(Classical) Gromov-Witten.

Idea: Count Curves. $f: \Sigma \to X$. (sm. proj)

e.g. 27 lines on cubic surface

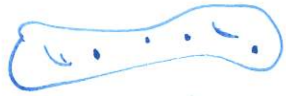
N_d = # ~~lines~~ degree d lines passing through $3d-1$ points.

$N_1 = 1$, $N_2 = 1$, $N_3 = 12$, $N_4 = 620$, $N_5 = 87304$ ...

$\beta \in H_2(X)$

discrete auto.

$\overline{\mathcal{M}}_{g,n}(X,\beta) := \mathcal{M}\{$ stable maps : $f: (\Sigma_g, p_1, \dots, p_n) \to X$, $f_*[C] = \beta$ $\}$



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Fact: $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a proper DM-stack (cpt orbifold)

e.g. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^m, H) \simeq Gr_{1,m}$.

~~$f: \Sigma \to X$.~~ dim $\overline{\mathcal{M}}$?

$f: \Sigma \to X$ $\qquad 0 \to T_\Sigma \to f^*T_X \to N_{\Sigma/X} \to 0$

$$\begin{array}{ccccccc} & \mathrm{Aut}(\Sigma). & & \mathrm{Def}(f). & & \mathrm{Def}(f,\Sigma). \\ 0 \to & H^0(\Sigma, T_\Sigma) & \to & H^0(\Sigma, f^*T_X) & \to & H^0(N_{\Sigma/X}) \\ \to & H^1(T_\Sigma) & \to & H^1(f^*T_X) & \to & H^1(N_\Sigma) = 0. \\ & \mathrm{Def}(\Sigma). & & \mathrm{Ob}(f). & & \mathrm{Ob}(f,\Sigma). \end{array}$$

$$\mathrm{vdim}\, \overline{M}(X,\beta) = h^0(N) - h^1(N).$$

$$= \cancel{h^0(\Sigma) - h^1(T)}\; h^0(f^*T_X) - h^1(f^*T_X)$$
$$+ h^1(T_\Sigma) - h^0(T_\Sigma).$$

$$\overset{RR.}{=} \langle c_1(T_X), \beta\rangle + \dim X(1-g)$$
$$+ 3g-3+n.$$

$$= \langle c_1(X), \beta\rangle + (\dim X - 3)(1-g) + n.$$

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e.g. ~~dim X = 3.~~ X: CY3.

$\rightsquigarrow \dim \overline{M}_{g,0}(X,\beta) = 0.$

$ev_i : \overline{M}_{g,n}(X,\beta) \to X.$

$$N_d = \int_{[\overline{M}_{0,3d}(\mathbb{P}^2, d)]} ev_1^*(p) \cup ev_2^*(p) \cup \cdots \cup ev_n^*(p).$$

Thm. $\exists$ well-defined.

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \in H_{2\,vdim}(X,\mathbb{Q})$$

"$\|$"

$$e(Ob) \cap [\overline{\mathcal{M}}_{g,n}(X,\beta)]$$

(e.g. $\beta = 0$, $\overline{\mathcal{M}}_{g,n}(X,\beta_0) \cong \overline{\mathcal{M}}_{g,n} \times X$)

Def. (GW). $\gamma_i \in H^*(X)$.

$$\langle \gamma_1, \dots, \gamma_n \rangle^X_{g,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n).$$

$$N_d = \langle [pt]^{3d-1} \rangle^{\mathbb{P}^2}_d.$$

Thm (Kontsevich).

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} N_{d_1}N_{d_2}\left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1}\right).$$

Rmk. $\varphi := \sum_{d=1}^{\infty} N_d \cdot \frac{z^{3d-1}}{(3d-1)!} e^{dy}$, ~~$N(1)=1$~~

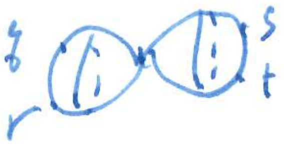
WDVV: $\varphi_{zzz} = \varphi_{yyz}^2 - \varphi_{yyy}\,\varphi_{yzz}$.

4. proof. Consider $\overline{M}_{0,3d}(\mathbb{P}^2, d)$.

$$\{1,2,\dots,n-4,\ q,r,\ s,t\}$$

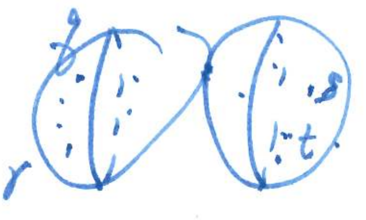
$$\pi : \overline{M}_{0,n}(\mathbb{P}^2, d) \to \overline{M}_{0,4} \simeq \mathbb{P}^1.$$

$$\pi^{-1}\, D(\{q,r\}|\{s,t\}) \sim D(\{q,s\}|\{r,t\})$$

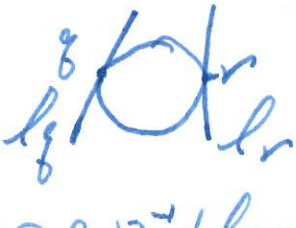


$$\|$$

$$\sum_{\substack{q,r\in A\\ s,t\in B\\ d_1+d_2=d}} D(A,d_1|B,d_2) = \sum_{\substack{q,s\in A\\ r,t\in B\\ d_1+d_2=d}} D(A,d_1|B,d_2)$$



$Y \subset \overline{M}_{0,n}(\mathbb{P}^2, d)$.



