Section 6: Definition of CEI

Joel Beimler

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Contents

1 Preliminaries and the shape of the invariants

1.1 Input data, conventions

Here we review the **non-constructive** definition of the CEI. As always, we work with a cyclic unital A_∞-algebra A and associate to it the data

$$(L = C_*(A)[d], b, B, \langle \cdot, \cdot \rangle_{\mathrm{Muk}}).$$

Here, L consists of the reduced Hochschild complex of A with its standard differential b, and a circle action given by the Connes operator $B : L \to L[-1]$. We further endow it with the Mukai pairing

$$\langle \cdot, \cdot \rangle_{\mathrm{Muk}} : L \otimes L \to \mathbb{K}$$

using the A_∞-operations of A. The circle operator B is self-adjoint with respect to this pairing.

Definition 1.1. We associate to L with its circle action the following chain complexes:

$$\begin{aligned} L^{\mathrm{Tate}} &:= (L((u)), b + uB), \\ L_+ &:= (L[[u]], b + uB), \\ L_- &:= (L[u^{-1}], b + uB). \end{aligned}$$

Denote $H := H_*(L)$, which we consider as a chain complex with trivial differential and circle action. Following the conventions of the paper, define

$$H^{\mathrm{Tate}} = H((u)), \quad H_+ = H[[u]], \quad H_- = H[u^{-1}].$$

Note that these are **not** the homologies of L^{tate}, $L_\pm$.

Definition 1.2. The Mukai pairing can be upgraded to the **higher residue** pairing

$$\langle\cdot,\cdot\rangle_{\mathrm{hres}} : L^{\mathrm{Tate}} \otimes L^{\mathrm{Tate}} \to \mathbb{K}((u)).$$

The **residue pairing** is the coefficient of u^1 in the higher residue pairing:

$$\langle xu^k, yu^l\rangle_{\mathrm{res}} = \begin{cases} (-1)^l \langle x,y\rangle_{\mathrm{Muk}}, & \text{if } k+l=1, \\ 0, & \text{else.} \end{cases}$$

It is defined on L^{Tate}, but is only nondegenerate on $L_+ \otimes L_-$.

Exercise 1.3. Check that $\langle\cdot,\cdot\rangle_{\mathrm{res}} : L_+ \otimes L_- \to \mathbb{K}((u))$ is nondegenerate.

Exercise 1.4. Both $\langle\cdot,\cdot\rangle_{\mathrm{Muk}}$ and $\langle\cdot,\cdot\rangle_{\mathrm{hres}}$ descend to homology.

Definition 1.5. A **splitting of the non-comutative Hodge filtration** of A is a graded map of $\mathbb{Z}_2$-graded vector spaces

$$s : H_*(L) \to H_*(L_+)$$

such that

- s splits the canonical projection $H_*(L_+) \to H_*(L)$ given by setting $u = 1$.
- $\langle s(x), s(y)\rangle_{\mathrm{hres}} = \langle x,y\rangle_{\mathrm{Muk}}$ for any $x, y \in H_*(L)$.

1.2 The nature of the invariants

The CEI we wish to define are elements

$$F^{A,s}_{g,n} \in \mathrm{Sym}^n H_*(L)_-,$$

which give rise to **numerical invariants** by using the extension of the residue pairing to

$$\langle\cdot,\cdot\rangle : \mathrm{Sym}^n(H_*(L)_-) \otimes \mathrm{Sym}^n(H_*(L)_+) \to \mathbb{K}$$

$$\langle \alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_n\rangle := \sum_{\sigma\in S_n} \prod_{i=1}^n \langle \alpha_i, \beta_{\sigma(i)}\rangle_{\mathrm{res}}.$$

Maybe one applies some Koszul sign rule to the factors. Now taking $\gamma_i \in H_*(L)$, elements of the form $\gamma_i u^i$ for $i \geq 0$ are in $H_*(L)_+$, so we obtain numbers by evaluating

$$\langle F^{A,s}_{g,n}, (\gamma_1 u^{k_1}) \cdots (\gamma_n u^{k_n})\rangle_{\mathrm{res}} \in \mathbb{K}.$$

