The combinatorics of the permutahedron diagonals

Guillaume Laplante-Anfossi, joint work with Bérénice Delcroix-Oger, Matthieu Josuat-Vergès, Vincent Pilaud and Kurt Stoeckl

The University of Melbourne

Copenhagen–Jerusalem Combinatorics Seminar - May 18th, 2023

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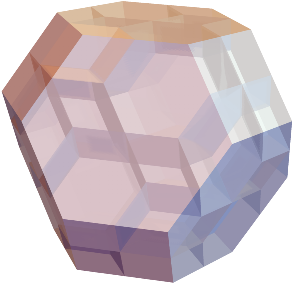

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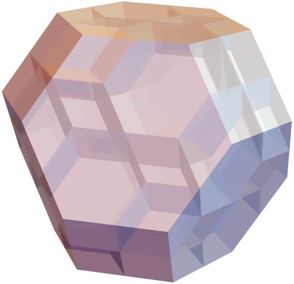

Cellular diagonals

Let P be a polytope in $\mathbb{R}^n$. The diagonal

$$\begin{array}{rcccc} \Delta & : & P & \to & P \times P \\ & & x & \mapsto & (x, x) \end{array}$$

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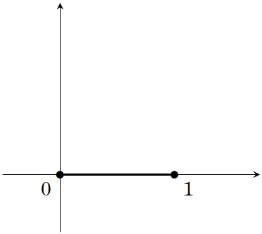

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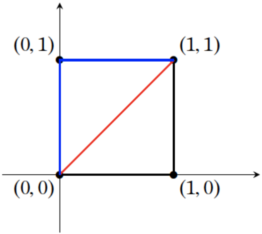
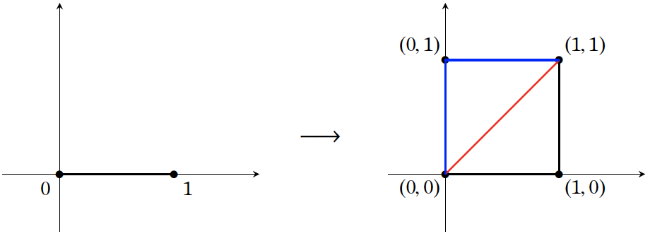


Cellular diagonals

Definition

A *cellular diagonal* of a polytope P is a continuous map $P \to P \times P$ such that

1. its image is a union of dim P-faces of $P \times P$ (i.e. it is *cellular*),
2. it agrees with the thin diagonal on the vertices of P, and
3. it is homotopic to the thin diagonal, relative to the image of the vertices.



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Example

- Simplices: Alexander–Whitney map (1935-38).
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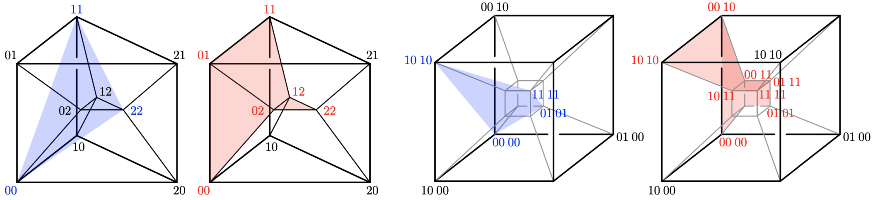

Picture: V. Pilaud.

Cellular diagonals

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- Associahedron: Saneblidze–Umble (2004), Markl–Shnider (2006), Masuda–Tonks–Thomas–Vallette (2021).
- Permutahedron: Saneblidze–Umble (2004), L.-A. (2022).

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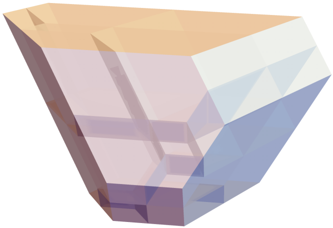

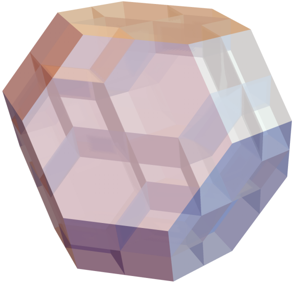

Cellular diagonals

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The *polytope of diagonals* $D_P := \Sigma(P \times P, P)$ is the fiber polytope of the projection $(x, y) \mapsto (x + y)/2$.

