserves quasi-isomorphisms. Given $\Phi \in \mathrm{O}-\mathrm{Alg}$, $\mathbb{L}i_!(\Phi)$ is obtained by first replacing Φ by a flat resolution, and then applying $i_!$.

Theorem 1 ([4, Thm. A])**.** *We have the following.*

(1) The category of open TCFTs is homotopy equivalent to the category of extended Calabi–Yau A_∞*-categories.*
(2) For any open TCFT Φ*, the homotopy universal extension* $\mathbb{L}i_!(\Phi)$ *is an open-closed TCFT.*
(3) Let Φ *be an open TCFT, and let* A *be its associated* A_∞*-category. Then, we have*

$$H_\bullet(j^*\mathbb{L}i_!(\Phi)) = HH_\bullet(A).$$

That is, the homology of the closed states of the homotopy universal open-closed extension of Φ *is equal to the Hochschild homology of* A*.*

In fact, Costello proves that $j^*\mathbb{L}i_!(\Phi)) = CH_\bullet(A)$, the Hochschild chains of A. This gives the following higher genus version of the Deligne conjecture.

Corollary 2.1 (Deligne conjecture)**.** *The Hochschild chains of any Calabi–Yau* A_∞*-category form a closed TCFT.*

2.1. **Fukaya application.** One major motivation for Costello's work comes from Gromov–Witten theory.

Conjecture 1. *For any symplectic manifold* X*, there is an open-closed TCFT whose open part is the Fukaya category of* X*, and whose closed part gives the Gromov–Witten theory of* X*.*

This has recently been made precise for the Fukaya algebra of a single embedded Lagrangian submanifold by Hirschi and Hugtenburg [6].

We will see in the next lecture (and study in depth in course of the semester) how one can use Costello's theorem above in order to compute Gromov–Witten type invariants from any Calabi–Yau A_∞-algebra together with a splitting of the Hodge filtration [1, 2].

3. The proof

The proof of Theorem 1 boils down to the construction of a model D of OC built out of disks and annuli with marked points and nodes on the boundary. We have that D is quasi-isomorphic to OC, and is given by cellular chains on a certain moduli space of nodal surfaces with a cell decomposition which respects the prop structure. There is moreover a natural sub-prop $\mathrm{D}_{\mathrm{open}} \subset \mathrm{D}$, quasi-isomorphic to O, obtained by restriction to disks with marked points on the boundary.

One can give an explicit presentation of D and $\mathrm{D}_{\mathrm{open}}$, and an explicit description of the differential, which allows one to identify the A_∞ relations (for $\mathrm{D}_{\mathrm{open}}$) as well as the Hochschild complex (for the closed part of D).

Remark 3.1. From next week on, we will consider a different model for the moduli space of open-closed Riemann surfaces due to Godin [5], which was shown to be equivalent to Costello's by Santander [7].

3.1. **Models of moduli spaces.** Costello considers first the moduli space $\mathscr{N}$ of Riemann surfaces with marked points and nodes on the boundary (see Fig. 2), obtained from open-closed Riemann surfaces via the following procedure:

- Discard surfaces with closed incoming boundaries, and for the others, replace the parametrization of a closed boundary by a single marked point.
- Replace each open boundary interval by a marked point.
- The free boundaries become the intervals between open marked points, and the boundary components with no marked points.

Each connected component of the surface must have at least one free boundary. Nodes are also allowed on the boundary, under specific conditions.

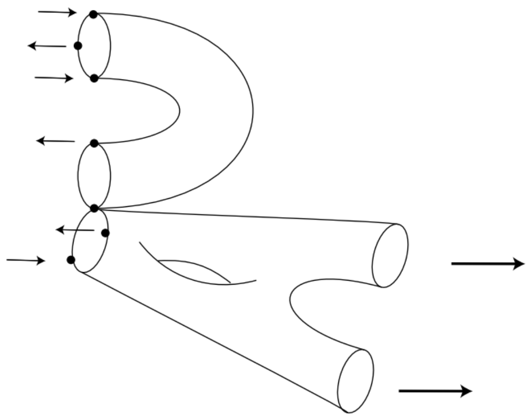

FIGURE 2. A Riemann surface with marked points and nodes on the boundary. The dots represent incoming or outgoing open boundaries; the boundaries with no dots are closed and outgoing. Illustration from [4].