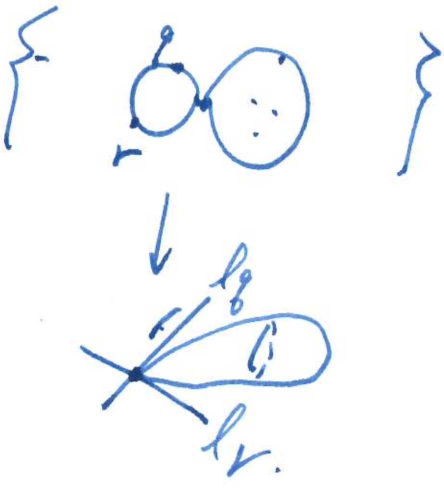
$$Y = ev_1^{-1}(z_1) \cap \cdots \cap ev_{n-4}^{-1}(z_{n-4}) \cap ev_q^{-1}(l_q)$$
$$\cap\, ev_r^{-1}(l_r) \cap ev_s^{-1}(z_s) \cap ev_t^{-1}(z_t)$$

$\dim Y = 1$.

$$Y \cap D(\{q,r\},0 \,|\, \{1,\dots,n-4,s,t\}, d) = N_d.$$

5.

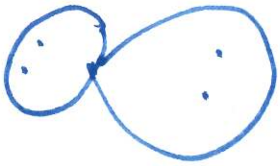
$\# \Upsilon \cap \pi^{-1} D_1(\cdots) = \# \Upsilon \cap \pi^{-1}(D_2(\cdots))$

$|\Upsilon \cap D_1(A, q | B, d)| =$ 

IS

N_d.

$1 \le d_1 \le d-1$.



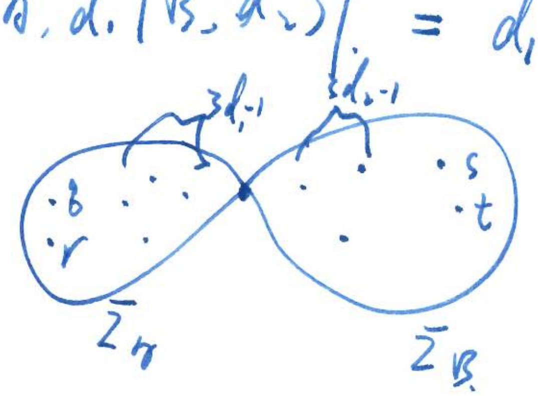
$|A| = 3d_1 + 1$

$|A| - 2 \ge 3d_1 - 1$ $\qquad |A| + |B| - 2 = n - 2 = 3d - 2$

$|B| \ge 3d_2 - 1$. $\qquad \Rightarrow |A| = 3d_1 + 1$

$\binom{3d-4}{3d_1-1}$ partitions. $\quad q, r \in A.$ $\quad s, t \in B.$ $\qquad |A| = 3d_1 + 1$

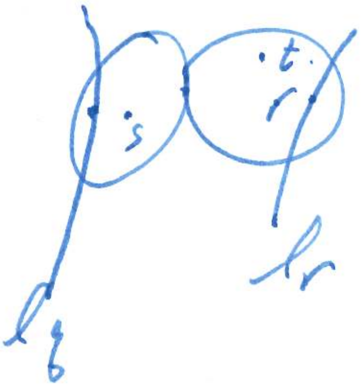
$\# |\Upsilon \cap D(A, d_1 | B, d_2)| = d_1^3 d_2 \cdot \binom{3d-4}{3d_1-1}$



$|\Sigma_A \cap l_q| = d_1$

$|\Sigma_A \cap l_r| = d_1$ $\qquad |f(\Sigma_1) \cap f(\Sigma_2)| = d_1 d_2$

$Y \cap D_2(A, d_1 \mid B, d_2)$
$\quad q, s \qquad r, t.$



$\underline{d_2 = 0.}$ $\quad Y \cap D_2(A, d_0 \mid B, 0) = \varphi$
$\qquad\qquad\qquad 0 \quad d_1 = \varphi.$

$1 \le d_1 \le d-1.$

$|A| - 1 \le 3d_1 - 1$
$|B| - 1 \le 3d_2 - 1$ $\rightsquigarrow \binom{3d-4}{3d_1-2}$ partition.