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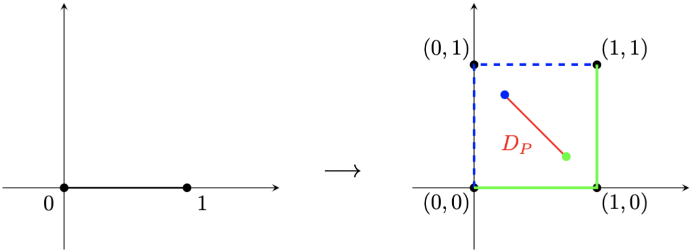

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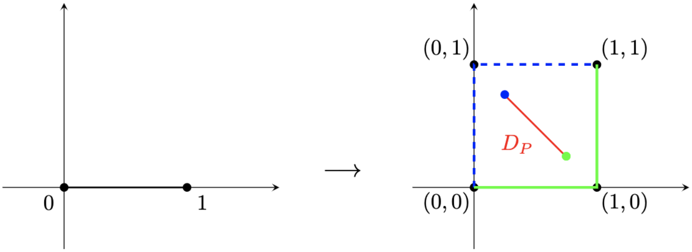

Each vertex of D_P, selected by a vector $\vec{v}$ in general position wrt P, defines a cellular diagonal $\triangle_{(P,\vec{v})}$.

Cellular diagonals

- Each cellular diagonal defines a (tight coh.) subdivision,
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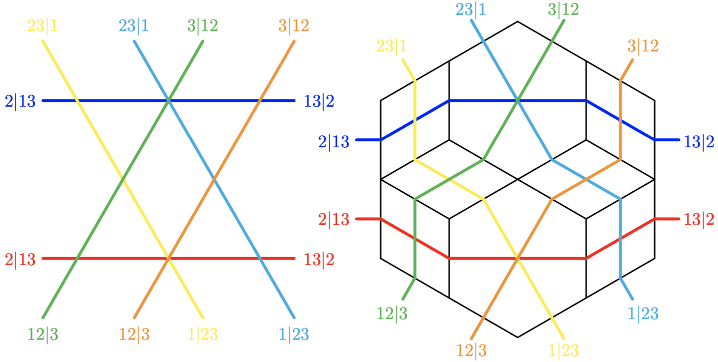

Picture: V. Pilaud

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THEOREM. *For $c \in A^p(X)$, $\tilde{c} \in A^q(X)$, the product $c \cup \tilde{c}$ in $A^{p+q}(X)$ is given by the Minkowski weight that assigns to a cone γ of codimension $p+q$ the value*

$$(c \cup \tilde{c})(\gamma) = \sum m^{\gamma}_{\sigma,\tau} \cdot c(\sigma) \cdot \tilde{c}(\tau).$$

The sum is over a certain set of cones σ and τ of codimension p and q that contain γ, determined by the choice of a generic vector v in N: σ and τ appear when $\sigma + v$ meets τ. The coefficient $m^{\gamma}_{\sigma,\tau}$ is the index $[N : N_\sigma + N_\tau]$ where $N_\sigma := \mathbf{Z}(N \cap \sigma)$ and $N_\tau := \mathbf{Z}(N \cap \tau)$.

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William Fulton and Bernd Sturmfels, *Intersection theory on toric varieties*, 1997.