Let us write the moduli space of open-closed Riemann surfaces $\mathscr{M}$.

Proposition 1. *The moduli spaces $\mathscr{N}$ and $\mathscr{M}$ are homotopy equivalent.*

Proof. One considers a bigger moduli space $\tilde{\mathscr{M}}$ of surfaces where open boundary intervals have a number $t \in [0,1/2]$ attached to them, with gluing performed along subintervals $[t,1-t]$. Then, $\mathscr{M}$ embeds at $t=0$, and $\mathscr{N}$ embeds at $t=1/2$, and both are homotopy equivalent to $\tilde{\mathscr{M}}$. $\square$

Then, one considers the subspace $\mathscr{G} \subset \mathscr{N}$ consisting of surfaces whose irreducible components are either a disc, or an annulus. One can also consider the restrictions $\mathscr{G}_{\text{open}}$ and $\mathscr{N}_{\text{open}}$ to the open parts.

Theorem 2. *The inclusion $\mathscr{G} \hookrightarrow \mathscr{N}$ is a weak homotopy equivalence.*

Proof. In the open case, this is the main result of [3]. One observes first that the moduli space of disks $\mathscr{G}_{\text{open}}$ is the modular envelope of the cyclic A_∞ operad, and that it embeds naturally in the boundary of $\mathscr{N}_{\text{open}}$. To this end, one uses the fact that the topological type of the surfaces in $\mathscr{G}_{\text{open}}$ and $\mathscr{N}_{\text{open}}$ can be described by a (ribbon) graph. Then, one is left to show that the moduli space $\mathscr{N}_{\text{open}}$ retracts unto its boundary. To this end, one uses the canonical hyperbolic metric on a given surface. $\square$

We thus get a sequence of homotopy equivalences

$$\mathscr{G} \overset{\sim}{\hookrightarrow} \mathscr{N} \sim \mathscr{M}. \tag{3.1}$$

Considering the restriction to the open parts of the moduli spaces, we get an analogous sequence of homotopy equivalences

$$\mathscr{G}_{\text{open}} \overset{\sim}{\hookrightarrow} \mathscr{N}_{\text{open}} \sim \mathscr{M}_{\text{open}}. \tag{3.2}$$

One then needs to define a cellular decomposition of the moduli space $\mathscr{G}$, which is compatible with the gluing operations. This is a subtle task and an absolutely crucial point.

One uses the fact that there is a unique holomorphic isomorphism from an annulus A to the cylinder $S^1 \times [0,1]$ which sends a chosen marked point to $(1,0)$. The inverse image of $1 \times [0,1]$ gives a cut on the annulus between the chosen point and another point on the boundary. One then defines the cell decomposition of a surface Σ as follows: the 0-skeleton consists of the nodes, marked points, and the places where the cut on an annulus intersect the boundary of the annulus; the one cells are the boundary $\partial\Sigma$, together with the cuts on the annuli; and the 2 skeleton is Σ.

Proposition 2. *The above defines a cell decomposition of $\mathscr{G}$, which is moreover compatible with gluing.*

We therefore obtain a prop in cellular spaces, to which we can apply the cellular chains functor.

Definition 5. We define the prop D (resp. $\mathrm{D}_{\mathrm{open}}$) as the cellular chains on $\mathscr{G}$ (resp. $\mathscr{G}_{\mathrm{open}}$).

Then, the sequence (3.1) gives the desired quasi-isomorphism of props

$$\mathrm{D} \simeq \mathrm{OC}, \tag{3.3}$$

while the sequence (3.2) gives the quasi-isomoprhism of props

$$\mathrm{D}_{\mathrm{open}} \simeq \mathrm{O}. \tag{3.4}$$

3.2. **Further describing the models.** Giving an explicit presentation of D and $\mathrm{D}_{\mathrm{open}}$ as props, together with a description of their differentials, is the only missing piece missing to proceed to the proof of Theorem 1. These amount to a description of the boundary in the corresponding moduli spaces. Without spelling out the full details, we have that:

- $\mathrm{D}_{\mathrm{open}}$ is generated by the disks, modulo unitality and cyclic symmetry.
- D is generated by disks and annuli, modulo some compositions of a disk with one outgoing marked point to an annuli, and unitality relations.