2 A 2d TCFT structure on L

Theorem 2.1 ([CT24, Theorem 6.1], due to [Cos06])**.** *The shifted reduced Hochschild chain complex $L = C_*(A)[d]$ of a cyclic A_∞-algebra A of CY-dimension d admits natural maps*

$$\rho^A_{g,k,l} : C^{\mathrm{comb}}_*(M^{\mathrm{fr}}_{g,k,l}) \to \mathrm{Hom}(L^{\otimes k}, L^{\otimes l})[d(2-2g-2k)]$$

whenever $g \geq 0$, $k \geq 1$, $l \geq 0$, and $2-2g-k-l < 0$.

*These are compatible with the sewing operations of framed Riemann surfaces. This is called a **structure of a positive boundary 2d TCFT**.*

Question 2.2. Can we understand the structure maps $\rho_{g,k,l}$ as a morphism of wheeled props?

Remark 2.3. These maps can be made explicit **up to sign** from [WW15]; i.e., their treatment is for **unshifted** Hochschild chains, which changes things in odd degrees. Note however that the degree shift of the maps is always even, so this has no bearing in the Z_2-graded case.

Remark 2.4. The **positive boundary condition** refers to the requirement $k \geq 1$. Corresponds to the fact that ribbon graphs must have at least one face. Also relevant becaues L is in general infinite-dimensional and thus cannot carry a non-degenerate bilinear pairing.

Question 2.5. In the discussion of [CT24], they mention that this is also related to the fact that L in general is infinite-dimensional, and thus cannot carry a non-degenerate bilinear pairing. Why is this related to the positive-boundary condition?

- Look at Wahl-Westerland for explicit model of what these maps and fatgraphs look like

3 Reducing to a map $\widehat{\mathfrak{g}}^{\mathrm{comb}} \to \widehat{\mathfrak{h}}_A$

3.1 Outline

The group $(S_k \times S_l) \ltimes (S^1)^{k+l}$ acts on both domain and target of the $\rho^A_{g,k,l}$, and we will take its homotopy quotient on both sides (after twisting by $\underline{\mathrm{sgn}}$ on the LHS). This will precisely recover the graded pieces of $\widehat{\mathfrak{g}}^{\mathrm{comb}}$ on the domain and $\widehat{\mathfrak{h}}_A$ on the target of the resulting induced maps.

Definition 3.1. Recall that

$$\widehat{\mathfrak{g}}^{\mathrm{comb}} = \left(\bigoplus_{\substack{g,k\geq 1\\ l\geq 0}} C^{\mathrm{comb}}_*(M^{\mathrm{fr}}_{g,k,l}, \underline{\mathrm{sgn}})_{hS} \right) [[\hbar, \lambda]][2].$$

Moreover, recall that

$$\widehat{\mathfrak{h}}_A = \bigoplus_{\substack{g,k\geq 1\\ l\geq 0}} \mathrm{Hom}^c(\mathrm{Sym}^k(L_+[1]), \mathrm{Sym}^l(L_-))[[\hbar, \lambda]].$$

Remark 3.2. The c-superscript refers to the requirement that the morphisms be continuous with respect to the topology on $\mathrm{Sym}^k(L_+[1])$ induced by the u-adic topology on $L_+ = L[[u]]$ and the discrete topology on $L_- = L[u^{-1}]$.

On $L[[u]]$, the neighbourhoods of zero in the u-adic topology are given by the ideals (u^n). Neighbourhoods of other points p are translations $p + U$ for U a neighbourhood of zero. The topology on the symmetric product is induced from the product/quotient topology.

Exercise 3.3. Characterize continuous functions with respect to these topologies.