Permutahedron diagonals

Definition

The $(n-1)$-dimensional *permutahedron* P_n is the convex hull of the points

$$(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n \ , \ \sigma \in \mathbb{S}_n \ .$$

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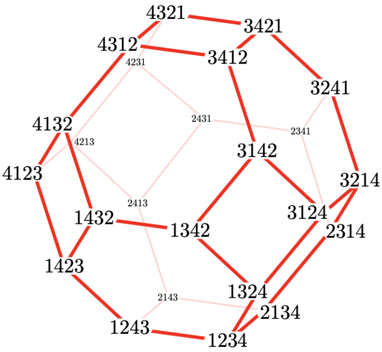

Permutahedron diagonals

The normal fan of the permutahedron is the *braid arrangement*

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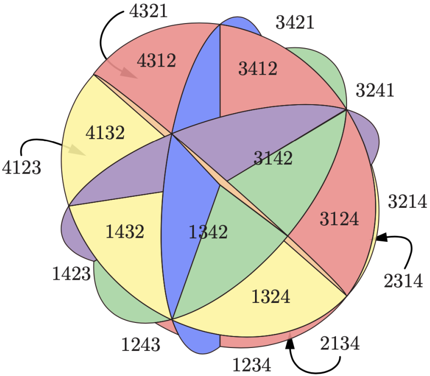

Iterated permutahedron diagonals

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For any integers $\ell, n \geq 1$, the (ℓ, n)-*braid arrangement* $\mathcal{B}_n^\ell$ is the arrangement obtained as the union of ℓ generically translated copies of the braid arrangement $\mathcal{B}_n$.

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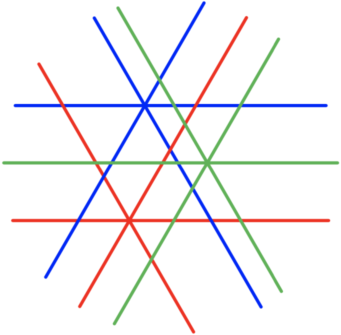

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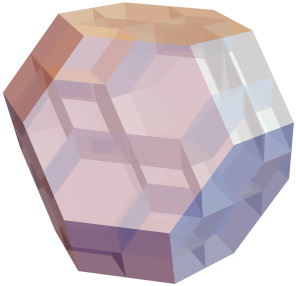

The (ℓ, n)-braid arrangement

We want to study the *flat poset* of the (ℓ, n)-braid arrangement $\mathcal{B}_n^\ell$.

Definition

A *flat* of an hyperplane arrangement $\mathcal{A}$ is a non-empty affine subspace of $\mathbb{R}^d$ that can be obtained as the intersection of some hyperplanes of $\mathcal{A}$. The *flat poset* of $\mathcal{A}$ is poset of flats of $\mathcal{A}$ ordered by reverse inclusion.

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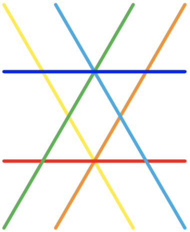

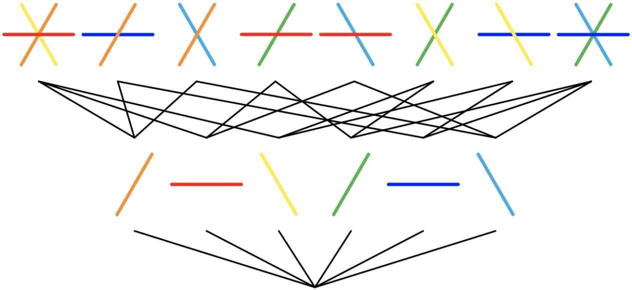

The (ℓ, n)-braid arrangement

Definition (Partition forest)

A (ℓ, n)-*partition forest* (resp. (ℓ, n)-*partition tree*) is a ℓ-tuple $(F_1, \ldots, F_\ell)$ of set partitions of $[n]$ whose intersection hypergraph is a hyperforest (resp. hypertree).

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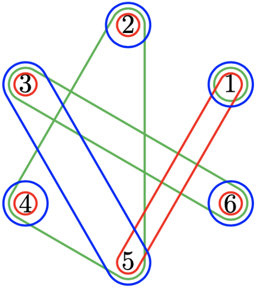

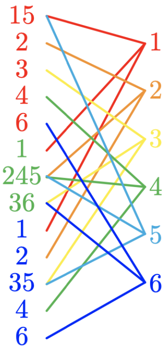

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Proposition

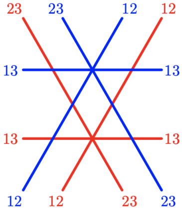
The flat poset of $\mathcal{B}_n^\ell$ is isomorphic to the (ℓ, n)-partition forest poset.

