NOTES ON CATEGORICAL ENUMERATIVE INVARIANTS

GUILLAUME LAPLANTE-ANFOSSI

ABSTRACT. The goal is to present Caldararu-Tu's construction of categorical enumerative invariants by introducing some basic definitions and sketching the proof strategy.

1. NON-COMMUTATIVE PRELIMINARIES

In this section we use homological degree conventions. We leave precise signs aside for the moment.

1.1. **Cyclic A-infinity algebras.** A shifted A_∞*-algebra* is a differential graded $\mathbb{K}$-vector space A endowed with multilinear maps

$$m_n : A^{\otimes n} \to A, \quad n \geqslant 2,$$

of degree $n-2$, which satisfy the A_∞*-relations*

$$\partial(m_n) = \sum_{\substack{p\geqslant 0, q\geqslant 2, r\geqslant 1 \\ p+q+r=n}} \pm m_{p+1+r}(\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r}).$$

Here, the boundary of a multilinear map is defined by $\partial(f) := d_A f - (-1)^{\deg f} f d_{A^{\otimes n}}$.

An A_∞-algebra A is *cyclic* if it is endowed with a non-degenerategraded symmetric bilinear form $\langle -,- \rangle : A \otimes A \to \mathbb{K}$ of fixed degree d, such that for all $a_0, \ldots, a_n \in A$ we have

$$\langle m_n(a_0, \ldots, a_{n-1}), a_n \rangle = \pm \langle m_n(a_1, \ldots, a_n), a_0 \rangle.$$

The degree d of the pairing is called the *Calabi–Yau dimension* of A.

1.2. **Hochschild chains.** We denote by $L := CH_\bullet(A)$ the *Hochschild chain complex* of A, i.e. the complex $\bigoplus_{n\geqslant 0} A \otimes A^{\otimes n}$ with differential $b : L \to L$ of degree -1 given by the formula

$$\begin{aligned} b(a_0 \otimes \cdots \otimes a_n) \quad := \quad & \sum_{k\geqslant 1} \sum_{i=0}^{n-k+1} \pm a_0 \otimes \cdots \otimes a_{i-1} \otimes m_k(a_i, \ldots, a_{i+k-1}) \otimes a_{i+k} \otimes \cdots \otimes a_n \\ & + \sum_{k\geqslant 1} \sum_{i=n-k+2}^{n} \pm m_k(a_i, \ldots, a_n, a_0, \ldots, a_{i+k-n-2}) \otimes a_{i+k-n-1} \otimes \cdots \otimes a_{i-1}. \end{aligned}$$

We have a circle action given by Connes operator $B : L \to L$, of degree $+1$, via the formula:

$$B(a_0 \otimes \cdots \otimes a_n) \quad := \quad \sum_{i=0}^{n} \pm 1 \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

The two satisfy the relations

$$b^2 = 0, \; B^2 = 0, \; (b+B)^2 = 0. \tag{1.1}$$

Adding a formal variable u of degree -2, we can then form the *periodic cyclic, negative cyclic* and *cyclic* mixed graded complexes

$$\begin{aligned} L^{\mathrm{Tate}} \quad &:= \quad (L((u)), \; b+uB), \\ L_+ \quad &:= \quad (L[[u]], \; b+uB), \\ L_- \quad &:= \quad (L[u^{-1}], \; b+uB) = L^{\mathrm{Tate}}/L_+. \end{aligned}$$

The natural u-adic filtration induces the *non-commutative Hodge filtration* in homology $H_\bullet(L_+)$. We say that A has the *Hodge-de Rham degeneration property* if the associated spectral sequence collapses at the first page.

1.3. **Splitting of the Hodge filtration.** Using the pairing and A_∞ operations of A [9, Prop. 5.22], one defines the (chain-level) *Mukai pairing*

$$\langle -,-\rangle_{\mathrm{Muk}} : L\otimes L \to \mathbb{K},$$

with respect to which the circle operator B is self-adjoint. This allows one to define the *higher residue pairing*

$$\langle -,-\rangle_{\mathrm{hres}} : L^{\mathrm{Tate}}\otimes L^{\mathrm{Tate}} \to \mathbb{K}((u))$$

via the formula

$$\left\langle \sum_k x_k u^k, \sum_l y_l u^l \right\rangle_{\mathrm{hres}} := \sum_{k,l} (-1)^l \langle x_k, y_l\rangle_{\mathrm{Muk}}\, u^{k+l}.$$

Since B is self-adjoint with respect to the Mukai pairing, both pairings descend to homology.