3.2 Twisting by $\underline{\mathrm{sgn}}$

Recall that $\underline{\mathrm{sgn}}$ is the sign representation on the **marked inputs** $(p_1, \ldots, p_k)$. Concretely,

Definition 3.4. Consider for $k \geq 1$ the local system $\underline{\mathrm{sgn}} := \underline{sgn}_k[-k]$, which is the rank-one local system over $M^{\mathrm{fr}}_{g,k,l}$ whose fibre over

$$(\Sigma, \mathbf{p}^k_1, \mathbf{q}^l_1, \phi^k_1, \psi^l_1)$$

is the sign representation of $S_k \to \mathrm{Gl}_1(\mathbb{K})$, i.e., it sends σ to $\mathrm{sgn}(\sigma)$. One only permutes the $p_1, \ldots, p_k$. The vector space has a natural basis vector we denote by $p_1 \wedge \ldots, \wedge p_k$.

Twisting the maps $\rho^A_{g,k,l}$ by $\underline{\mathrm{sgn}}$ yields new maps

$$\begin{aligned} \rho^{A,\mathrm{tw}}_{g,k,l} : C^{\mathrm{comb}}_*(M^{\mathrm{fr}}_{g,k,l}, \underline{\mathrm{sgn}}) &\longrightarrow \mathrm{Hom}((L[1])^{\otimes k}, L^{\otimes l}) \\ \Gamma \otimes p_1 \wedge \ldots \wedge p_k &\longmapsto \rho^A_{g,k,l}(\Gamma) \circ s^{\otimes k}, \end{aligned}$$

where $\Gamma \in C^{\mathrm{comb}}_*(M^{\mathrm{fr}_{g,k,l}})$ and $s : L[1] \to L$ is the shift map

$$s(a) = (-1)^{|a|} a.$$

Observe that the $-k$-shift induced from tensoring by $\underline{\mathrm{sgn}}$ is compensated for by tensoring with $s^{\otimes k}$, so the maps have the same degree. One shows by a computation that the $\rho^{A,\mathrm{tw}}_{g,k,l}$ are still chain maps.

Exercise 3.5. Prove that the $\rho^{A,\mathrm{tw}}_{g,k,l}$ are chain maps.

Hint: [CT24, Section after Remark 6.4].

Question 3.6. Is the sign representation here related to Koszul duality of wheeled prop?

3.3 Homotopy Quotient

There is an action of $(S_k \times S_l) \ltimes (S^1)^{k+l}$ on both domain and target: On the domain, the action of $S_k \times S_l$ on $C^{\mathrm{comb}}_*(M^{\mathrm{fr}}_{g,k,l}, \underline{\mathrm{sgn}})$ is induced from that on $M^{\mathrm{fr}}_{g,k,l}$ by permuting labels of inputs and outputs, respectively. The $(S^1)^{k+l}$-factor acts by rotating the framing maps ϕ_i and ψ_j.

Question 3.7. What is the action on the target?

The maps descend to chain maps between the homotopy quotients on both sides:

$$\rho^{A,\mathrm{tw}}_{g,k,l} : C^{\mathrm{comb}}_*(M^{\mathrm{fr}}_{g,k,l}, \underline{\mathrm{sgn}})_{\mathrm{hS}} \to \mathrm{Hom}^c(\mathrm{Sym}^k(L_+[1]), \mathrm{Sym}^l L_-)$$

Question 3.8. Why is the homotopy quotient of the RHS equivalent to $\mathrm{Hom}^c(\mathrm{Sym}^k(L_+[1], \mathrm{Sym}^l(L_-)))$?

- Give description of this LHS space

Proposition 3.9 ([CT24, Proposition 6.7])**.** *The twisted maps $\{\rho^{A,\mathrm{tw}}_{g,k,l}\}$ induce a morphism of DGLA's*

$$\rho^{A,\mathrm{tw}} : \widehat{\mathfrak{g}}^{\mathrm{comb}} \to \widehat{\mathfrak{h}}_A.$$

On $\hbar$ and λ, $\rho^{A,\mathrm{tw}}$ acts as the identity.

Exercise 3.10. Show that any morphism of wheeled props induces a morphism of Lie algebras between their totalizations.

Question 3.11. Why is the adjective "twisted" important here?