$|l_q \cap \Sigma_A| = d_1$ $\qquad |\Sigma_A \cap \Sigma_B| = d_1 d_2.$
$|l_r \cap \Sigma_B| = d_2$

$\Rightarrow \# Y \cap \pi^{-1} D_2 = N_{d_1} \cdot N_{d_2} \cdot d_1^2 d_2^2 \binom{3d-4}{3d_1-2}.$

7. Axioms for GW.

$\underset{\sim}{V} = X$. $\quad B \subset H_2(V, \mathbb{Z})$. Kähler cone.

Def. A (tree level) system of GW:

$$I^V_{g,n,\beta} : H^*(V, \mathbb{Q})^{\otimes n} \to H^*(\overline{M}_{g,n}, \mathbb{Q})$$

(tree: $g=0$)

~~2.2.0~~.

Axiom 0. $\quad I^V_{g,n,\beta} = 0. \quad \beta \notin B.$

1. S_n - invariant.

2. $\left| I^V_{g,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n) \right| = \sum |\gamma_i| + 2(K_V \cdot \beta)$
$+ (2g-2) \dim_{\mathbb{C}} V$

Basic:

$$I^V_{0,3,\beta}(\gamma_1 \otimes \gamma_2 \otimes \gamma_3).$$

$$I^V_{1,1,\beta}(\gamma). \quad I^V_{g,0,\beta}(1). \quad g \geq 2.$$

$$\langle I^V_{g,n,\beta} \rangle (\gamma_1 \otimes \cdots \otimes \gamma_n) := \int_{\overline{M}_{g,n}} I^V_{g,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n)$$

codim 0.

Axiom 3. $\pi_n : \overline{M}_{g,n} \to \overline{M}_{g,n-1}$

$$\langle I^\vee_{0,3,\beta} \rangle (\gamma_1 \otimes \gamma_2 \otimes e_\nu^0) = \begin{cases} 0, & \beta \neq 0 \\ \int_V \gamma_1 \wedge \gamma_2, & \beta = 0. \end{cases}$$

$$I^\vee_{g,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{n-1} \otimes e_\nu^0) = \pi_n^* I^\vee_{g,n-1,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{n-1})$$

Axiom 4. (Divisor). $|\gamma_n| = 2.$

$$\langle I^\vee_{g,n,\beta} \rangle \langle \gamma_1 \otimes \cdots \otimes \gamma_n \rangle = \int_\beta \gamma_n \cdot \langle I^\vee_{g,n-1,\beta} \rangle \cdot (\gamma_1 \otimes \cdots \otimes \gamma_{n-1}).$$

codim 0

$$\pi_{n*}(I^\vee_{g,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n)) = \int_\beta \gamma_n \cdot I^\vee_{g,n-1,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{n-1})$$

Axiom 5. $$I^\vee_{0,n,0}(\gamma \cdots) = \int_V \gamma_1 \wedge \cdots \wedge \gamma_n \cdot e^0_{\overline{M}_{0,n}}$$

Axiom 6. Splitting. $S_1 \sqcup S_2 = \{1, \cdots, n\}$

$$\varphi_S^*(I^\vee) = \varepsilon(S) \sum I^\vee_{g_1,n_1+1,\beta_1}((\otimes_{j \in S_1} \gamma_j) \otimes \Delta_a)\, g^{ab} \otimes I^\vee_{g_2,n_2+1,\beta}(\Delta_b \otimes_{j \in S_2} \gamma_j)$$

Thm. Basic $\Rightarrow$ all.

Quantum coh.

$$\langle \gamma_1 * \gamma_2, \gamma_3 \rangle \simeq \sum_\beta \langle \gamma_1, \gamma_2, \gamma_3 \rangle^X_\beta \, e^{-\beta \cdot \omega}$$

Thm. $*$ is associative.

$$\Phi(\gamma) = \sum_{n,\beta} e^{-\beta} \cdot \frac{1}{n!} \langle I^V_{0,n,\beta} \rangle (\gamma^{\otimes n})$$

$$\gamma = \sum t_i T_i .$$

· $$C_{ijk} = \frac{\partial^3 C}{\partial t_i \partial t_j \partial t_k}$$

$$\langle T_i \cdot T_j \rangle = g_{ij} .$$

$$T_i * T_j = \sum C_{ije} g^{ef} T_f .$$

WDVV. $\Leftrightarrow$ $*$ associative $\Leftrightarrow$ $(H^*(V), \langle\,\rangle)$ is a Frobenius mfd.

$$C_{ije} g^{ef} C_{fkl} = C_{ile} g^{ef} C_{fjk} .$$