Definition 1. A *splitting* of the non-commutative Hodge filtration of A is a graded map of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces

$$s : H_\bullet(L) \to H_\bullet(L_+)$$

such that

(1) it splits the canonical projection $H_\bullet(L_+) \to H_\bullet(L)$, and
(2) for any a,b in $H_\bullet(L)$, we have $\langle a,b\rangle_{\mathrm{Muk}} = \langle s(a), s(b)\rangle_{\mathrm{hres}}$.

2. Definition of categorical enumerative invariants

In this section, we let A be a $\mathbb{Z}/2\mathbb{Z}$-graded cyclic A_∞-algebra over a field of characteristic zero, and we suppose further that A is smooth, finite dimensional, unital, and satisfies the Hodge-de Rham degeneration property.

Let $\mathscr{M}_{g,k,l}$ denote the moduli space of Riemann surfaces of genus g with $k+l$ marked points, k labeled as inputs and l labeled as outputs. In order to define categorical enumerative invariants, Caldararu–Tu consider the ribbon graphs model $M_{g,k,l}$ for $C_\bullet(\mathscr{M}_{g,k,l})$, see [10, 8, 6, 7, 2]. This is a bicolored dg-prop, whose totalisation is a Lie algebra[1]

$$\mathfrak{g} := \mathrm{Tot}(M_{g,k,l}). \tag{2.1}$$

Theorem 1 ([3])**.** *There is a canonical element $\upsilon \in \mathfrak{g}$ which satisfies the Maurer–Cartan equation*

$$d\upsilon + \frac{1}{2}[\upsilon,\upsilon] = 0,$$

and is unique up to homotopy (i.e. gauge equivalence between Maurer–Cartan elements).

This element is called the *string vertex*, and its genus g, k,l-points components $\upsilon_{g,k,l}$ are referred to as *string vertices*.

It is built out of the fundamental classes of the moduli spaces $\mathscr{M}_{g,k,l}$, and can be computed recursively. Indeed, according to Caldararu–Tu [1, Sec. 5.4] the Maurer–Cartan equation determines completely $\upsilon_{g,k,l}$ from previously computed string vertices; it thus suffices to fix $\upsilon_{0,1,2}$ to specify υ uniquely.

Theorem 2 ([2, 10])**.** *The Hochschild chains of any cyclic A_∞-algebra A form an open-closed TCFT. That is, we have a morphism of props $C_\bullet(\mathscr{M}_{g,k,l}) \to \mathrm{End}_L$, or equivalently a morphism of props*

$$\rho : M_{g,k,l} \to \mathrm{End}_L\,. \tag{2.2}$$

Denoting by

$$\mathfrak{h}_A := \mathrm{Tot}(\mathrm{End}_L), \tag{2.3}$$

the totalisation Lie algebra of End_L, we get a Lie algebra morphism

$$\mathfrak{g} \xrightarrow{\rho} \mathfrak{h}_A\,. \tag{2.4}$$

[1]Note that I am using here notations that differ slightly from the ones in [1].

The image of the string vertex υ under this morphism encodes the Gromov–Witten type invariants. In order to obtain a power series in Hochschild (or cyclic) homology classes, Caldararu–Tu use further a splitting of the non-commutative Hodge filtration of A in order to define a formality L_∞ quasi-isomorphism

$$\mathfrak{h}_A \overset{s}{\rightsquigarrow} \mathfrak{h}_A^{\mathrm{triv}} \tag{2.5}$$

between the $\mathfrak{h}_A$ and itself, with trivial bracket.

Definition 2. *Categorical enumerative invariants* are the genus g, n-point components $F_{g,n}^{A,s}$ of the image of the string vertex $\upsilon \in \mathfrak{g}$ under the composite $\mathfrak{g} \xrightarrow{\rho} \mathfrak{h}_A \overset{s}{\rightsquigarrow} \mathfrak{h}_A^{\mathrm{triv}}$.

Caldararu–Tu do not provide an explicit description of the morphism ρ, and rely on its existence, as proved in [2]. For the present purposes, only the images of the string vertices are needed, and they will be simply defined as is in Theorem 3 below. However, as shown in [10, Sec. 6.2], the action ρ can be made explicit following the Kontsevich–Soibelman recipe [7, pp. 58.62].