4 Opaque definition of the CEI

4.1 The Maurer-Cartan $\beta^A \in \mathfrak{h}_A$

The CEI are obtained essentially as the image of the combinatorial string vertices under $\rho^{A,\mathrm{tw}}$. Recall:

Theorem 4.1. *There exists a degree -1-element $\widehat{\mathcal{V}}^{\mathrm{comb}} \in \widehat{\mathfrak{g}}^{\mathrm{comb}}$, unique up to homotopy, of the form*

$$\widehat{\mathcal{V}}^{\mathrm{comb}} = \sum_{\substack{g,k\geq 1\\ l\geq 0}} \widehat{\mathcal{V}}^{\mathrm{comb}}_{g,k,l} \hbar^g \lambda^{2g-2+k+l},$$

satisfying some conditions. The $\widehat{\mathcal{V}}^{\mathrm{comb}}_{g,k,l} \in C^{\mathrm{comb}}_*(M^{\mathrm{fr}}_{g,k,l}, \underline{\mathrm{sgn}})$ are called the ***combinatorial string vertices****, and they are $\mathbb{Q}$-linear combinations of ribbon graphs.*

Definition 4.2. Define

$$\widehat{\beta}^A := \rho^{A,\mathrm{tw}}(\widehat{\mathcal{V}}^{\mathrm{comb}}) = \sum_{g,k\geq 1, l\geq 0} \underbrace{\rho^{A,\mathrm{tw}}(\widehat{\mathcal{V}}^{\mathrm{comb}}_{g,k,l})}_{=:\widehat{\beta}^A_{g,k,l}} \hbar^g \lambda^{2g-2+k+l}.$$

The $\widehat{\beta}^A_{g,k,l} \in \mathrm{Hom}^c(\mathrm{Sym}^k(L_+[1]), \mathrm{Sym}^l L_-)$ will feature in the computation of the CEI. They depend only on A and not on any splitting s.

Definition 4.3. Recall that

$$\mathfrak{h}_A^+ = \left(\bigoplus_{k \geq 1} \mathrm{Sym}^k L_- \right) [[\hbar, \lambda]][1],$$

i.e., all strictly positive symmetric powers L_-. The scalar part $\mathbb{K}[[\hbar, \lambda]][1]$ is in the centre of $\mathfrak{h}_A$, so $\mathfrak{h}_A^+$ should be a quotient of $\mathfrak{h}_A$, and there is a short exact sequence

$$0 \to \mathbb{K}[[\hbar, \lambda]][1] \to \mathfrak{h}_A \to \mathfrak{h}_A^+ \to 0.$$

The sequence splits, whence

$$\mathfrak{h}_A = \mathbb{K}[[\hbar, \lambda]][1] \oplus \mathfrak{h}_A^+.$$

They construct a quasi-isomorphism

$$\begin{aligned} \bar{\imath} : \mathfrak{h}_A^+ &\to \widehat{\mathfrak{h}}_A \\ \alpha &\mapsto (-1)^{|\alpha|} \imath(\alpha); \end{aligned}$$

For this, consider the map $\imath$ defined by

$$\begin{aligned} \imath : \mathrm{Hom}^c(\mathrm{Sym}^k(L_+[1]), \mathrm{Sym}^l L_-) &\to \mathrm{Hom}^c(\mathrm{Sym}^{k+1}(L_+[1]), \mathrm{Sym}^{l-1} L_-) \\ \imath(\Phi)(\beta_1 \cdots \beta_{k+1}) &:= \sum_{j=1}^{k+1} \pm C_{\beta_j} \Phi(\beta_1 \cdots \widehat{\beta}_j \cdots \beta_{k+1}). \end{aligned}$$

The C_{β_j} in turn are defined, for $\beta \in L_+[1]$, as contraction operators $C_\beta : \mathrm{Sym}^l L_- \to \mathrm{Sym}^{l-1} L_-$ by

$$C_\beta(\alpha_1 \cdots \alpha_l) = \sum_{i=1}^{l} \pm \langle u\beta, \alpha_i \rangle_{\mathrm{res}} \cdot \alpha_1 \cdot \widehat{\alpha}_i \cdots \alpha_l.$$