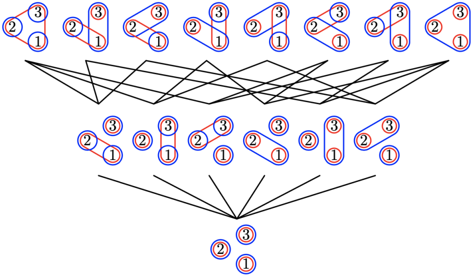
Proof.

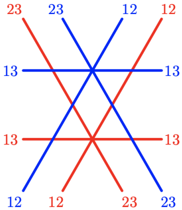
1. Each set partition corresponds to a flat of $\mathcal{B}_n$,
2. Since the ℓ copies of $\mathcal{B}_n$ are in generic position, acyclic intersection hypergraphs correspond to flats of $\mathcal{B}_n^\ell$,
3. Moreover, refinement of flats is given by componentwise refinement of partitions.

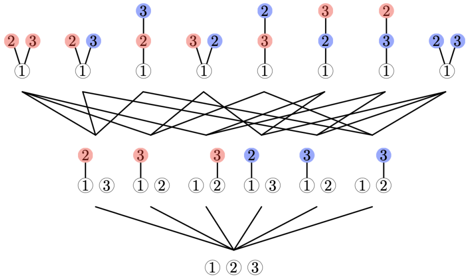
□

Partition forests and rainbow forests









Facets of the diagonal – Vertices of the arrangement

Theorem

The number of vertices of the (ℓ, n)-braid arrangement $\mathcal{B}_n^\ell$ is

$$f_0(\mathcal{B}_n^\ell) = \ell\big((\ell - 1)n + 1\big)^{n-2}.$$

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$n \backslash \ell$	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8
3	1	8	21	40	65	96	133	176
4	1	50	243	676	1445	2646	4375	6728
5	1	432	3993	16384	46305	105456	208537	373248
6	1	4802	85683	521284	1953125	5541126	13119127	27350408
7	1	65536	2278125	20614528	102555745	362797056	1029059101	2500000000
8	1	1062882	72412707	976562500	6457339845	28500625446	96889010407	274371577992

Facets of the diagonal – Vertices of the arrangement

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For any $k_1, \ldots, k_\ell$ such that $0 \leq k_i \leq n-1$ for $i \in [\ell]$ and $\sum_{i=1}^{\ell} k_i = n-1$, the number of vertices v of the (ℓ, n)-braid arrangement $\mathcal{B}_n^\ell$ such that the smallest flat of the ith copy of $\mathcal{B}_n$ containing v has dimension $n - k_i - 1$ is given by

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Use a colored Prüfer code to count (ℓ, n)-rainbow trees with k_i nodes colored by i. □

Vertices of the diagonal – Regions of the arrangement

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The numbers of regions and of bounded regions of the (ℓ, n)-braid arrangement $\mathcal{B}_n^\ell$ are given by

$$f_{n-1}(\mathcal{B}_n^\ell) = n!\,[z^n]\exp\left(\sum_{m\geq 1}\frac{F_{\ell,m}\,z^m}{m}\right)$$

and

$$b_{n-1}(\mathcal{B}_n^\ell) = (n-1)!\,[z^{n-1}]\exp\left((\ell-1)\sum_{m\geq 1}F_{\ell,m}\,z^m\right).$$

where $F_{\ell,m} := \frac{1}{(\ell-1)m+1}\binom{\ell m}{m}$ *is the Fuss-Catalan number.*

Vertices of the diagonal – Regions of the arrangement

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1. use Zaslavsky's theorem (1975), expressing the f-polynomial of the arrangement in terms of the Möbius function of the flat poset, and

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3. determine the characteristic polynomial of $\mathcal{B}_n^\ell$,
4. and conclude using generating functionology.