The L_∞ morphism s is defined as a sum-over-graphs formula (the graphs which appear below in Theorem 3, which are distinct from the graphs defining $M_{g,k,l}$ and $\upsilon_{g,k,l}$). It uses a chain-level lift of the splitting of the non-commutative Hodge filtration obtained via the homotopy transfer theorem, as well as the integration of a family of Lie algebras according to a method of Fukaya [5] and Fiorenza [4]. Note that Caldararu–Tu provide the formula for s, and prove that it is indeed a L_∞ morphism, without providing the derivation of it from Fukaya's method.

3. An explicit formula for categorical enumerative invariants

The main contribution of the Caldararu–Tu paper is to give a sum-over-graphs formula for $F_{g,n}^{A,s}$. Our goal now is to describe it as precisely as possible in Theorem 3 below.

3.1. **Graphs.** Let $\mathrm{Gra}(g,1,n-1,m)$ denote the set of *stable partially directed graphs*, that is graphs G with n leaves (1 labeled incoming, $n-1$ labeled outgoing) and m vertices, equipped with

- a genus labeling function $g : V(G) \to \mathbb{Z}_{\geq 0}$,
- a subset of directed edges,
- a spanning tree T,

such that

- there is no directed loop,
- each vertex has at least one incoming half-edge,
- for every vertex, the genus g and the valency η satisfy the relation $2g-2+\eta > 0$.

We denote by $\mathrm{Aut}(G)$ the size of the automorphism group of G.

3.2. **Graph weight.** To such a graph G, Caldararu–Tu associate a rational number $w(G)$ as follows. An edge e of the spanning tree T of G is *contractable* if the graph G/e obtained by contracting all the directed edges connecting the two endpoints of e is again a stable partially directed graph. Denote by T^{cont} the set of contractable edges of T, and by E^{nl} the set of non-loop edges of G. Then, the weight $w(G)$ is defined inductively by the conditions

- if G has only one vertex, $w(G) := 1$,
- otherwise $w(G) := \frac{1}{|E^{\mathrm{nl}}|} \sum_{e \in T^{\mathrm{cont}}} w(G/e)$.

3.3. **Graph decorations.** Finally, given a cyclic A_∞-algebra A endowed with a splitting s of the non-commutative Hodge filtration, Caldararu–Tu associate a certain linear map to any vertex (resp. edge or leg) of a stable partially directed graph G. The graph then indicates how these different "contributions" should be concatenated in order to obtain an element in $\mathrm{Hom}(L_-, \mathrm{Sym}^{n-1} L)$.

Using the homotopy transfer theorem, Caldararu–Tu fix an arbitrary chain-level lift S of the splitting s, as well as an inverse S^{-1}. For $\alpha = \alpha_0 + \alpha_1 u^{-1} + \cdots \in L_-$, they denote by $\Theta : L_- \to L_+[1]$ the circle action map

$$\Theta(\alpha) := B(\alpha_0)$$

They then define a bounding homotopy F for Θ, using S and S^{-1}. Analogously, there is a bounding homotopy H for the circle action $\Omega : \mathrm{Sym}^2 L_- \to \mathfrak{h}_A$,

$$\Omega(x,y) := \langle B(x_0), y_0 \rangle_{\mathrm{Muk}}$$

defined by using the maps S and S^{-1}. Finally, there is a bounding homotopy $\delta : L_- \otimes L_- \to \mathbb{K}$ which measures the failure of H to be symmetric.

We are now ready to describe each of the contributions.

3.3.1. *Vertices.* A vertex v is decorated by the string vertex corresponding to its genus and valency:

$$\mathrm{Cont}(v) := \rho(\upsilon_{g(v),k(v),l(v)})$$

It depends only on A.

3.3.2. *Legs.* The legs contributions depend on the splitting of the non-commutative Hodge filtration.

$$\mathrm{Cont}(l) := \begin{cases} S & l \text{ is incoming,} \\ S^{-1} & l \text{ is outgoing.} \end{cases}$$

3.3.3. *Directed edges.* The directed edges contributions depend on the splitting of the non-commutative Hodge filtration, and correspond to the (homotopy) circle action.

$$\mathrm{Cont}(e) := \begin{cases} F & e \text{ is in the spanning tree,} \\ \Theta & e \text{ is not in the spanning tree.} \end{cases}$$

3.3.4. *Undirected edges.* The undirected edges contributions also depend on the splitting of the non-commutative Hodge filtration.

$$\mathrm{Cont}(e) := \begin{cases} \delta & e \text{ is in the spanning tree,} \\ H & e \text{ is not in the spanning tree.} \end{cases}$$

Let us denote by $G(\upsilon, F, \Theta, \delta, H)$ the composition of the operators performed according to the graph G, in order to produce an element in $\mathrm{Hom}(L_-, \mathrm{Sym}^{n-1} L) \subset \mathfrak{h}_A^{\mathrm{triv}}$.