Moreover, the differentials are illustrated in Fig. 3.

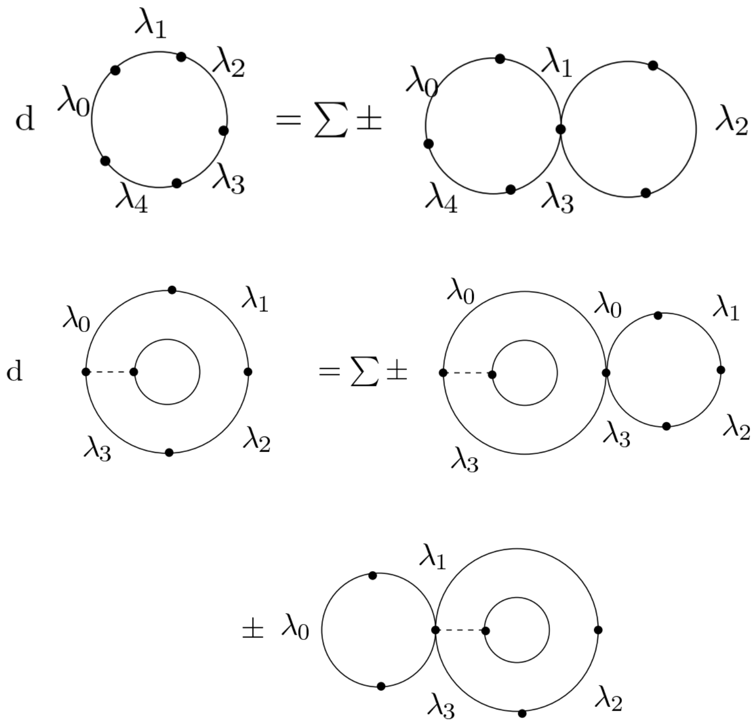

FIGURE 3. The differential of a chain given by marked points on a disk/annulus. The interior circle of the annulus is a closed outgoing boundary, the marked points on the exterior may be incoming or outgoing open. Illustration from [4].

Proof of Theorem 1. We have

(1) Given (3.4), one needs to identify (weak) $\mathrm{D}_{\mathrm{open}}$-algebras with (extended) Calabi–Yau A_∞-algebras. This is immediate from the presentation of $\mathrm{D}_{\mathrm{open}}$ and the explicit formula for its differential.
(2) Given (3.3), one needs to show the result for weak D-algebras. This follows from the presentation of D, using a natural filtration on the generators which allows one to reduce to proving the result at the level of the associated graded. One sets the identity elements to be in filtration degree 0, and annuli with $n+1$ marked points on the boundary in filtration degree n.
(3) One can always strictify A_∞-categories, therefore it is sufficient to identify the usual Hochschild complex of an associative algebra. This is straightforward using the presentation of D and the formula for its differential.

□

Remark 3.2. Point (1) above says that the (wheeled) prop $\mathrm{D}_{\mathrm{open}}$ is the (wheeled) propic envelope of the cyclic A_∞ operad.

4. EXERCISES

4.1. **Wheeled props.** We give an alternative definition of prop and of algebra over a prop. We also give the definition of a *wheeled prop*.

We work in the category of chain complexes over a field $\mathbb{K}$ of characteristic zero.

Definition 6. A *prop* is a family of $(\mathbb{S}_m, \mathbb{S}_n)$-bimodules $\{\mathrm{P}(m,n)\}_{m,n\geqslant 0}$ endowed with horizontal composition

$$\otimes : \mathrm{P}(m_1,n_1)\otimes \mathrm{P}(m_2,n_2) \to \mathrm{P}(m_1+m_2, n_1+n_2)$$

vertical composition

$$\circ : \mathrm{P}(m,n)\otimes \mathrm{P}(n,k) \to \mathrm{P}(m,k),$$

together with a unit $e\in \mathrm{P}(1,1)$, which satisfy associativity, unitality and equivariance axioms. A *wheeled prop* is further equipped with contraction maps

$$\xi^i_j : \mathrm{P}(m,n) \to \mathrm{P}(m-1,n-1),\ 1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n,$$

which satisfy compatibility axioms.