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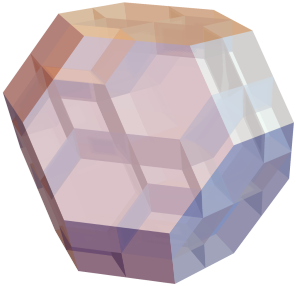

Operadic diagonals

Recall that each face $A_1|\ldots|A_k$ of the permutahedron $P_{|A_1|+\cdots+|A_k|-1}$ is isomorphic to the product $P_{|A_1|-1} \times \cdots \times P_{|A_k|-1}$ of lower dimensional permutahedra, via the isomorphism

$$\begin{array}{rccc} \Theta & : & \mathbb{R}^{|A_1|} \times \cdots \times \mathbb{R}^{|A_k|} & \\ & & (x_1, \ldots, x_{|A_1|}) \times \cdots \times (x_{|A_1|+\cdots+|A_{k-1}|+1}, \ldots, x_{|A_1|+\cdots+|A_k|}) & \\ & \xrightarrow{\cong} & \mathbb{R}^{|A_1|+\cdots+|A_k|} & \\ & \mapsto & (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(|A_1|+\cdots+|A_k|)}) & \end{array}$$

where σ is the $(|A_1|, \ldots, |A_k|)$-shuffle sending the increasingly ordered elements of $A_1 \cup \ldots \cup A_k$ to the block-by-block increasingly ordered elements of $A_1|\ldots|A_k$.

Operadic diagonals

Definition

A diagonal of the permutahedra $\triangle$ is *operadic* if for every face $A_1|\ldots|A_k$ of the permutahedron $P_{|A_1|+\cdots+|A_k|-1}$, the map Θ induces a topological cellular isomorphism

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2. *one that recovers the Saneblidze–Umble diagonal (2004) at the cellular level.*

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- $\mathrm{LA}(n) := \{(I, J) \mid \{I, J\} \in U(n),\ \min(I \cup J) = \min I\}$, and by
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Proposition

The two operadic diagonals are given by the family of vectors $\vec{v} = (v_1, \ldots, v_n)$ which satisfy

$$\sum_{i\in I} v_i > \sum_{i\in J} v_j \ , \ \forall (I, J) \in \mathrm{LA}(n) \ , \ \textit{resp.}\ \mathrm{SU}(n) \ .$$

Operadic diagonals

Theorem

There are exactly two operadic diagonals of the permutahedra, which respect the weak Bruhat order on the vertices.

Proof.

Construct inductively the family of diagonals

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2. Only one choice in dimension 1,
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Construct inductively the family of diagonals

1. Only one choice in dimension 0,
2. Only one choice in dimension 1,
3. Only one choice in dimension 2,
4. Two choices in dimension 3,
5. From dimension 4 on, the operadic property forces one to stick with one family of vectors.

□

Operahedra

Definition

An *operahedron* of dimension $k \geq 0$ is a polytope P_t whose face lattice is isomorphic to the lattice of nestings of a planar tree t with $k+2$ vertices.

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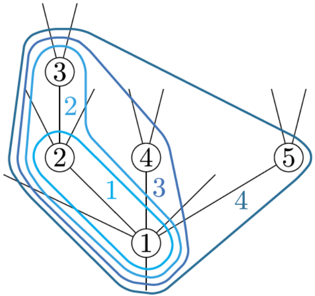

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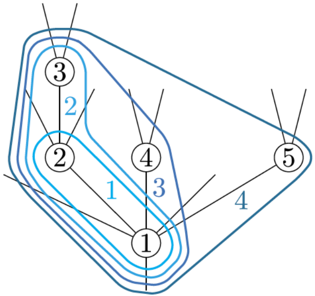