Theorem 3 ([1, Theorem 1.3])**.** *For any $g \geqslant 0$ and $n \geqslant 1$ such that $2g-2+n > 0$, the categorical enumerative invariant $F_{g,n}^{A,s}$ is given by the formula*

$$F_{g,n}^{A,s} = \sum_{\substack{m \geqslant 1 \\ G \in \mathrm{Gra}(g,1,n-1,m)}} (-1)^{m-1} \frac{w(G)}{\mathrm{Aut}(G)} G(\upsilon, F, \Theta, \delta, H) \tag{3.1}$$

4. Questions

Question 1. Check the relations (1.1). Note that you can use the cyclic bar construction to check $b^2 = 0$ more conceptually.

Question 2. Specialize the formula of [9, Prop. 5.22] to the case of a cyclic A_∞-algebra, and express the Mukai pairing purely in terms of the A_∞ operations and cyclic pairing.

Question 3. Show that the Mukai and higher residue pairing descend to cyclic homology.

Question 4. What is the precise statement or Theorem 2? Is it L which has the structure of an open-closed TCFT, or the pair (A, L)? Seems like L has the structure of a closed TCFT [2], while the pair is considered in [10].

Question 5. How is the formula for the L_∞ morphism derived from Fukaya [5] and Fiorenza [4] works? For the definition of partially directed graphs, Caldararu–Tu reference Getzler–Kapranov's "Modular operads" paper. Is there a deeper connection to their work?

Question 6. Can one compute the values of the string vertices given in [1, Sec. 5.4] from the choice of value of the string vertex $\upsilon_{0,1,2}$? How is this latter value chosen in the first place?

Question 7. The *genus* of a graph G is defined to be

$$g(G) := \sum_{v \in V(G)} g(v) + \operatorname{rank} H_1(G).$$

Show that all genus zero graphs have weight 1.

Question 8. Where does the graph weight come from, conceptually?

Question 9. Explain the origins of the names "non-commutative Hodge filtration", "Hodge–de Rham degeneration property", "Mukai pairing", , and "splitting of the Hodge filtration". In what sense are all these notions "non-commutative analogues" of classical notions?

Question 10. Where does the name "Tate complex" come from?

Solution (Bingyu). TBW □

Question 11 (Mingyuan)**.** Great to have Theorem 3, but how does one compute a number from it?

REFERENCES

[1] Andrei Caldararu and Junwu Tu. Effective Categorical Enumerative Invariants, April 2024. arXiv:2404.01499 [math].

[2] Kevin Costello. Topological conformal field theories and Calabi–Yau categories. *Advances in Mathematics*, 210(1):165–214, March 2007.

[3] Kevin Costello. The partition function of a topological field theory. *Journal of Topology*, 2(4):779–822, 2009.

[4] Domenico Fiorenza. Sums over graphs and integration over discrete groupoids. *Applied Categorical Structures*, 14(4):313–350, 2006.

[5] Kenji Fukaya. Cyclic symmetry and adic convergence in lagrangian floer theory. *Kyoto Journal of Mathematics*, 50(3):521–590, 2010.

[6] Veronique Godin. Higher string topology operations, February 2008. arXiv:0711.4859 [math].

[7] Maxim Kontsevich and Yan Soibelman. Notes on A_∞-algebras, A_∞-categories and non-commutative geometry. I. In Anton Kapustin, Maximilian Kreuzer, and Karl-Georg Schlesinger, editors, *Homological Mirror Symmetry: New Developments and Perspectives*, volume 757 of *Lecture Notes in Physics*, pages 153–219. Springer, Berlin, 2008.

[8] Daniela Egas Santander. Comparing fat graph models of moduli space, August 2015. arXiv:1508.03433 [math].

[9] Nick Sheridan. Formulae in noncommutative hodge theory. *Journal of Homotopy and Related Structures*, 15(1):249–299, 2020.

[10] Nathalie Wahl and Craig Westerland. Hochschild homology of structured algebras. *Adv. Math.*, 288:240–307, January 2016.