Question 1. Show that the datum of a prop in the sense of Definition 6 is equivalent to the datum of a prop in the sense of Definition 1.

Question 2. What is the analogue of Definition 1 in the case of a wheeled prop?

Question 3. Give a proof of Lemma 1, i.e. show that OC is indeed a prop. Use your favourite definition.

Example 4.1. The *endomorphism prop* of a chain complex V is given in arity (m,n) by the chain complex

$$\mathrm{End}_V(m,n) := \mathrm{Hom}(V^{\otimes m}, V^{\otimes n}),$$

with unit the identity map, horizontal composition given by the tensor product of chain maps and vertical composition given by ordinary composition of chain maps. If V is finite-dimensional, the *trace* $\mathrm{Hom}(V,V)\cong V^*\otimes V\to\mathbb{K}$ generalizes to contraction maps

$$\xi^i_j : \mathrm{End}_V(m,n)\cong (V^*)^{\otimes m}\otimes V^{\otimes n}\to (V^*)^{\otimes(m-1)}\otimes V^{\otimes(n-1)}\cong \mathrm{End}_V(m-1,n-1)$$

given by evaluating the ith form at the jth input, which makes End_V into a wheeled prop. More generally, this makes sense whenever V is equipped with a symmetric non-degenerate bilinear form $\langle -,-\rangle$.

A *morphism* of wheeled props is a morphism of $(\mathbb{S},\mathbb{S})$-bimodules $\mathrm{P}\to\mathrm{Q}$ which preserves the horizontal and vertical composition maps, unit, and contraction maps.

Definition 7. An *algebra* V over a (wheeled) prop P is a morphism of (wheeled) props $\mathrm{P}\to\mathrm{End}_V$.

Question 4. Show that the datum of an algebra over a (wheeled) prop in the sense of Definition 7 is equivalent to the datum of a strong algebra over a (wheeled) prop in the sense of Definition 2 (use Question 2 for the wheeled case).

4.2. **A-infinity categories.**

Definition 8. An *operad* is a family of $\mathbb{S}_n$-modules $\{\mathrm{P}(n)\}_{n\geqslant 1}$ endowed with with composition maps

$$\circ_i : \mathrm{P}(m)\otimes\mathrm{P}(n)\to\mathrm{P}(m+n-1),\ 1\leqslant i\leqslant m,$$

and a unit $\mathrm{id}\in\mathrm{P}(1)$, which satisfy associativity, unitality and equivariance.

A *non-symmetric operad* is a family of chain complexes $\{\mathrm{P}(n)\}_{n\geqslant 1}$ endowed with with composition maps $\circ_i$ and a unit $\mathrm{id}\in\mathrm{P}(1)$, which satisfy associativity and unitality.

Definition 9. The (shifted) A_∞ *operad* is the free operad on generators μ_n, $n\geqslant 2$ of degree (-1), with differential defined by

$$-d(\mu_n) = \sum_{\substack{p\geqslant 0, q\geqslant 2, r\geqslant 1\\ p+q+r=n}} \mu_{p+1+r}\circ_{p+1}\mu_q.$$

The *endomorphism operad* of a chain complex A is defined by $\mathrm{End}_A(n) := \mathrm{Hom}(A^{\otimes n}, A)$.

An *algebra* A over an operad P is a morphism of operads $\mathrm{P}\to\mathrm{End}_A$.

Definition 10. An A_∞-*algebra* is a chain complex A endowed with operations $m_n : A^{\otimes n}\to A$, $n\geqslant 2$ of degree (-1), which satisfy the relations

$$-\partial(m_n) = \sum_{\substack{p\geqslant 0, q\geqslant 2, r\geqslant 1\\ p+q+r=n}} m_{p+1+r}(\mathrm{id}^{\otimes p}\otimes m_q\otimes \mathrm{id}^{\otimes r}).$$

Here, the boundary of a multilinear map is defined by $\partial(f) := d_A f - (-1)^{\deg f} f d_{A^{\otimes n}}$.