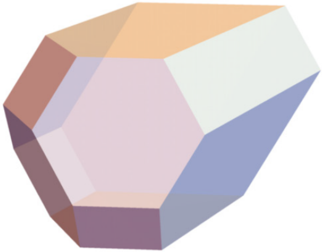

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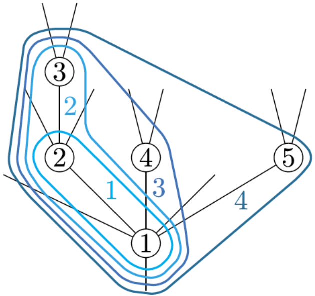

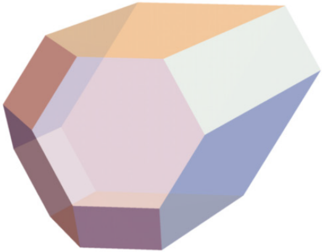

The operahedra generalize the associahedra and encode the notion of homotopy operad.

Operadic diagonals of the operahedra

Definition

An *operadic diagonal* for the operahedra is a choice of diagonal $\triangle_t$ for each operahedron P_t, such that $\triangle := \{\triangle_t\}$ commutes with the map Θ.

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Theorem

There are exactly

1. *two operadic diagonals of the Loday operahedra, therefore exactly*
2. *two colored topological cellular operad structures on the Loday operahedra, and incidentally exactly*
3. *two universal tensor products of homotopy operads,*

which agree with the generalized Tamari order on fully nested trees.

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which agree with the generalized Tamari order on fully nested trees.

Proof.

To have an operad structure on the operahedra, we need the same choice of diagonal for each subtree of a given tree. Now suppose one chooses LA for t and SU for t'. One can then find a bigger tree t'' which has both t and t' has subtrees, a contradiction. □

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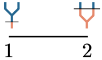

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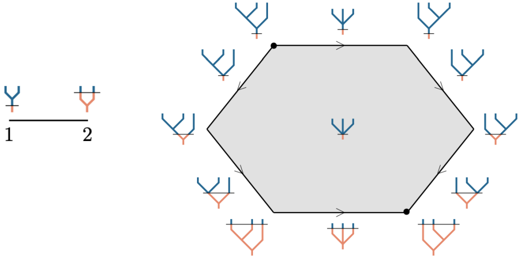

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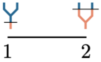

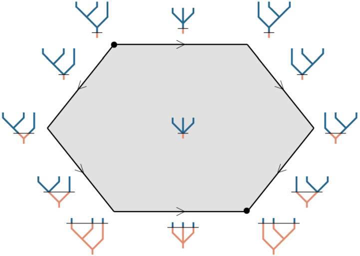

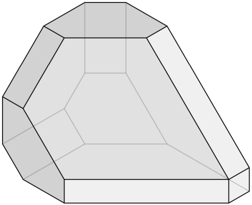

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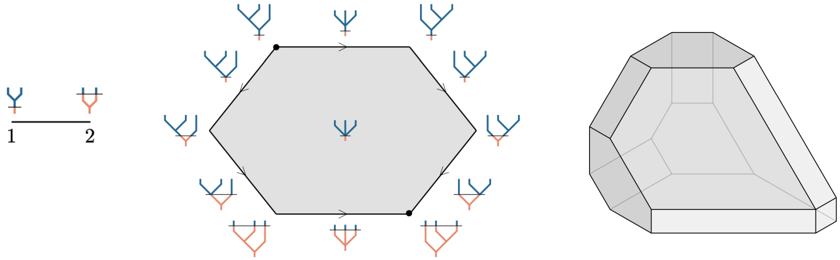

The multiplihedra encode the notion of A_∞-morphism between A_∞-algebras.

Operadic diagonals of the multiplihedra

Theorem

There are exactly

- *two operadic diagonals of the Forcey–Loday multiplihedra, therefore exactly*
- *two topological cellular operadic bimodule structures (over the Loday associahedra) on the Forcey–Loday multiplihedra, and incidentally exactly*
- *two compatible universal tensor products of A_∞-algebras and A_∞-morphisms,*

which agree with the Tamari(-type) order on (2-colored) planar trees.

Proof.

Similar to the preceding one. □

Conclusion

Thank you for your attention!