To obtain a map $\imath : \mathfrak{h}_A^+ \to \widehat{\mathrm{h}}_A$, consider the sequence

$$\mathrm{Sym}^{k+l}(L_-) = \mathrm{Hom}^c(\mathbb{K}, \mathrm{Sym}^{k+l} L_-) \to \mathrm{Hom}^c(\mathrm{Sym}^1(L_+[1]), \mathrm{Sym}^{k+l-1}(L_-)) \to \cdots \to \mathrm{Hom}^c(\mathrm{Sym}^k(L_+[1]), \mathrm{Sym}^l(L_-)),$$

where the horizontal map is always $\imath$ as defined above. Then define $\imath : \mathfrak{h}_A^+ \to \widehat{\mathfrak{h}}_A$ to be the sum

Proposition 4.4 ([CT24, Lemma 4.4])**.** *The map $\bar{\imath}$ is a quasi-isomorphism.*

Definition 4.5. Since $\bar{\imath}$ is a quasi-isomorphism, there exists a Maurer-Cartan element $\beta^A \in \mathfrak{h}_A^+$ defined by the property that $\bar{\imath}(\beta^A)$ is gauge equivalent to $\widehat{\beta}^A$.

From the decomposition

$$\mathfrak{h}_A = \mathbb{K}[[\hbar, \lambda]][1] \oplus \mathfrak{h}_A^+,$$

we may take β^A to live in $\mathfrak{h}_A$ by taking the $\mathbb{K}[[\hbar, \lambda]][1]$-component to be zero (it has to be checked separately that this prescription still satisfies the Maurer-Cartan equation, which is done in [CT24, Lemma 7.2]).

Exercise 4.6. Make a formal statement from the previous definition, and give a proof.

Question 4.7. Why is it necessary to introduce $\mathfrak{h}_A^+$, when one already has a definition of the string vertex via the maps from $\widehat{\mathfrak{g}}^{\mathrm{comb}}$?

4.2 Abstract descendent potentials $\mathcal{D}$

One can relate Maurer-Cartan elements to certain closed elements in a completion of the Weyl algebra. We introduce this completion.

Definition 4.8. The **Weyl algebra** is here defined to be

$$\mathcal{W}(L^{\text{Tate}}) = \left(\bigoplus_{n\geq 0}(L^{\text{Tate}})^{\otimes n}\right)[[\hbar, \lambda]]/\sim,$$

quotiented by the relation

$$\alpha \otimes \beta - (-1)^{|\alpha||\beta|}\beta \otimes \alpha \sim \hbar \langle \alpha, \beta\rangle_{\text{res}}.$$

The variables $\hbar$ and λ are of even degree.

Next, take the completion in the "λ-adic topology" and localize at $\hbar$ to get

$$\widehat{\mathcal{W}}_{\hbar}(L^{\text{Tate}}) = \lim_{\overleftarrow{n}} \mathcal{W}(L^{\text{Tate}})[\hbar^{-1}]/(\lambda^n).$$

Similarly denote

$$\widehat{\text{Sym}}_{\hbar}(L_-) = \left(\bigoplus_{n\geq 0}\text{Sym}^n(L_-)\right)[[\hbar^{\pm}]]((\lambda))/\sim,$$

for the analogous equivalence relation.

In these completed localizations, we can take formal exponentials of any element:

$$\exp(\beta) = \sum_{n\geq 0}\frac{\beta^n}{n!}.$$

Lemma 4.9 ([CT24, Lemma 6.11])**.** *An element $\beta \in \lambda\mathfrak{h}_A[1]$ of even degree satisfies the Maurer-Cartan equation*

$$(b + uB + \hbar\Delta)\beta + \frac{1}{2}\{\beta, \beta\} = 0$$

if and only if $\mathcal{D} = \exp(\beta/\hbar) \in \widehat{\text{Sym}}_{\hbar}(L_-)[[\hbar, \lambda]]$ is $(b + uB + \hbar\Delta)$-closed.

Moreover, two such MC-elements $\beta_{1,2}$ are gauge equivalent iff the corresponding elements $\mathcal{D}_{1,2}$ are homologous. All identities here hold in $\widehat{\text{Sym}}_{\hbar}L_-[[\hbar, \lambda]]$.