Definition 11. The *totalisation* of an operad P is the direct sum
$$\mathrm{Tot}(\mathrm{P}) := \bigoplus_{n\geqslant 1} \mathrm{P}(n).$$

The totalisation of an operad is a Lie algebra, with bracket given on $\mu \in \mathrm{P}(m)$ and $\nu \in \mathrm{P}(n)$ by
$$[\mu,\nu] := \sum_{i=1}^{m} \mu \circ_i \nu - (-1)^{\deg\mu\deg\nu} \sum_{j=1}^{n} \nu \circ_j \mu.$$

Question 5. Let A be an A_∞-algebra. Show that $\alpha := d_A + \sum_{n\geqslant 2} m_n$ is a Maurer–Cartan element in the totalisation Lie algebra of the endomorphism operad End_A.

Given a dg Lie algebra $\mathfrak{g}$ and a Maurer–Cartan element $\alpha \in \mathfrak{g}$ (i.e. a degree (-1) element such that $d\alpha + \frac{1}{2}[\alpha,\alpha] = 0$), one can twist the differential: setting $d_\alpha := d + [-,\alpha]$ defines a new dg Lie algebra structure on $\mathfrak{g}$ denoted $\mathfrak{g}^\alpha$.

Definition 12. The *Hochschild complex* of A is the chain complex $\mathrm{Tot}(\mathrm{End}_A)^\alpha$.

Question 6. Unravel the definitions and give an explicit description of the Hochschild complex. Check that you recover the usual explicit definition.

4.3. **Remaining interrogations.**

Question 7. How does one prove Theorem 2 in full generality? Why can one reduce to the open case?

Question 8. The proof of Point (3) in Theorem 1 uses strictification (i.e. "every A_∞-algebra is quasi-isomorphic to a dg algebra"). How would the proof look like without it? Is it just a matter of keeping track of the signs, as Costello's says?

Question 9. What is the difference between a Calabi–Yau A_∞-category as defined by Costello, and a cyclic A_∞-category?

Question 10 (Bingyu)**.** Why does Costello's theorem hold over $\mathbb{Q}$? Why not over $\mathbb{Z}$?

4.4. **Extra material.** Given a cooperad C and an operad P, one defines the *convolution operad* $\mathrm{Hom}_{\mathbb{S}}(\mathrm{C},\mathrm{P})$ as follows. In arity n, it is given by the space of $\mathbb{S}$-module maps
$$\mathrm{Hom}_{\mathbb{S}}(\mathrm{C},\mathrm{P})(n) := \mathrm{Hom}_{\mathbb{S}_n}(\mathrm{C}(n),\mathrm{P}(n)).$$
Partial composition maps are defined by the convolution product
$$f \circ_i g := \gamma_{\mathrm{P}}^{(i)}(f\otimes g)\Delta_{\mathrm{C}}^{(i)},$$
where $\Delta_{\mathrm{C}}^{(i)}$ (resp. $\gamma_{\mathrm{P}}^{(i)}$) is the infinitesimal decomposition (resp. composition) map of C (resp. P). The unit is given by the composite $\mathrm{C} \to I \to \mathrm{P}$ of the counit of C with the unit of P.

The associative operad As is the non-symmetric operad which has a single generator μ_n in each arity. Composition is defined by $\mu_m \circ \mu_n := \mu_{m+n-1}$. Denote by As^* its linear dual, which is a cooperad.

Proposition 3. *The datum of an* A_∞*-algebra structure on* A *is equivalent to a Maurer–Cartan element in the totalisation of the convolution operad*
$$\mathrm{Tot}(\mathrm{Hom}(\mathrm{As}^*,\mathrm{End}_A)).$$

Question 11 (Extra)**.** Show that for any non-symmetric operad P, there is an isomorphism of operads
$$\mathrm{P} \cong \mathrm{Hom}(\mathrm{As}^*,\mathrm{P}).$$
Deduce that the datum of an A_∞-algebra structure on A is in fact equivalent to the datum of a Maurer–Cartan element in $\mathrm{Tot}(\mathrm{End}_A)$.

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