Exercise 4.10. Give a proof of this Lemma.

Recall that the bracket here measures the failure of Δ to be a derivation:

$$\{x, y\} := \Delta(xy)(\Delta x)y - (-1)^{|x|}x(\Delta y).$$

Remark 4.11. Equivalently, $\mathfrak{h}_A$ is a BV algebra?!

Thus by applying this lemma to the MC-element $\beta_A \in \mathfrak{h}_A$, we obtain a $(b + uB + \hbar\Delta)$-closed element

$$\exp(\beta^A/\hbar) \in \widehat{\text{Sym}}_{\hbar}L_-[[\hbar, \lambda]].$$

Denote its homology class by

$$\mathcal{D}^A_{\text{abs}} \in H_*(\widehat{\text{Sym}}_{\hbar}L_-[[\hbar, \lambda]], \quad b + uB + \hbar\Delta).$$

This element depends only on A since the string vertex is unique up to homotopy.

4.3 Enter the splitting

It turns out that the object $\mathcal{D}^A_{\mathrm{abs}}$ lives in the homology of the right space, but with respect to the differential $b + uB + \Delta$ instead of just b. One can thus use a splitting to construct an isomorphism

$$H_*(\widehat{\mathrm{Sym}}_\hbar L_-[[\hbar,\lambda]], b + uB + \hbar\Delta) \cong H_*(\widehat{\mathrm{Sym}}_\hbar L_-[[\hbar,\lambda]], b) = \widehat{\mathrm{Sym}}_\hbar H_-[[\hbar,\lambda]].$$

Lemma 4.12. *A splitting $s : H_*(L) \to H_*(L_+)$ as above extends to an isomorphism of symplectic vector spaces*

$$s : H_*(L)^{\mathrm{Tate}} \to H_*(L^{\mathrm{Tate}})$$

(the symplectic form is the alternating nondegenerate residue pairing $\langle \cdot,\cdot\rangle_{\mathrm{res}}$).

This in turn induces an isomorphism on the Weyl algebras

$$\Phi^s : \widehat{\mathcal{W}}_\hbar(H_*(L)^{\mathrm{Tate}}) \to \widehat{\mathcal{W}}_\hbar(H_*(L^{\mathrm{Tate}})).$$

Exercise 4.13. Prove this.

Now we have maps

$$\begin{array}{ccc} \widehat{\mathcal{W}}_\hbar(H_*(L)^{\mathrm{Tate}}) & \xrightarrow{\Phi^s} & \widehat{\mathcal{W}}_\hbar(H_*(L^{\mathrm{Tate}})) = H_*(\widehat{\mathcal{W}}_\hbar(L^{\mathrm{Tate}})) \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle p} \\ \widehat{\mathrm{Sym}}_\hbar H_*(L)_-[[\hbar,\lambda]] & \xrightarrow{\Psi^s} & H_*(\widehat{\mathrm{Sym}}_\hbar L_-[[\hbar,\lambda]]) \end{array}$$

The vertical map i is the inclusion, and p is a quotient map ("canonical quotient map to Fock space"). The lower horizontal map is defined to be the composition of the other three. Then $\mathcal{D}^A_{\mathrm{abs}}$ lives in the lower-right corner.

Question 4.14. Why is there such an inlusion map i? What is this canonical quotient map?

Definition 4.15 (The CEI)**.** The **total descendent potential** $\mathcal{D}^{A,s} \in \widehat{\mathrm{Sym}}_\hbar H_*(L)_-[[\hbar,\lambda]]$ of the pair (A, s) is

$$\mathcal{D}^{A,s} = (\Psi^s)^{-1}(\mathcal{D}^A_{\mathrm{abs}}).$$

Then the CEI $F^{A,s}_{g,n} \in \mathrm{Sym}^n H_*(L)_-$ are defined by the identity

$$\sum_{g,n} F^{A,s}_{g,n} \hbar^{g-1} \lambda^{2g-2+n} = \ln(\mathcal{D}^{A,s}) =: F^{A,s}.$$

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